Calabi Yau Manifolds

Arithmetic & modularity

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KITP: Workshop on modularity in quantum systems geometry of CY (and that of their moduli spaces)

physics

arithmetic of families of CY and modularity properties (of X_{φ_*})

eg • Black hole solns in string theory

↑ radial coord in space time

• entropy → L-values

$\boxed{1}$ CY manifolds

$\boxed{2}$ Arithmetic I

$\boxed{3}$ Arithmetic II:

singularities, mirror symmetry, modularity

$\boxed{4}$ Attractor mechanism and modularity

$\boxed{1}$ CY manifolds: compact Kähler with $c_1 = 0$

(This talk $d = 3$)

- CY manifolds have parameters X_φ

- It is a theorem that
  $\exists!$ (up to a constant) $(n,0)$-form Ω
  which is holomorphic $(d\Omega = 0)$

$$\dim H^{(n,0)} = \dim H^{(0,n)} = 1$$

Example: quintic in $\mathbb{P}^4$

$$P(X_\psi) = \sum_{i=1}^{5} x_i^5 - \psi^{-1} x_1 x_2 x_3 x_4 x_5$$

$\{x_i\}$ homogenous coords $\mathbb{P}^4$

$h^{11} = 1$ $\qquad$ $h^{21} = 101$

$b_3 = 2(1+101) = 204$

mirror quintic $\qquad h^{12} = 1 \qquad h^{11} = 101$

(quotient by a group and blow up) *

Example: Hulek + Verril

$$P(X_\varphi) = \left(\sum_{i=1}^{5} X_i\right)\left(\sum_{i=1}^{5} \frac{1}{X_i}\right) - \varphi^{-1}$$

$$\{X_1, \dots, X_5\} \in \mathbb{P}_4^*$$

Take the quotient by:

$$X_i \longrightarrow X_{i+1}, \quad X_i \longrightarrow \frac{1}{X_i}$$

(Blow up singularities where some X_i vanish to give a smooth CY)

Hodge numbers : HV

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 4k+1 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 4k+1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\boxed{b_3 = 4}$

$k=2 \quad G = \mathbb{Z}/5\mathbb{Z}$

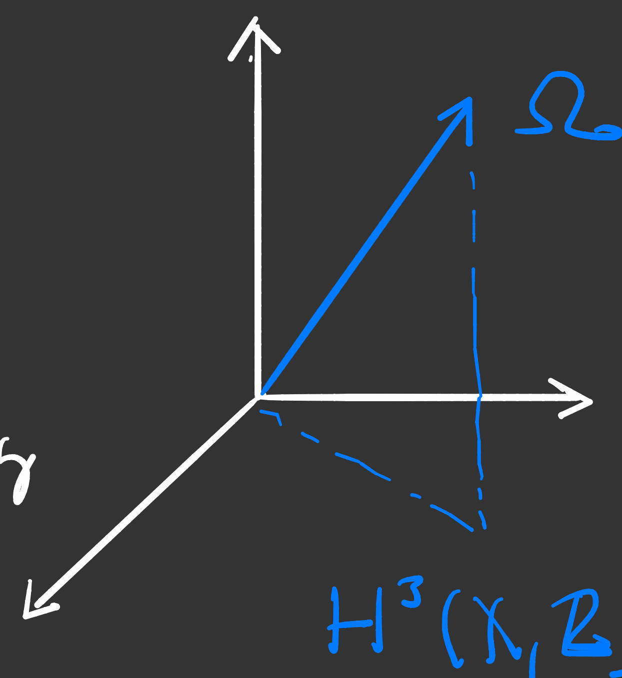
$k=1 \quad G = \mathbb{Z}/10\mathbb{Z}$

Periods and the complex structure

- there is a canonical way to give coordinates on the space of complex structures

Ω is defined up to a scale but is otherwise unique

$\Rightarrow$ defines a $\boxed{\text{line}}$ in $H^3(X,\mathbb{Z})$



- study the variations of the complex structure by studying how Ω varies in H^3
- The coordinates on this $\boxed{\text{line}}$ are the periods

That is:

$H^3(X, \mathbb{Z})$ $\quad (\alpha_a, \beta^b)$ symplectic basis

$H_3(X, \mathbb{Z})$ $\quad (A^a, B_b)$

$\int_X \alpha \wedge \beta = \delta$

So

$$\Omega = z^a \alpha_a - \mathcal{F}_a(z)\beta^a$$

Periods:

$$z^a(\varphi) = \int_{A^a} \Omega(\varphi)$$

homogeneous coords on moduli space

$$\mathcal{F}_a(\varphi) = \int_{B_a} \Omega(\varphi) = \partial_a \mathcal{F}$$

prepotential

* Periods determine the geometry of the moduli space

Fact: The periods satisfy a differential eq

Intuitive reason:

$$\Omega, \frac{\partial \Omega}{\partial \psi}, \frac{\partial^2 \Omega}{\partial \psi^2}, \frac{\partial^3 \Omega}{\partial \psi^3}, \frac{\partial^4 \Omega}{\partial \psi^4}, \ldots$$

all d-closed forms

But at most b_3 are linearly independent so there must be linear relations between the first b_3+1 $\Rightarrow$ system of linear differential equations of order b_3

$$\mathcal{L}\Omega \sim 0 \text{ in cohomology}$$

$b_3 = 4$ (one parameter families) $\deg \mathcal{L} = 4$

Since the periods are taken over a fixed basis

$$d\,\mathcal{Z}^a = 0, \qquad d\,\mathcal{F}_a = 0 \quad \text{(exactly)}$$

This system is known as the Picard-Fuchs eq.

- solutions around a singularity are series

$$\sum_{n=-r}^{\infty} a_n \varphi^n$$

expansions which end

and logarithmic solutions are allowed

That is: only regular singularities

$b_3 = 4$ $\qquad X_\varphi \quad , \quad P(\underline{X}, \varphi) = 0$

$$\mathcal{L} = S_4 \vartheta^4 + S_3 \vartheta^3 + S_2 \vartheta^2 + S_1 \vartheta + S_0 , \quad \vartheta = \varphi \frac{d}{d\varphi}$$

S_i = polynomials in φ

Mirror quintic : $\mathcal{L}$ has 3 regular singularities

$\varphi = 0$, $\quad \varphi = \infty \qquad \varphi = 1/5$

singular $\nearrow$ $\qquad$ isolated singularity

LCS L $\qquad$ "conifold"

$\boxed{\text{maximal unipotent monodromy}}$

X_φ is smooth for $\varphi \neq 0, 1/5$

$\boxed{\text{HV}}$ $S_4 = (\varphi - 1)(9\varphi - 1)(25\varphi - 1)$

5 singularities

$\varphi = 1, 1/9, 1/25$ conifold type

$\varphi = 0$ LCSL

X_φ is $\boxed{\underline{\text{smooth}}}$ for $\varphi \neq 1, 1/9, 1/25, 0, \infty$

$\boxed{12}$ Arithmetic I : The theta function

Let X_φ a family of algebraic varieties st X_φ is a CY hypersurface with defining polynomial $P(\underline{x}, \varphi)$

Let $\varphi \in \mathbb{Q}$

Questions :

* how many solutions of $P(\underline{x}, \varphi) = 0$ are there over $\mathbb{Q}$ (ie $x_i \in \mathbb{Q}$)
* how does this number vary with φ ?

