Asymptotic properties of modular type objects

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Outline

1. Modular forms

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2. Mock modular forms

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Modularity

Definition:

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight $k \in \mathbb{Z}$ if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

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Weakly holomorphic: f holomorphic, linear exponential growth at cusps

Notation: $M_k^!$

Fourier expansion

Holomorphic: bounded at cusps
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Fourier expansion ($q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$)

$$f(\tau) = \sum_{n\in\mathbb{Z}} c_f(n) q^n$$

Examples

- Dedekind η-function:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$$

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- Theta function:

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$$

Identity of Gauss

Write

$$\Theta(\tau)^3 =: \sum_{n\geq 0} r(n)q^n.$$

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Then

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with the Hurwitz class numbers

$$H(n) := \#\{\text{equivalence classes of integral binary quadratic forms of discriminant } n\}.$$

Partitions

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Generating function: (Euler)

$$P(q) := \sum_{n \geq 0} p(n) q^n = \prod_{n \geq 1} \frac{1}{1-q^n}$$

L. Euler

Example

Example: Partitions of 4

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

so $p(4) = 5$.

Further values:

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$$p(50) = 204226$$

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$$p(50) = 204226$$

$$p(100) = 190569292$$

Fibonacci numbers

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but

$$p(5) = 7 \qquad\qquad F_6 = 8$$

L. Fibonacci

Pentagonal Number Theorem

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$$\prod_{n\geq 1}(1-q^n)=\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n-1)}{2}}$$

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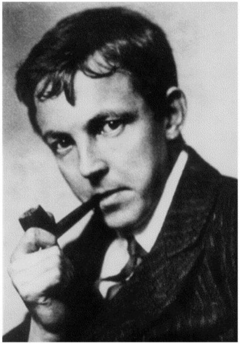
Recursion:

$$\begin{aligned} p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)\\ +p(n-12)+p(n-15)-p(n-22)-\dots \end{aligned}$$

Asymptotic behavior

Asymptotic behavior: (Hardy–Ramanujan)

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \qquad (n \to \infty)$$



G. Hardy

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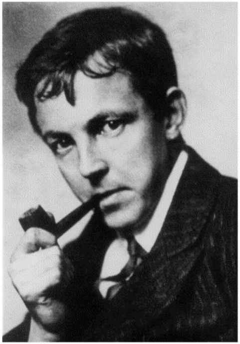
$$A_k(n) := \sum_{h \pmod{k}^*} \omega_{h,k} e^{-\frac{2\pi i n h}{k}}$$



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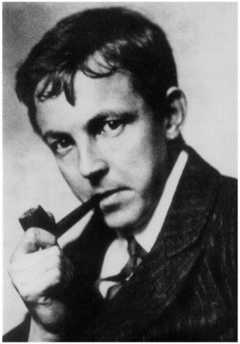

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Exact formula

Bessel function of order α:

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Rademacher formula:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k\geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

H. Rademacher

Idea of proof

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$$a(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{A(q)}{q^{n+1}} dq$$

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$\mathcal{C}$ path inside unit circle, surrounding 0 counterclockwise.
If $A(q)$ is modular can approximate it near roots of unity.

General statement

Kloosterman sums:

$$K(m,n) = \sum_{d \pmod{c}^*} e^{\frac{2\pi i}{c}\left(m\overline{d}+nd\right)}$$

with $d\overline{d} \equiv 1 \pmod{c}$

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Theorem (Rademacher–Zuckerman)

Let $f(\tau) = \sum_n c_f(n) q^n \in M_k^! \ (k \in -2\mathbb{N}_0)$. Then for $n \in \mathbb{N}$

$$c_f(n) = 2\pi(-1)^{\frac{k}{2}} \sum_{m \leq -1} c_f(m) \left(\frac{|m|}{n}\right)^{\frac{1-k}{2}} \sum_{c \geq 1} \frac{K(m,n;c)}{c} I_{1-k}\left(\frac{4\pi\sqrt{|m|n}}{c}\right).$$

Asymptotic main term

Corollary

$$c_f(n) \sim \frac{|n_0|^{\frac{1}{4}-\frac{k}{2}}}{2\sqrt{2}\pi} n^{\frac{k}{2}-\frac{3}{4}} e^{4\pi\sqrt{|n_0|n}},$$

where $n_0 < 0$ *is minimal with* $c_f(n_0) \neq 0$.