TOO HARD !!!

It makes sense however to "reduce mod p"
(p a prime number)
(that is work over finite fields $\mathbb{F}_{p^k}$ $k=1,2,\dots$)
* We can still learn $\boxed{\text{a lot}}$

The fundamental quantities of interest are

N_k = number of solns of $P(\underline{x}, \varphi)=0$ over $\mathbb{F}_{p^k}$

Generating function

zeta-function

$$\zeta_X(T) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k \, T^k}{k}\right)$$

Note : N_k depends on p $\boxed{\text{and}}$ φ

"simple" case: X a point

$$N_k = 1 \qquad \forall k$$

$$\sum_{k=1}^{\infty} \frac{1}{k} N_k T^k = \sum_{k=1}^{\infty} \frac{1}{k} T^k = -\log(1-T)$$

$$\Rightarrow \quad \zeta_{\text{point}}(T) = \frac{1}{1-T}$$

Remark: $$\prod_p \zeta_{\text{point}}(p^{-s}) = \prod_p \frac{1}{1-p^{-s}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} = \underline{\underline{\zeta_{\mathbb{R}}(s)}}$$

enter: L-functions!

Weil Conjectures (theorems!) X alg variety

- $\zeta(T)$ is a rational function (Dwork)
- Functional equation (Groethendieck)

X smooth

$$\zeta\left(\frac{1}{p^d T}\right) = \pm\, p^{d\chi/2}\, T^{\chi}\, \zeta(T)$$

- Riemann hypothesis (Deligne)

$$\zeta(T) = \frac{R_1(T)\,R_3(T)\cdots R_{2d-1}(T)}{R_0(T)\,R_2(T)\cdots R_{2d}(T)}$$

$R_i(T)$: polynomial (with coeffs in $\mathbb{Z}$), $\deg R_i = b_i$

$R_0 = 1-T$, $R_{2d}(T) = 1-p^d T$, $R_i(T) = \prod_{i=1}^{b_i}(1-\alpha_{ij}T)$, $|\alpha_{ij}| = p^{i/2}$

CY
dim 1

Example: Elliptic curve $\mathcal{E}$

$$\rightarrow \quad \zeta(T) = \frac{1 + a_p T + p T^2}{(1-T)(1-pT)} \qquad \boxed{a_p = -N_1}$$

$p \neq$ prime of bad reduction

$\mathcal{E}$ associated to a modular form

Taniyama-Shimura conjecture
proof: Wiles, Breuil, Conrad, Diamond, Taylor

L function

$$\prod_p \zeta(p^{-s}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$a_n \to$ n-th coeff of a modular form of weight 2 of $\Gamma_0(N) \subset SL(2,\mathbb{Z})$

N = conductor (its prime factors ~ primes of bad reduction)

For a smooth CY 3-fold X

$b_1 = 0$
$b_5 = 0$

$$\zeta_X(T) = \frac{R_3(T)}{(1-T)\,R_2(T)\,R_4(T)\,(1-p^3T)}$$

$\deg R_3(T) = b_3$

$\deg R_2(T) = \deg R_4(T) = h^{11}$

For the mirror quintic & #V : $b_3 = 4$

For X_φ smooth

$$\zeta_X(T) = \frac{R_3(T)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)}$$

$$R_3(T) = 1 + a_p(\varphi)T + b_p(\varphi)pT^2 + a_p(\varphi)p^3T^3 + p^6T^4$$

$$a_p(\varphi) = -N_1(\varphi)$$ ← points over $\mathbb{F}_p$

$$2p\,b_p(\varphi) = N_1^2(\varphi) - N_2(\varphi)$$ ← points over $\mathbb{F}_{p^2}$

a, b can be computed in terms of the periods

$\boxed{\text{Frobenius}}$: $X_{/\mathbb{F}_p}$ alg variety over $\mathbb{F}_p$
with defining poly $P(\underline{x})$
with coeff over $\mathbb{F}_p$

Facts: $c^p \equiv c \mod p$ $(3^5 = 243 = 3 \mod 5)$

If $c \in \mathbb{F}_{p^k}$ $c^{p^k} \equiv c$

If $c_1, c_2 \in \mathbb{F}_{p^k}$ $(c_1 + c_2)^p \equiv c_1^p + c_2^p$

let $P(\underline{x}) = \sum c_m \underline{x}^{\underline{m}}$, $\underline{x}^{\underline{m}} = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$

with $c_m \in \mathbb{F}_p$, $x_i \in \mathbb{F}_{p^k}$

Thm $P(x) = 0 \Longleftrightarrow P(x)^p = 0$ $\boxed{x \leftrightarrow x^p \quad \text{Frob map}}$

$\Longleftrightarrow P(x^p) = 0$

$\uparrow$ $\underline{x}^p$ satisfies same eq as $\underline{x}$

* Frob is automorphism, fixed points = rational points over $\mathbb{F}_p$

Then $P(x) = 0 \Longleftrightarrow P(x^p) = 0$

$\hookrightarrow$ $\underline{x}^p$ satisfies same eq as $\underline{x}$

$x \leftrightarrow x^p$ Frob map

Frob $\rightsquigarrow$ automorphism

$\boxed{\text{fixed points} \leftrightarrow \text{precisely the rational points over } \mathbb{F}_p}$

p-adic version of the Lefshetz fixed point theorem from topology

$$N_1 = \sum_{k \geq 0} (-1)^k \operatorname{Tr}(\mathrm{Frob}_* H_k(X))$$

linear map induced by Frob on the homology of X

Back to the zeta function:

$$\zeta_X(T) = \frac{R_3(T)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)}$$

$$R_3(T) = 1 + a_p(\varphi)T + b_p(\varphi)pT^2 + a_p(\lambda)p^3T^3 + p^6T^4$$

$$\boxed{= \det(\mathbb{1} - T\,\mathrm{Frob}_3^{-1})}$$

$\mathrm{Frob}_3 : H^3 \longrightarrow H^3$

4x4 matrix constructed from the periods

$\boxed{R_3 \text{ can be "quickly" computed for } \varphi \in \mathbb{F}_p \; (\varphi = 0, 1, \ldots, p-1) \text{ for many } p.}$

$\boxed{3}$ Arithmetic II

$\boxed{3.1}$ Singular cases

$\boxed{3.2}$ Mirror symmetry

$\boxed{3.3}$ L-function.

3.1 Singularities: example $\sum x_i^5 - \psi^{-1} x_1 \cdots x_5 = 0$

Mirror quintic at conifold singularities ($\psi = 1/5$)

The teta function is still a rational function. However

$$R_3 \longrightarrow (1 - \epsilon p T)\underbrace{(1 - a_p T + p^3 T^2)}$$

$\epsilon = \left(\frac{5}{p}\right) = \pm 1$ $\longrightarrow$ form for a rigid CY

a_p = p-th coeff in q-expansion of the eigenform g found by Schoen (unique cusp form of weight 4 of $\Gamma_0(25)$)

$$
\begin{aligned}
g = \eta(q^5)^4 \big[\eta(q)^4 + 5\eta(q)^3\eta(q^{25}) + 20\eta(q)^2\eta(q^{25})^2 \\
+ 25\eta(q)\eta(q^{25})^3 + 25\eta(q^{25})^4\big]
\end{aligned}
$$

$$
\begin{aligned}
&= q + q^2 + 7\,q^3 - 7\,q^4 + 7\,q^6 + 6\,q^7 - 15\,q^8 + 22\,q^9 - 43\,q^{11} \\
&\quad - 49\,q^{12} - 28\,q^{13} + 6\,q^{14} + 41\,q^{16} + 91\,q^{17} + 22\,q^{18} \\
&\quad - 35\,q^{19} + 42\,q^{21} - 43\,q^{22} + 162\,q^{23} - 105\,q^{24} - 28\,q^{26} \\
&\quad - 35\,q^{27} - 42\,q^{28} + 160\,q^{29} + 42\,q^{31} + \cdots
\end{aligned}
$$

Gouvêa & Yui: modularity of rigid CY 3-folds defined over $\mathbb{Q}$.

study behaviour at singularities!

3.2 Mirror symmetry

As defined the ζ-function $\boxed{\text{does not}}$ respect MS

$$\zeta_X(T) = \frac{R_3(T)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)} \quad \begin{matrix} \leftarrow \text{deg } b_3 = 2h^{21}+2 \\ \leftarrow \text{deg } 2h^{11}+2 \end{matrix}$$

$$R_3(T) = 1 + a_p(\psi)T + b_p(\psi)pT^2 + a_p(\lambda)p^3T^3 + p^6T^4$$

$\zeta_X(T)$ "knows" there are h^{11} KC parameters but it doesn't depend on these

For the quintic Y and its mirror X:

$$\zeta_X(T,\varphi) = \frac{R_3(T,\varphi)}{(1-T)(1-pT)^{101}(1-p^2T)^{101}(1-p^3T)}$$

$$\zeta_Y(T,\psi) = \frac{R_{204}(T,\psi)}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}$$

where

$$R_3(T,\varphi) = (1-T)(1-pT)(1-p^2T)(1-p^3T) + \mathcal{O}(5^2)$$

In fact:
$$\zeta_X = \frac{1}{\zeta_Y} + \mathcal{O}(5^2)$$

In fact: we have the expansion

(PC, XC, FRV 2002)

$$\frac{R_3(T,\varphi)}{(1-T)(1-pT)(1-p^2T)(1-p^3T)} = \sum_{n=0}^{\infty} r_n \left(\frac{5^2 T}{(1-pT)(1-p^2T)} \right)$$

$r_0 = 1 \quad ; \quad r_n \in \mathbb{Z}$

In progress (PC+XC): implement the mirror map:

$$q = e^{2\pi i t} \quad , \quad t = \frac{1}{2\pi i} \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}$$

$\boxed{3.3}$ L-function & modularity

consider: $$L(s,\varphi) = \prod_p R_3(p^{-s},\varphi)^{-1}$$

where, except for the bad primes

$$R_3(T,\varphi) = 1 + \underline{a_p(\varphi)}\,T + \underline{b_p(\varphi)}\,p\,T^2 + a_p(\varphi)\,p^3 T^3 + p^6 T^4$$

<u>Example</u>: mirror quintic (Butt, Cohen, PC, XD, FRV, Watkins, 2009)

$p=5$ $\qquad R_3(T,\varphi) = 1$

$p \mid \varphi^5$ $\qquad R_3(T,\varphi) = 1-T$

$p \mid (5\varphi)^5 - 1$ $\qquad R_3(T,\varphi) = 1 - a_p T + p^3 T^2$

What are the properties of this object?
What about a_φ & b_φ?

recall a_φ & b_φ depend on φ!
so how do a_φ & b_φ vary with φ?

Modularity:

Compare the complete L-function

$$\Lambda(s,\varphi) = N^{s/2}(2\pi)^{-2s}\,\Gamma(s)\,\Gamma(1-s)\,L(s,\varphi)$$

with the spinor L-function for a genus 2 weight 3, spinor Siegel modular form of $Sp(4)$

Numerical tests for a functional eq:

$$\Lambda(4-s,\varphi) = \pm\,\Lambda(s,\varphi)$$

- there are similar tests for other CY varieties

14 Arithmetic and the attractor mechanism

Consider solutions of type IIB supergravity with a spherically symmetric BPS black hole in 4 dim and a CY 3-fold in the extra 6 dimensions