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Proof Use $I_\ell(x) \sim \frac{e^x}{\sqrt{2\pi x}}$ (as $x \to \infty$).

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$F : \mathbb{H} \to \mathbb{C}$ real-analytic is a weight k harmonic Maass form if it is modular of weight k and

J. Bruinier

J. Funke

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$F : \mathbb{H} \to \mathbb{C}$ real-analytic is a weight k harmonic Maass form if it is modular of weight k and

$$\Delta_k(F) = 0$$

with $(\tau = \tau_1 + i\tau_2)$

$$\Delta_k := -\tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) + ik\tau_2 \left(\frac{\partial}{\partial \tau_1} + i \frac{\partial}{\partial \tau_2} \right)$$

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plus growth condition

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Notation: $H_k^!$

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Examples

- weight 2 Eisenstein series:

$$\widehat{E}_2(\tau) := \underset{\substack{\uparrow \\ \text{quasimodular form}}}{E_2}(\tau) - \frac{3}{\pi\tau_2}$$

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- Class number generating function:

$$\widehat{\mathcal{H}}(\tau) := \sum_{\substack{n \geq 0 \\ n \equiv 0,3 \pmod 4}} H(n) q^n + \frac{i}{8\sqrt{2}\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta(w)}{(-i(\tau + w))^{\frac{3}{2}}} dw$$

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Natural splitting

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$$\underset{\substack{\uparrow\\ \text{holomorphic}\\ \text{part}\\ \text{(mock modular form)}}}{F = F^+} + \underset{\substack{\uparrow\\ \text{non-holomorphic}\\ \text{part}}}{F^-}$$

with

$$F^+(\tau) := \sum_{n \gg -\infty} c_F^+(n) q^n$$

$$F^-(\tau) := \sum_{n \geq 1} c_F^-(n) \Gamma(k-1; 4\pi |n| \tau_2) q^n$$

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Alternative representation

The non-holomorphic part has the shape

$$\int_{-\bar{\tau}}^{i\infty} f(w)(\tau + w)^{2-k} dw.$$

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$\uparrow$

modular, weight $2-k$ (shaddow)

Alternative representation

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$$\int_{-\bar{\tau}}^{i\infty} \underset{\substack{\uparrow \\ \text{modular, weight } 2-k \text{ (shaddow)}}}{f(w)} (\tau + w)^{2-k} dw.$$

ξ-operator: $\xi_k := 2i\tau_2^k \overline{\frac{\partial}{\partial\overline{\tau}}} \qquad H_k \to M_{2-k}$

Ramanujan's last letter

"I am extremely sorry for not writing you a single letter up to now. I recently discovered very interesting functions which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions they enter into mathematics as beautifully as the theta functions. I am sending you with this letter some examples."

S. Ramanujan

Mock theta functions

These mock theta functions are 22 peculiar q-series.

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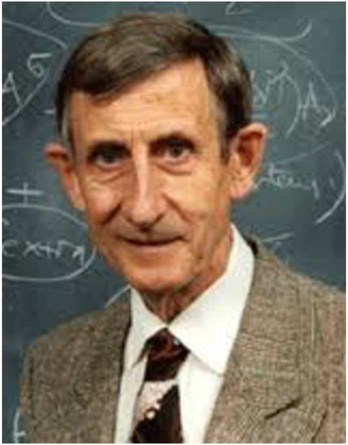
Example:

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = \sum_{n\geq 0} \alpha(n) q^n$$

with

$$(a;q)_n = (a)_n := \prod_{m=0}^{n-1} (1 - aq^m)$$

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains a challenge for the future..."



F. Dyson

Mock modularity of $f(q)$

Theorem (Zwegers)

The function $f(q)$ is a mock modular form.

S. Zwegers

Asymptotics for $f(q)$

Ramanujan's claim:

$$\alpha(n) \sim \frac{(-1)^{n+1}}{2\sqrt{n}} e^{\pi\sqrt{\frac{n}{6}}} \qquad (n \to \infty)$$

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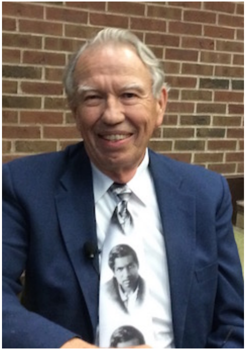

G. Andrews

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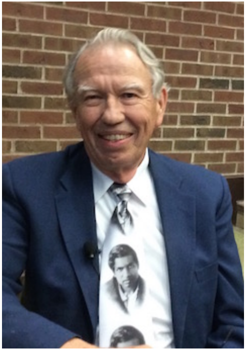
Key: Bound

$$J(\alpha) := \int_0^\infty \frac{\sinh(\alpha t)}{\sinh\left(\frac{3\alpha t}{2}\right)} e^{-\frac{3\alpha t^2}{2}} dt \quad (\mathrm{Re}(\alpha) > 0).$$



G. Andrews

Andrews–Dragonette Conjecture:

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k\geq 1} \frac{(-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor}}{k} A_{2k}\left(n - \frac{k\left(1+(-1)^k\right)}{4}\right) \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12n}\right)$$

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K. Ono

Theorem 1 (B.–Ono)

The Andrews-Dragonette Conjecture is true.

A general formula

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Idea of proof: For simplicity assume $k \in -2\mathbb{N}_0$. Let $F \in H_k$.

A general formula

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Poincaré series: $m \in -\mathbb{N}$

$$F_{k,m} :\doteq \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \phi_{k,m}\big|_k \gamma,$$

Idea of proof

where

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

$$f\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right),$$

$$\phi_{k,m}(\tau) := \tau_2^{-\frac{k}{2}} M_{-\frac{k}{2}, \frac{1-k}{2}}(4\pi|m|\tau_2) e^{2\pi i m \tau_1}$$

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Proof cont.

Let

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$$F = G.$$

Fourier expansion:

$$F_{k,m}(\tau) = \left(1 - \frac{\Gamma(1-k; 4\pi|m|\tau_2)}{\Gamma(1-k)}\right) q^m + \sum_{n\geq 0} b_m^+(n) q^n + \sum_{n\geq 1} b_m^-(n)\Gamma(1-k; 4\pi n\tau_2) q^{-n}$$

Proof cont.

Let

$$G(\tau) := \sum_{m<0} c_F^+(m) F_{k,m}(\tau).$$

Then

$$F = G.$$

Fourier expansion:

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In particular for $m \in \mathbb{N}$

$$b_m^+(n) = 2\pi(-1)^{1-\frac{k}{2}} \left|\frac{m}{n}\right|^{\frac{1-k}{2}} \sum_{c\geq 1} \frac{K(m,n;c)}{c} I_{1-k}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

Outline

1. Modular forms
2. Mock modular forms
3. Mixed mock modular forms
4. (Mixed) False theta functions
5. Meromorphic modular forms
6. Non-modular forms

Mixed mock modular forms

Definition: A mixed harmonic Maass form has the shape

$$H = \sum_j \underbrace{f_j}_{\in M_k^!} \underbrace{F_j}_{\in H_\ell^!}.$$

Mixed mock modular forms

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Mixed mock modular form: holomorphic part

$$H^+ := \sum_j f_j F_j^+$$

Algebraic geometry: an example

Recall

$$\mathcal{H}(\tau) = \sum_n H(n) q^n.$$

Generating function for Euler numbers:

$$F(\tau) := \sum_{n \geq 0} \beta(n) q^n := \frac{\mathcal{H} \big| U_4(\tau)}{\eta(\tau)^6},$$

where for $f(\tau) = \sum_n a(n) q^n$ the U-operator is

$$f \big| U_\ell(\tau) := \sum_n a(\ell n) q^n$$

Notation

Notation:

For $k \in \mathbb{N}$, $g \in \mathbb{Z}$, $t \in \mathbb{R}$

$$f_{k,g}(t) := \begin{cases} \frac{\pi^2}{\sinh^2\left(\frac{\pi t}{k} - \frac{\pi i g}{2k}\right)} & \text{if } 2k \nmid g, \\ \\ \frac{\pi^2}{\sinh^2\left(\frac{\pi t}{k}\right)} - \frac{k^2}{t^2} & \text{if } 2k \mid g. \end{cases}$$

Notation (cont.)