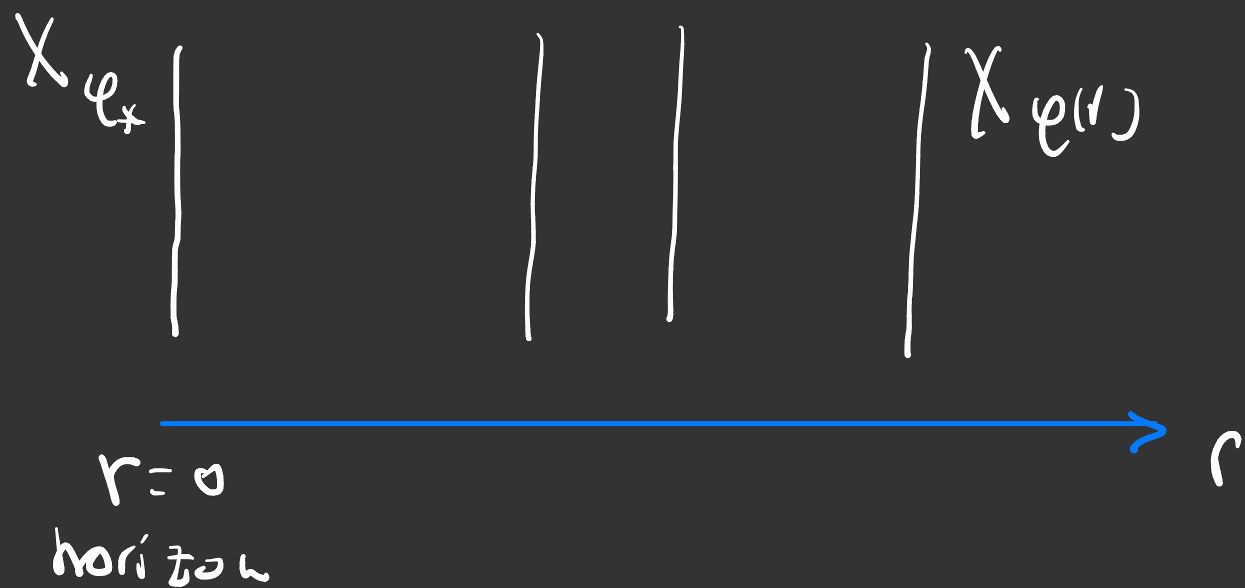
$$ds^2 = -e^{2U(r)}\, dt^2 + e^{2U(r)}\, d\underline{x}^2 \quad , \quad \underline{x} = (x, y, z)$$

- asymptotically flat (Minkowski)

  $r \to \infty \qquad u(r) \to 0$

- horizon $r = 0 \qquad u(r) \longrightarrow -\infty$

10 dim space



complex structure φ varies with r

Type IIB SUGRA is gravity with extra $u(1)$ gauge fields (extra copies of EM)

So, the BH has electric & magnetic charges

$$Q = \begin{pmatrix} q_a \\ p^b \end{pmatrix} \qquad a, b = 1, \ldots, h^{2,1}(X)+1$$

These are integers

$$\Gamma = p^a \alpha_a - q_b \beta^b \in H^3(X, \mathbb{Z})$$

dual $\gamma = q_a A^a - p^b B_b \in H_3(X, \mathbb{Z})$

supersymmetry preserving BH solutions satisfy the attractor equations :

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} d\underline{x}^2 \qquad (U \to 0 \text{ as } r \to \infty)$$

$$\frac{dU(\rho)}{d\rho} = -e^{U} |Z_\gamma(\rho)|$$

$$\boxed{\rho = \frac{1}{r}} \qquad \frac{d\varphi(\rho)}{d\rho} = -2 e^{U} g^{\varphi\bar{\varphi}} \partial_{\bar{\varphi}} |Z_\gamma(\rho)|$$

non-linear dynamical systems on the $\mathbb{C}$-structure moduli space with flow parameter ρ

$g_{\varphi\bar{\varphi}}$ = metric on CS moduli space $= \partial_\varphi \partial_{\bar{\varphi}} K$

$e^{-K} = i \int \Omega \wedge \bar{\Omega}$

$Z_\gamma(\rho)$ = central charge $= e^{K/2} \int_\gamma \Omega$

Variation of the $\mathbb{C}$-structure with r determined by these equations

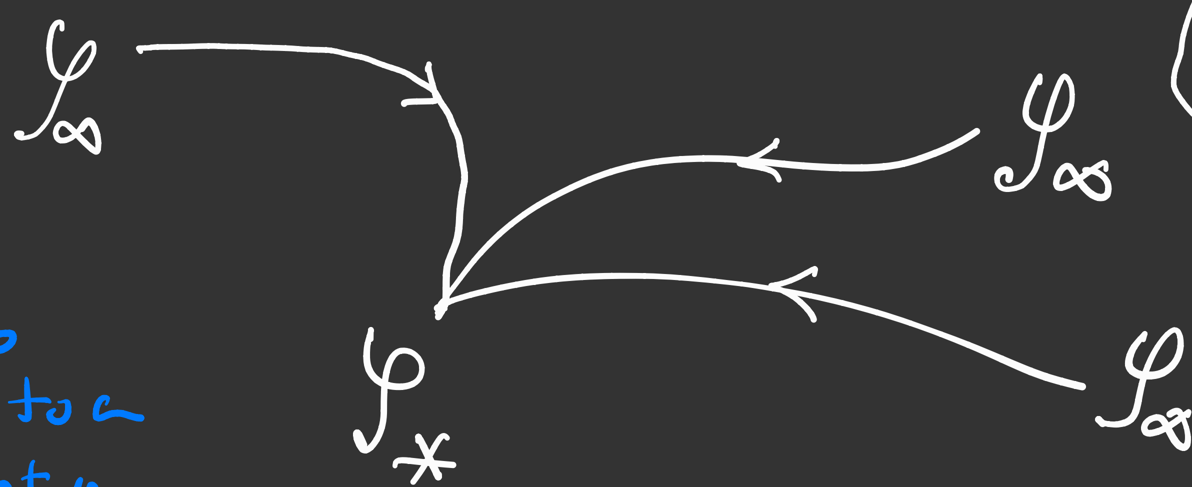
$$\frac{d u(\rho)}{d\rho} = - e^{u} \, |Z_\gamma(\rho)|$$

e^{-u} monotonically increasing as $r \to 0$

$$\frac{d\varphi(\rho)}{d\rho} = -2 e^{u} g^{\varphi\bar{\varphi}} \partial_{\bar{\varphi}} |Z_\gamma(\rho)|$$

eq for Z (gradient flow)

$\mathbb{C}$-structure parameters $\boxed{\text{flow}}$ to an attractor point φ_* (which depends only on the charges of the BH) where $|Z|$ reaches a minimum.