Kloosterman sums:

$$\mathcal{K}_\ell(n,m;k) := \sum_{h \pmod{k}^*} \psi_\ell\left(h,h',k\right) e^{-\frac{2\pi i}{k}\left(hn+\frac{h'n}{4}\right)}$$

with $hh' \equiv -1 \pmod{k}$

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with $hh' \equiv -1 \pmod{k}$

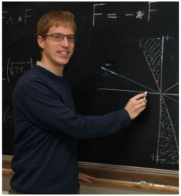
Bessel function integral:

$$\mathcal{I}_{k,g}(n) := \int_{-1}^{1} f_{k,g}\left(\frac{t}{2}\right) I_{\frac{7}{2}}\left(\frac{\pi}{k}\sqrt{(4n-1)\left(1-t^2\right)}\right)\left(1-t^2\right)^{\frac{7}{4}} dt$$

An exact formula

Theorem 3 (B.–Manschot)

$$\beta(n) = -\frac{\pi}{6(4n-1)^{\frac{5}{4}}} \sum_{k\geq 1} \frac{\mathcal{K}_0\left(n,0;k\right)}{k} I_{\frac{5}{2}}\left(\frac{\pi}{k}\sqrt{4n-1}\right)$$



J. Manschot

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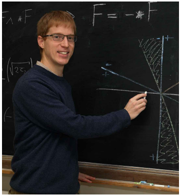
$$+ \frac{1}{\sqrt{2}(4n-1)^{\frac{3}{2}}} \sum_{k\geq 1} \frac{\mathcal{K}_0\left(n,0;k\right)}{\sqrt{k}} I_3\left(\frac{\pi}{k}\sqrt{4n-1}\right)$$



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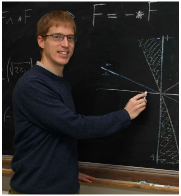
$$- \frac{1}{8\pi(4n-1)^{\frac{7}{4}}} \sum_{k\geq 1} \sum_{\substack{\ell\in\{0,1\} \\ -k<g\leq k \\ g\equiv \ell \pmod 2}} \frac{\mathcal{K}_\ell\left(n,g^2;k\right)}{k^2} \mathcal{I}_{k,g}(n)$$



J. Manschot

Sketch of proof

Circle Method.

Sketch of proof

Circle Method.

Bound integrals of the shape

$$\mathcal{I}_{k,g,b}(w) := e^{\frac{2\pi b}{kw}} w^{\frac{5}{2}} \int_{-\infty}^{\infty} f_{k,g}(t) e^{-\frac{2\pi t^2}{kw}} dt.$$

Negligibable for $b \leq 0$. "Principal part integrals" for $b > 0$.

Main term

Corollary

We have as $n \to \infty$

$$\beta(n) = \left(\frac{1}{96} n^{-\frac{3}{2}} - \frac{1}{32\pi} n^{-\frac{7}{4}} + O\left(n^{-2}\right) \right) e^{2\pi\sqrt{n}}.$$

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Rogers false theta functions

Wrong sgn-factors prevent modularity.

Rogers false theta functions

Wrong sgn-factors prevent modularity.

Example:

$$\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\left(n+\frac{1}{2}\right)^2}$$

Unimodal sequences

Definition:

Sequence $\{a_j\}_{j=1}^s$ with

$$a_1 \leq a_2 \leq \ldots \leq \underset{\substack{\uparrow \\ \text{peak}}}{a_k} \geq a_{k+1} \geq \ldots \geq a_s$$

and $a_1 + \ldots + a_s = n$ is a unimodal sequence (stack).

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and $a_1 + \ldots + a_s = n$ is a unimodal sequence (stack).

Let

$$u(n) := \#\text{unimodal sequences of size } n.$$

Generating function

Generating function:

$$U(q) := \sum_{n\geq 0} u(n)q^n = \frac{1}{(q;q)_\infty^2} \sum_{n\geq 1} (-1)^{n+1} q^{\frac{n(n+1)}{2}}$$

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Define

$$\psi(\tau) := i \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}.$$

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Note:

$$U(q) = \frac{i}{2} q^{-\frac{1}{2}} \frac{\psi(\tau)}{\eta(\tau)^2} + \frac{q^{\frac{1}{12}}}{\eta(\tau)^2}$$

Completion

Idea:

"Complete" ψ to obtain a function transforming like a modular form.

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Completion: $w \in \mathbb{H}$

$$\widehat{\psi}(\tau, w) := i \sum_{n \in \mathbb{Z}} \mathrm{erf}\left(-i\sqrt{\pi i(w-\tau)}\left(n+\frac{1}{2}\right)\right)(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}$$

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$\uparrow$
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$$\widehat{\psi}(\tau, w) := i \sum_{n\in\mathbb{Z}} \underset{\substack{\uparrow\\ \text{error function}}}{\operatorname{erf}} \left(-i\sqrt{\pi i(w-\tau)}\left(n+\frac{1}{2}\right)\right)(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}$$

Note that for $\varepsilon > 0$

$$\lim_{t\to\infty} \widehat{\psi}(\tau, \tau + it + \varepsilon) = \psi(\tau).$$

Modularity

Remark

Motivation for changing sgn into error function taken from false theta functions.

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Theorem 4 (B.–Nazaroglu)

The function $\widehat{\psi}$ transforms like a modular form.

C. Nazaroglu

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Theorem 4 (B.–Nazaroglu)

The function $\widehat{\psi}$ transforms like a modular form.

C. Nazaroglu

Sketch of proof:

Poisson summation.

Asymptotics: (Auluck, Wright)

$$u(n) \sim \frac{1}{8 \cdot 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{2\pi\sqrt{\frac{n}{3}}} \qquad (n \to \infty)$$

Asymptotics

Asymptotics: (Auluck, Wright)

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Notation: $K_k(n)$, $K_k(n, r)$ Kloosterman sums

An exact formula

Theorem 5 (B.–Nazaroglu)

We have

$$u(n) = \frac{2\pi}{12n-1} \sum_{k\geq 1} \frac{K_k(n)}{k} I_2\left(\frac{\pi}{3k}\sqrt{12n-1}\right)$$

An exact formula

Theorem 5 (B.–Nazaroglu)

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$$u(n) = \frac{2\pi}{12n-1} \sum_{k\geq 1} \frac{K_k(n)}{k} I_2\left(\frac{\pi}{3k}\sqrt{12n-1}\right)$$

$$- \frac{\pi}{2^{\frac{3}{4}}\sqrt{3}(24n+1)^{\frac{3}{4}}} \sum_{k\geq 1} \sum_{r \pmod{2k}} \frac{K_k(n,r)}{k^2}$$

$$\times \int_{-1}^{1} \left(1-x^2\right)^{\frac{3}{4}} \cot\left(\frac{\pi}{2k}\left(\frac{x}{\sqrt{6}} - r - \frac{1}{2}\right)\right) I_{\frac{3}{2}}\left(\frac{\pi}{3\sqrt{2}k}\sqrt{\left(1-x^2\right)\left(24n+1\right)}\right) dx.$$

Idea of proof

Write

$$U(q) = -q^{-\frac{1}{24}} f(\tau) + q^{\frac{1}{12}} g(\tau)$$

with

$$f(\tau) := -\frac{i}{2} \frac{\psi(\tau)}{\eta(\tau)^2}, \quad g(\tau) := \frac{1}{\eta(\tau)^2}.$$

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Set for $\varrho \in \mathbb{Q}$

$$\mathcal{E}_\varrho(\tau) := \int_\varrho^{\tau+i\infty+\varepsilon} \frac{\eta(z)^3}{\sqrt{i(z-\tau)}} dz,$$

where the integration path avoids the branch-cut.

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Lemma

We have, for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$,

$$f(\tau) = e^{\frac{\pi i}{4}} \nu_\eta(M)^{-1} \sqrt{-i(c\tau+d)} \left(f\left(\tfrac{a\tau+b}{c\tau+d}\right) - \frac{1}{2} g\left(\tfrac{a\tau+b}{c\tau+d}\right) \mathcal{E}_{\frac{a}{c}}\left(\tfrac{a\tau+b}{c\tau+d}\right)\right).$$

Mordel-type integrals

For $\mathrm{Re}(V) > 0$

$$\mathcal{E}_\varrho\left(\varrho + iV\right) = -\frac{i}{\pi}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\left(n+\frac{1}{2}\right)^2\varrho}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{e^{-\pi Vx^2}}{x-\left(n+\frac{1}{2}\right)(1+i\varepsilon)}dx.$$