φ evolves smoothly to a fixed point φ_* at $r=0$

95 Ferrara, Kallosh, Strominger
......

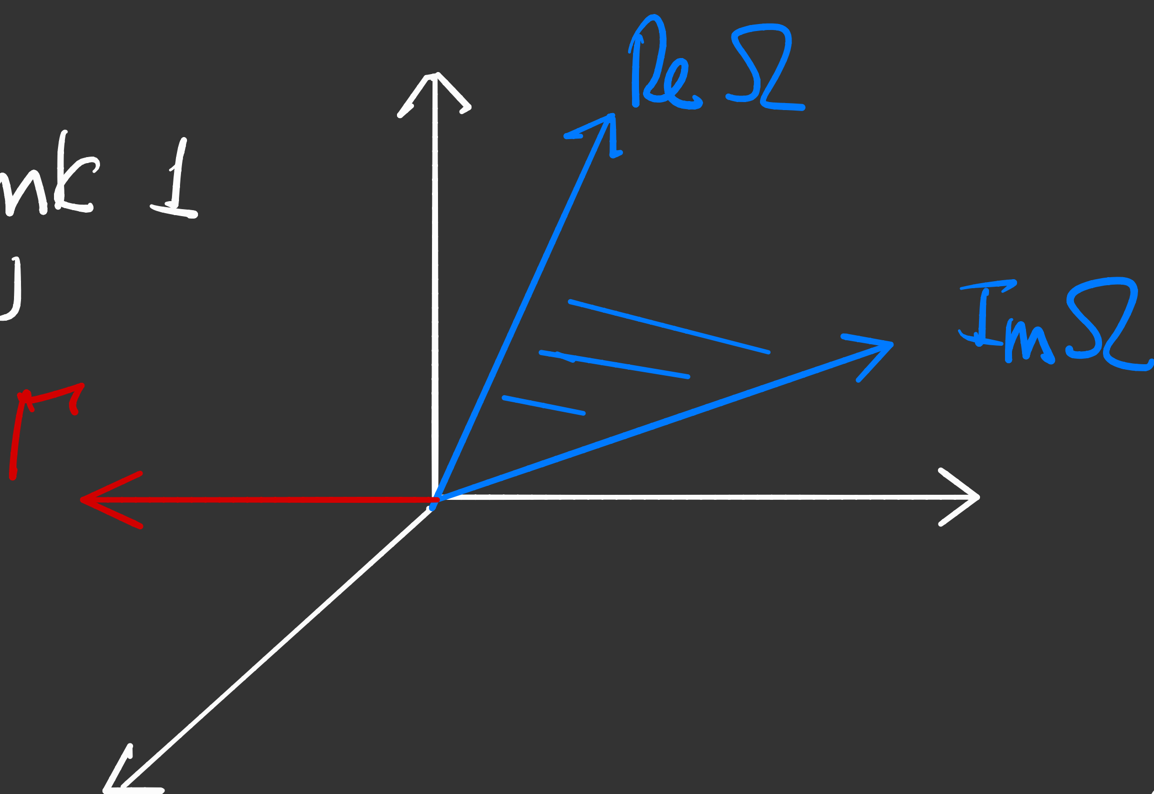
Moore ~98
↳ arithmetic nature of attractor varieties

φ_* and $X_* = X(\varphi_*)$ have special properties

$$\boxed{\begin{aligned} &\Gamma = p^a \alpha_a - q_a \beta^a \in H^{3,0} \oplus H^{0,3} \\ &\text{ie} \quad \Gamma^{(2,1)} = \Gamma^{(1,2)} = 0 \end{aligned}}$$

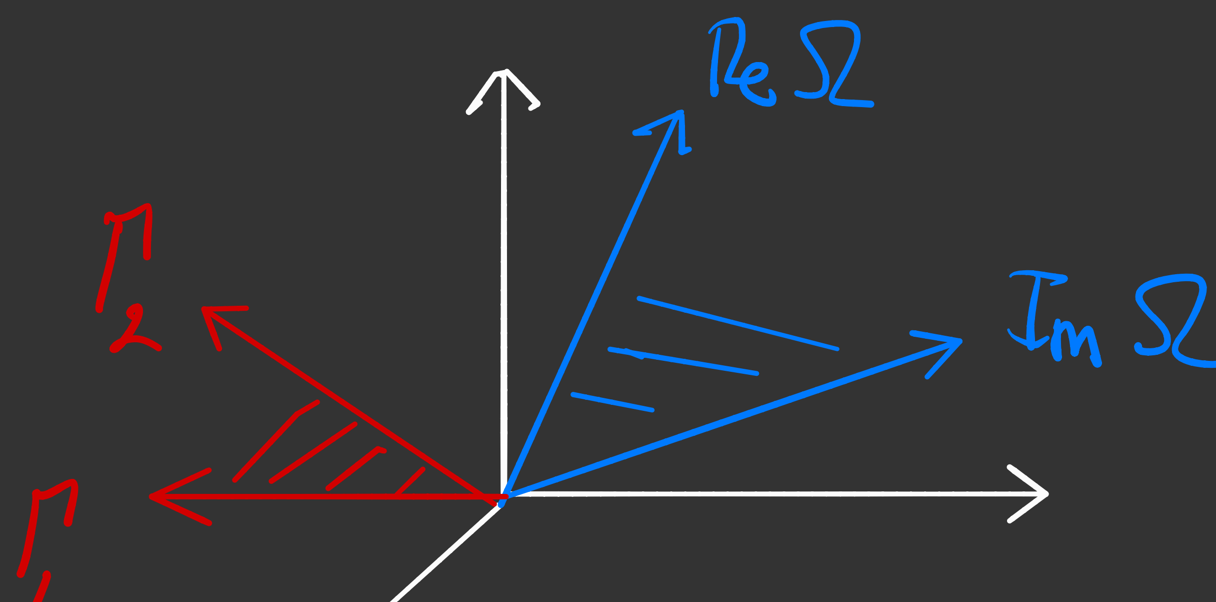
- can solve attractor eqs for charges γ st φ_* is an attractor point but the result generically is that γ is not integral!

rank 1
(common)



$V_{\mathbb{R}}(\varphi)$ = plane spanned over $\mathbb{R}$ by Re Ω & Im Ω
$\to$ $V_{\mathbb{R}}(\varphi)$ moves with φ as φ varies

Inside $H^3(X,\mathbb{R})$
$\to$ lattice of charge vectors
$\Gamma \in H^3(X,\mathbb{Z})$
(fixed lattice)



rank 2
(rare)

HV manifold has <u>rank 2</u> attractor points but <u>not</u> the mirror quintic

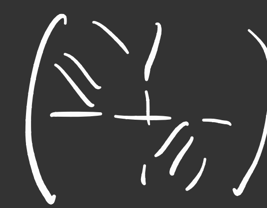
[Candelas, de la Ossa, Elmi, van Straten Dec 2019]

In the rank 2 case:

When $X = X_*$ is a rank two attractor variety

Hodge structure of $H^3(X,\mathbb{Z})$ splits

In turn: the splitting becomes apparent in the arithmetic structure of X

Frob^{-1} 

$$R(T) = \det\left(\mathbb{1} - T\,\mathrm{Frob}_p^{-1}\right)$$

$$= (1 - p\alpha T + p^3T^2)(1 - \beta T + p^3 T^2)$$

factors over $\mathbb{Z}$

$H^{1,2} \oplus H^{2,1}$ $\qquad$ $H^{3,0} \oplus H^{0,3}$

$$R(T) = \underbrace{(1 - p\alpha T + p^3T^2)}_{H^{1,2} \oplus H^{2,1}} \underbrace{(1 - \beta T + p^3 T^2)}_{H^{3,0} \oplus H^{0,3}}$$

$(1 - \alpha(pT) + p(pT)^2)$

recall elliptic curve

form for a rigid CY

attached to modular forms of specific weight and conductor

(Serre's conjectures)

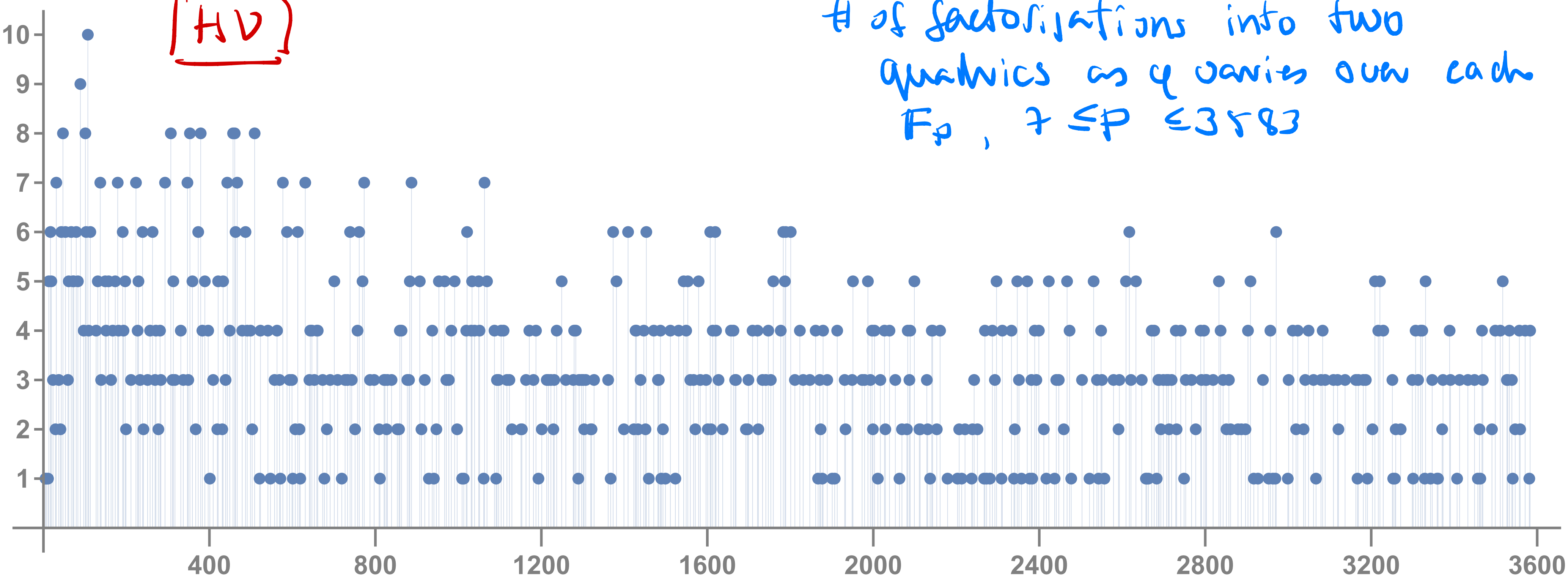
(Arithmetic) strategy:

make tables of $R(T, q)$ for many p & q
and look for persistent factorisations

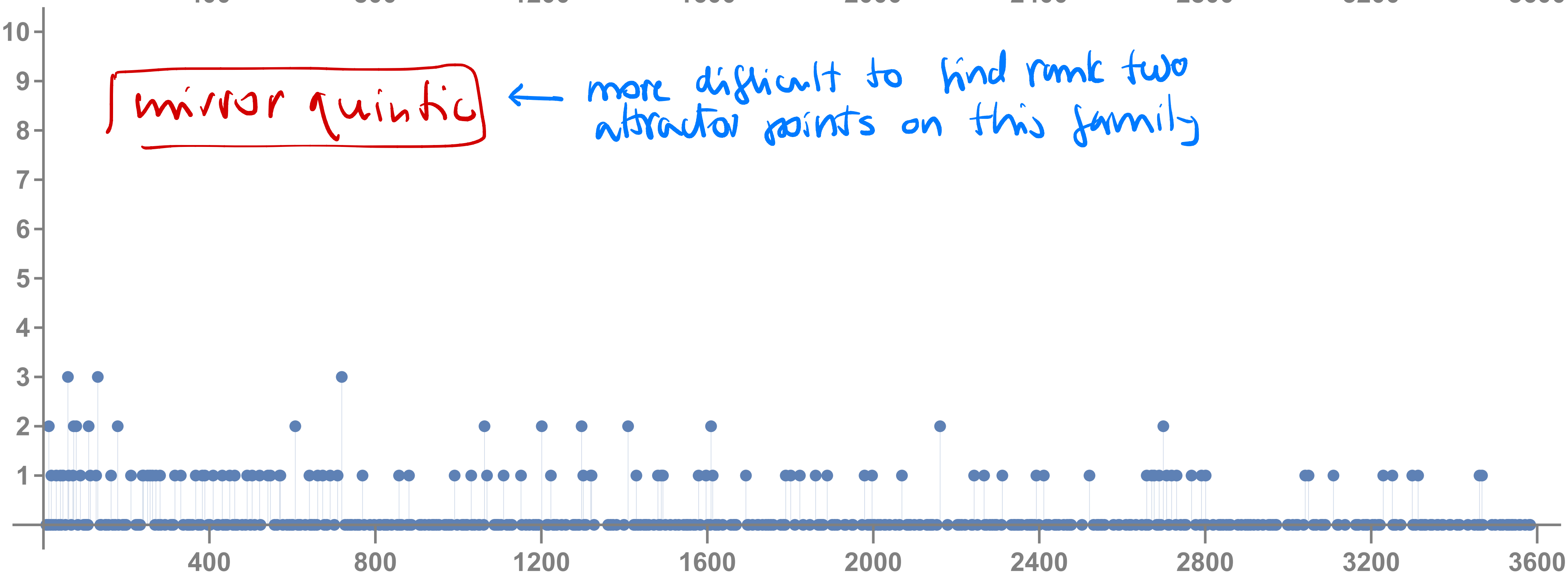
↳ factorisations which occur
of some polynomial $G[q]$
with integer coeffs.