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Write

$$e^{2\pi dV}\mathcal{E}_{\frac{h'}{k}}\left(\frac{h'}{k}+iV\right) = \mathcal{E}^{*}_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) + \mathcal{E}^{\mathrm{e}}_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right),$$

Mordel-type integrals (cont.)

where

$$\mathcal{E}^{*}_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) := -\frac{i}{\pi}e^{2\pi dV}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\left(n+\frac{1}{2}\right)^2\frac{h'}{k}}$$

$$\times\lim_{\varepsilon\to 0^+}\int_{-\sqrt{2d}}^{\sqrt{2d}}\frac{e^{-\pi Vx^2}}{x-\left(n+\frac{1}{2}\right)(1+i\varepsilon)}dx,$$

$$\mathcal{E}^{\mathrm{e}}_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) := -\frac{i}{\pi}e^{2\pi dV}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\left(n+\frac{1}{2}\right)^2\frac{h'}{k}}$$

$$\times\lim_{\varepsilon\to 0^+}\int_{|x|\geq\sqrt{2d}}\frac{e^{-\pi Vx^2}}{x-\left(n+\frac{1}{2}\right)(1+i\varepsilon)}dx.$$

Mordel-type integrals (cont.)

Error bounds: For $0 \leq d < \frac{1}{8}$

$$\mathcal{E}^{\mathrm{e}}_{\frac{h'}{k},d}\left(\frac{h'}{k} + iV\right) = O\left(\log(k)\right).$$

Mordel-type integrals (cont.)

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Write

$$\mathcal{E}^{*}_{\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) = -\frac{ie^{2\pi dV}}{2k}\sum_{r \pmod{2k}}(-1)^r e^{\pi i\left(r+\frac{1}{2}\right)^2\frac{h'}{k}} \times \int_{-\sqrt{2d}}^{\sqrt{2d}} \cot\left(\frac{\pi}{2k}\left(x-r-\frac{1}{2}\right)\right) e^{-\pi Vx^2}dx.$$

Enter the Circle Method.

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Ramanujan (Bialek)

For $z_2 \gg 1$:

$$\frac{1}{E_4(z)} = \sum_{n \geq 0} \beta_n e^{2\pi i n z}$$

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Here

$$\beta_n := (-1)^n \frac{3}{E_6\left(e^{\frac{\pi i}{3}}\right)} \sum_{\lambda} \sum_{(c,d)} \frac{h_{(c,d)}(n)}{\lambda^3} e^{\frac{\pi \sqrt{3} n}{\lambda}},$$

$\frac{1}{E_4}$ (cont.)

where

- $\lambda \in \mathbb{N}$ has the form
  $\lambda = 3^a \prod_{j=1}^{r} p_j^{a_j} \quad (a \in \{0,1\}, p_j = 6m+1 \text{ prime}, \; a_j \in \mathbb{N}_0),$

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- (c,d) distinct solution to $\lambda = c^2 - cd + d^2$,
- $h_{(1,0)}(n) := 1, \ h_{(2,1)}(n) := (-1)^n$, for $\lambda \geq 7$

$$h_{(c,d)}(n) := 2\cos\left((ad+bc-2ac-2bd+\lambda)\frac{\pi n}{\lambda} - 6\arctan\left(\frac{\sqrt{3}c}{2d-c}\right)\right).$$

Asymptotic main term

Asymptotic main term:

$$\beta_n \sim \frac{3(-1)^n}{E_6\left(e^{\frac{\pi i}{3}}\right)} e^{\pi\sqrt{3}n} \qquad (n \to \infty)$$

Very rapid growth!

Building blocks

Define

$$f_{k,j,r}(z) := \sum_{m\geq 0} \sum_{\mathfrak{b}\subseteq\mathbb{Z}[i]}{}^{*} \frac{c_{4k}(\mathfrak{b}, m)}{N(\mathfrak{b})^{\frac{k}{2}-j}} (4\pi m)^r e^{\frac{2\pi m}{N(\mathfrak{b})}} e^{2\pi imz},$$

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Meromorphic cusp forms: decay like cusp form

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Meromorphic cusp forms: decay like cusp form