(Candelas, XD, A Thorne, v Straten, —)

Candelas + XD + v Straten: to appear



\# of factorizations into two quadrics as q varies over each $\mathbb{F}_p$, $7 \leq p \leq 3583$



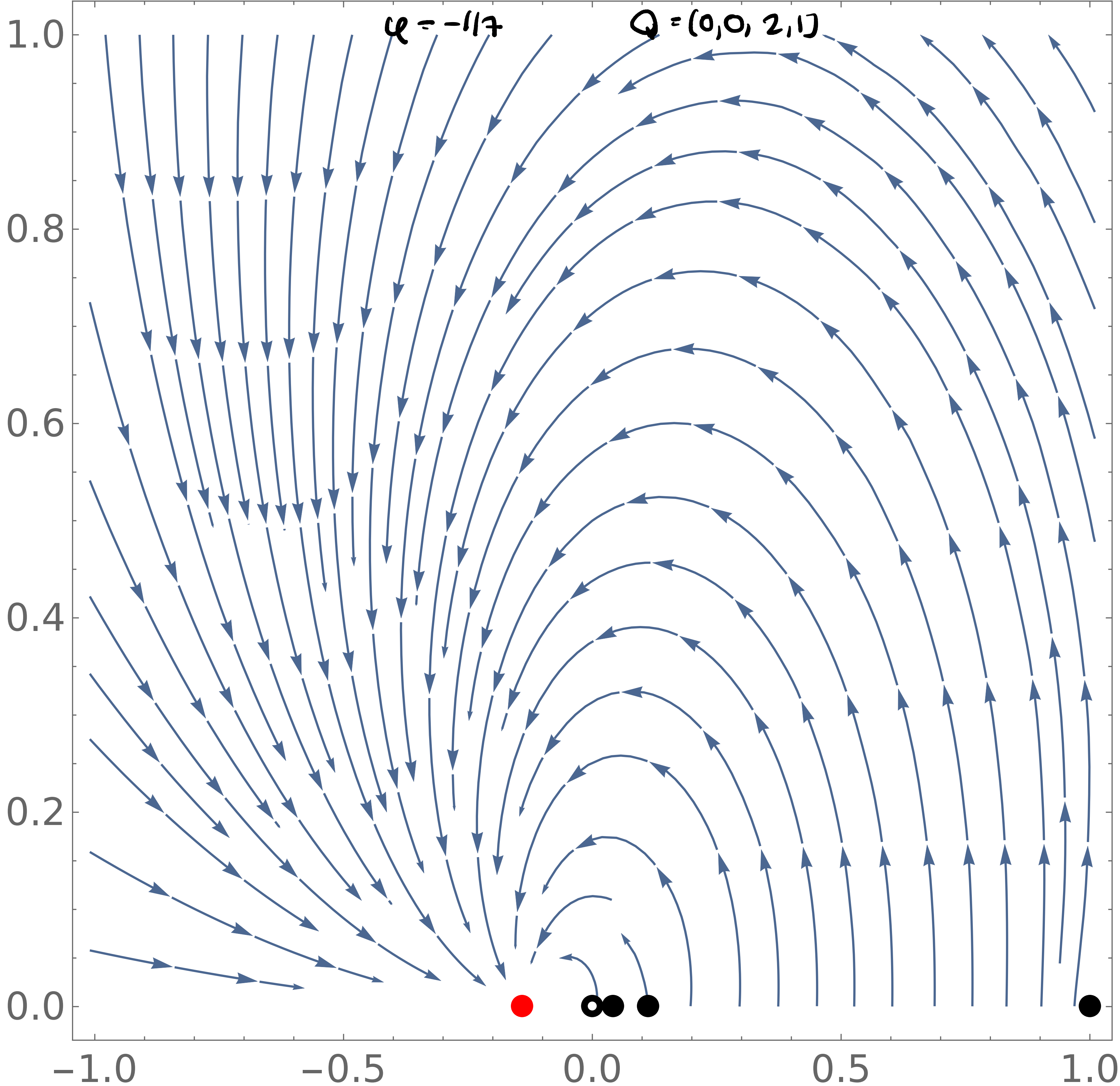
← more difficult to find rank two attractor points on this family