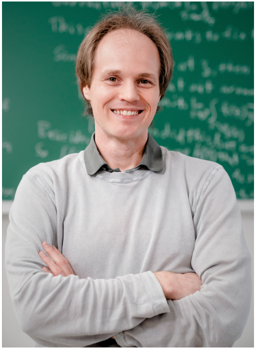
Notation: $\mathbb{S}_k$

General theorem

Theorem 6 (B.–Kane)

Let $f \in \mathbb{S}_{2-2k}$ with $k > 0$ with only pole in $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ in i. Then

$$f(z) \doteq \sum_{n \geq 0} a_n \sum_{j=0}^{n} \frac{(2k+n-1)!}{(2k+n-1-j)!} \times \binom{n}{j} f_{2k+2n,j,n-j}(z).$$



B. Kane

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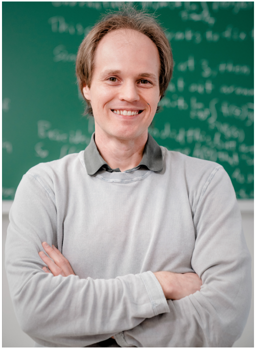

B. Kane

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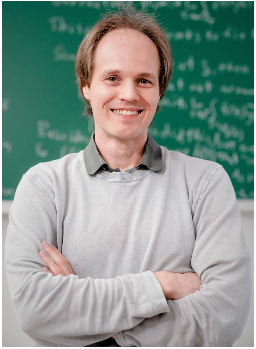

B. Kane

Remark

Similar for poles at other points.

Polar harmonic Maass forms

Definition:

$F : \mathbb{H} \to \mathbb{C}$ transforming modular of weight is a polar harmonic Maass form of weight k if:

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Polar harmonic Maass forms

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Notation: $\mathbb{H}_k$

Building blocks

Let

$$H_{2k}(\mathfrak{z}, z) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{1}{1 - e^{2\pi i (z - \mathfrak{z})}}\bigg|_{2k,\mathfrak{z}} \gamma,$$

$$\widehat{H}_{2k}(\mathfrak{z}, z) := H_{2k}(\mathfrak{z}, z) + \sum_{r=0}^{2k-2} \frac{(2iz_2)^r}{r!} \frac{\partial^r}{\partial \bar{z}^r} H_{2k}(\mathfrak{z}, \bar{z}).$$

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Lemma 7

(1) $\mathfrak{z} \mapsto \widehat{H}_{2k}(\mathfrak{z}, z) \in \mathbb{S}_{2k}$

(2) $z \mapsto \widehat{H}_{2k}(\mathfrak{z}, z) \in \mathbb{H}_{2-2k}$

A key lemma

Lemma 8

Let $F \in \mathbb{H}_{2-2k}, k < 0$ with $\xi_{2-2k}(F) \in S_k$. Then

$$F(z) = \sum_{n=0}^{n_\ell} a_n \left[R^n_{2k,\mathfrak{z}} \left(\mathcal{H}_{2k}(\mathfrak{z}, z) \right) \right]_{\mathfrak{z}=i},$$

where

$$R_{\kappa,\mathfrak{z}} := 2i \frac{\partial}{\partial \mathfrak{z}} + \frac{\kappa}{\mathfrak{z}}.$$

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$$F(z) = \sum_{n=0}^{n_\ell} \underset{\substack{\uparrow \\ \text{explicit}}}{a_n} \left[R^n_{2k,\mathfrak{z}}\left(\mathcal{H}_{2k}(\mathfrak{z}, z)\right)\right]_{\mathfrak{z}=i},$$

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Outline

1. Modular forms
2. Mock modular forms
3. Mixed mock modular forms
4. (Mixed) False theta functions
5. Meromorphic modular forms
6. Non-modular forms

Unimodal sequences

Recall:

Sequence $\{a_j\}_{j=1}^{s}$ with

$$a_1 \leq a_2 \leq \ldots \leq a_k \geq a_{k+1} \geq \ldots \geq a_s$$

and $a_1 + \ldots + a_s = n$ is a unimodal sequence (stack).

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

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Let

$$ss(n) := \#\text{shifted stacks of size } n.$$

Shifted stacks

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Thus $ss(4) = 3$.

Generating function

Auluck:

$$\mathcal{S}_s(q) := \sum_{n\geq 0} ss(n)q^n = 1 + \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q)_{n-1}^2(1-q^n)}$$

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(log) asymptotic behavior (Wright)

$$\log(ss(n)) \sim 2\pi\sqrt{\frac{n}{5}} \qquad\qquad (n\to\infty)$$

Generating function

Theorem 9 (B.–Mahlburg)

We have

$$ss(n) \sim \frac{\phi^{-1}}{2\sqrt{25}^{\frac{3}{4}}\, n} e^{2\pi\sqrt{\frac{n}{5}}} \qquad (n \to \infty)$$

with ϕ the Golden Ratio.

K. Mahlburg

Key idea of the proof

- Embed into the modular world:

$$\mathcal{S}_s(q) = 1 + \mathrm{CT}_{[\zeta]}\left(\sum_{r\geq 0}\frac{\zeta^{-r}q^{\frac{r^2-r}{2}}}{(q)_r}\sum_{m\geq 1}\frac{\zeta^m q^m}{(q)_{m-1}}\right)$$

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  CT is the constant term
- Use Jacobi triple product formula

$$\left(-\zeta^{-1}\right)\infty\left(-\zeta q\right)_\infty(q)_\infty = -q^{-\frac{1}{8}}\zeta^{-\frac{1}{2}}\vartheta\left(z+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right),$$

  where

$$\vartheta(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{\pi i n^2\tau + 2\pi i n\left(z+\frac{1}{2}\right)}.$$

Proof continued . . .

- Modular inversion:

$$\vartheta\left(z+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right)=i\sqrt{\frac{2\pi}{\varepsilon}}e^{-\frac{2\pi^2}{\varepsilon}\left(z+\frac{1}{2}\right)^2}\vartheta\left(\frac{2\pi\left(z+\frac{1}{2}\right)}{i\varepsilon};\frac{2\pi i}{\varepsilon}\right)$$

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$$\mathrm{Li}_2\left(e^{-B\varepsilon}\zeta;e^{-\varepsilon}\right)=\frac{1}{\varepsilon}\mathrm{Li}_2(\zeta)+\left(B-\frac{1}{2}\right)\log(1-\zeta)+O(\varepsilon).$$

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Ranks and cranks

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where

$$o(\lambda) := \#\text{ of 1s in } \lambda,$$

$$\mu(\lambda) := \#\text{parts } > o(\lambda).$$

Crank generating function

Let (basically)

$$M(m, n) := \# \text{ of partitions of } n, \text{ crank } m,$$
$$N(m, n) := \# \text{ of partitions of } n, \text{ rank } m.$$

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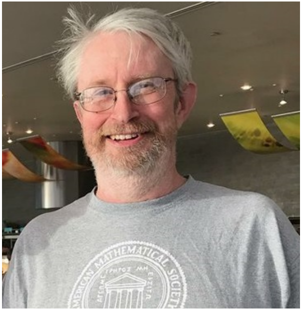
Andrews–Garvan:

$$C(\zeta;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} M(m,n)\zeta^m q^n$$
$$= \frac{1-\zeta}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-\zeta q^n}$$



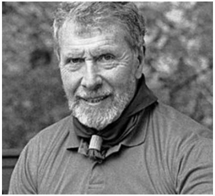
F. Garvan

Rank generating function

Atkin–Swinnerton-Dyer:

$$R(\zeta; q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} N(m,n)\zeta^m q^n$$

$$= \frac{1-\zeta}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-\zeta q^n}$$



A. Atkin

P. Swinnerton-Dyer

Moments

Crank and rank moments ($r \in \mathbb{N}_0$) (Atkin–Garvan)

$$M_r(n) := \sum_{m\in\mathbb{Z}} m^r M(m,n)$$

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Note: $N_{2r+1}(n) = M_{2r+1}(n) = 0$

Asymptotics and inequalities

Theorem 10 (B.–Mahlburg–Rhoades, Garvan)

(1) *As* $n \to \infty$

$$M_{2k}(n) \sim N_{2k}(n).$$

R. Rhoades

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Important tool: modularity (differentiate with respect to Jacobi variable)

Positive moments

Positive crank and rank moments ($r \in \mathbb{N}$) (Andrews–Chan–Kim)

$$M_r^+(n) := \sum_{m\in\mathbb{N}} m^r M(m,n)$$
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Note:

$$M_{2k}^+(n) = \frac{1}{2} M_{2k}(n)$$

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- Circle Method


Thank
You