We find that for the HV manifold there is always a factorisation when

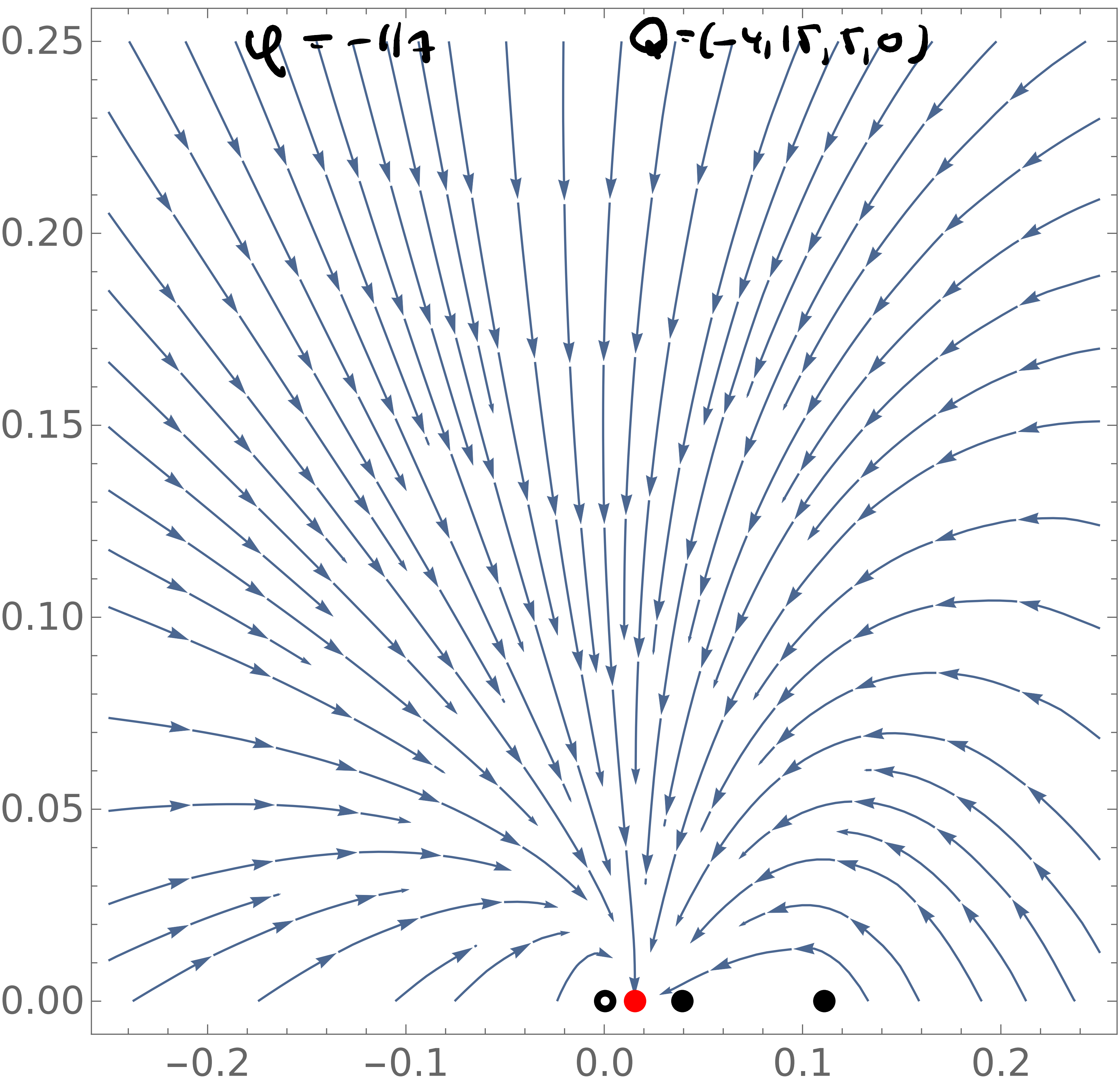
- $7\varphi+1=0$
- $\varphi^2-66\varphi+1=0$: $\varphi_\pm = 33 \pm 8\sqrt{17}$

For $p=19$ say

$-\frac{1}{7} \equiv 8$, $\varphi_\pm \equiv 4, 5$



φ = -1/7
Q = (0,0,2,1]
1.0
0.8
0.6
0.4
0.2
0.0
-1.0
-0.5
0.0
0.5
1.0



φ = -1/7
Q=(-4,15,5,0)
0.25
0.20
0.15
0.10
0.05
0.00
−0.2
−0.1
0.0
0.1
0.2

$p = 19$			
φ	smooth/sing.	singularity	$R(T)$
1	singular	1	$(1 - pT)(1 - 20T + p^3T^2)$
2	smooth		$1 + 4pT + 2pT^2 + 4p^4T^3 + p^6T^4$
3	smooth		$1 - 8T + 242pT^2 - 8p^3T^3 + p^6T^4$
4	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
5	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
6	smooth		$1 + 8T - 318pT^2 + 8p^3T^3 + p^6T^4$
7	smooth		$1 - 44T - 238pT^2 - 44p^3T^3 + p^6T^4$
8	smooth		$(1 - 2pT + p^3T^2)(1 - 80T + p^3T^2)$
9	smooth		$(1 + 4pT + p^3T^2)(1 - 160T + p^3T^2)$
10	smooth		$1 + 12T + 562pT^2 + 12p^3T^3 + p^6T^4$
11	smooth		$(1 + 4pT + p^3T^2)(1 - 140T + p^3T^2)$
12	smooth		$1 + 12T + 82pT^2 + 12p^3T^3 + p^6T^4$
13	smooth		$1 + 178T + 1082pT^2 + 178p^3T^3 + p^6T^4$
14	smooth		$1 + 12T - 158pT^2 + 12p^3T^3 + p^6T^4$
15	smooth		$1 + 42T - 2p^2T^2 + 42p^3T^3 + p^6T^4$
16	singular	$\frac{1}{25}$	$(1 - pT)(1 + 76T + p^3T^2)$
17	singular	$\frac{1}{9}$	$(1 - pT)(1 - 20T + p^3T^2)$
18	smooth		$1 - 54T + 322pT^2 - 54p^3T^3 + p^6T^4$

Table 1: *The R-factors for $\varphi \in \mathbb{F}_{19}$. Note the factorisations into two quadrics for the five values $\varphi = 4, 5, 8, 9, 11$.*

$\varphi_\pm$ -117

There is more information in the tables: there are modular forms

$$R = (1 - p\,\alpha_p T + p^2 T^2)(1 - \beta_p T + p^3 T^2)$$

proof of Serre conjecture (generalizing Shimura Taniyama)

"motives" of length two are modular

For $\varphi = -1/7$:

α_a, β_q are Fourier coefficients of a modular form for $\Gamma_0(14)$

$f_2 = \sum \alpha_n q^n$ weight 2 LMFDB 14.2.a.a.

$f_4 = \sum \beta_n q^n$ weight 4 14.4.a.a

Similarly: for $\ell_{\pm}$

$\alpha_p, \beta_p \longrightarrow$ coeffs of modular forms

for $\Gamma_1(34) \subset \Gamma_0(34)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 34$$

$f_2 \longrightarrow$ 34.2.b.a

$f_4 \longrightarrow$ 34.4.b.a

Associated with these modular forms are L-functions

$$L_j(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy\, y^{s-1} f_j(iy)$$

$$q = e^{-2\pi y}$$

At φ_* the periods of Ω are given by simple multiples of

$L_4(1)$ and $L_4(2)$

critical values for L_4

(expected: Deligne conjecture)

See Wen-zhe Yang math.AG

Area of the horizon

Charges:

$$Q_{k\ell} = k(4K, -15K, -5, 0) + \ell(0, 0, 2, 1)$$ $K = 1, 2$

let $$V_* = \frac{7}{\pi} \frac{L_4(2)}{L_4(1)}$$

Then

$$\boxed{A(\varphi_*) = 14\pi \left\{ k^2 V_* + \left(\ell - \frac{5k}{2}\right)^2 \frac{1}{V_*} \right\}}$$

- BH entropy $\propto A$
- Mirror symmetry and instanton numbers : $q = -1/7$

$$t_* = t\left(-\frac{1}{7}\right) = \frac{1}{12} + \frac{5\pi i}{28} \frac{L_4(1)}{L_4(2)}$$

- What makes a CY an attractor variety?

Forthcomming : . Candelas, Kuusela, McGovern
. Candelas, XD, vanStraten

So far: just the tip of the iceberg

Thank you!