Example 1.

Definition.

For \( m \geq N \): 
\[
\sigma_m(n) := \sum_{d \mid n, \ d > 0} d^m.
\]
Example 1.

**Definition.** For $m \in \mathbb{N}_0 := \{0, 1, 2, \ldots \}$, $n \in \mathbb{N} := \{1, 2, 3, \ldots \}$, the *$m$th power divisor function* is defined by

$$
\sigma_m(n) := \sum_{d \mid n, d > 0} d^m.
$$
Modular forms

A generating function:

\[
\sum_{n=1}^{\infty} \sigma_m(n) q^n
\]
Modular forms

A generating function:

\[ \sum_{n=1}^{\infty} \sigma_m(n)q^n = \sigma_m(1)q + \sigma_m(2)q^2 + \sigma_m(3)q^3 + \sigma_m(4)q^4 + \cdots \]
Modular forms

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\[ = q + (1 + 2^m)q^2 + (1 + 3^m)q^3 + (1 + 2^m + 4^m)q^4 + \cdots \]
Let \( m = 2k - 1, \ k \in \{2, 3, 4, 5, \ldots \} \).
Let $m = 2k - 1$, $k \in \{2, 3, 4, 5, \ldots \}$.

**Definition.** The **weight $2k$ Eisenstein series** is defined by

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$
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\]

\( B_\kappa \) := \( \kappa \)-th Bernoulli number.
Modular forms

The modular group

\[ \Gamma = \text{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \]
The modular group

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\[ \Gamma \text{ is generated by} \]

\[ T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
Modular forms

Γ acts on \( \mathbb{H} \) by

\[
( \begin{pmatrix} a & b \\ c & d \end{pmatrix} ) \cdot \tau := \frac{a \tau + b}{c \tau + d}
\]

Ex. \( T \cdot \tau = \tau + 1 \), \( S \cdot \tau = \frac{1}{\tau} \)

\( E_2 \) is "symmetric" with respect to the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \).
Modular forms

Γ acts on \( \mathbb{H} \) by

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$E_{2k}$ is “symmetric” with respect to the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$...
That is, for all $\tau \in \mathbb{H}$,

\begin{align*}
E_{2k}(\tau + 1) &= E_{2k}(\tau), \\
E_{2k}(-1/\tau) &= \tau^{2k}E_{2k}(\tau),
\end{align*}
That is, for all $\tau \in \mathbb{H}$,

$$E_{2k}(\tau + 1) = E_{2k}(\tau),$$
$$E_{2k}(-1/\tau) = \tau^{2k}E_{2k}(\tau),$$

and in general, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$,

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k}E_{2k}(\tau).$$
Modular forms
A weight $k \geq 1$ modular form $f$: $\mathcal{H} \to \mathcal{C}$ on $\Gamma'$ of $\text{SL}_2(\mathbb{Z})$ satisfies $f$ is holomorphic on $\mathcal{H}$, $f(\tau + \frac{a}{c}\tau + \frac{b}{d}) = \varepsilon \gamma f(\tau)$, $\gamma \varepsilon j = 1$, $8\tau^2 \mathcal{H}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma' f$ is holomorphic or meromorphic in the cusps.

E.g., $f$ has a Fourier expansion of the shape $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$, where $q = e^{2\pi i \tau}$, $a_n = a f(n^2 \mathbb{C})$, $m = m f^2 \mathbb{Z}$, $h = h f^2 \mathbb{N}$. 
A weight $k \in \frac{1}{2}\mathbb{Z}$ modular form $f : \mathbb{H} \to \mathbb{C}$ on $\Gamma' \subseteq \text{SL}_2(\mathbb{Z})$ satisfies
Modular forms

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A weight \( k \in \frac{1}{2}\mathbb{Z} \) modular form \( f : \mathbb{H} \to \mathbb{C} \) on \( \Gamma' \subseteq \text{SL}_2(\mathbb{Z}) \) satisfies

- \( f \) is holomorphic on \( \mathbb{H} \),

- \( f \left( \frac{a\tau + b}{c\tau + d} \right) = \varepsilon_\gamma (c\tau + d)^kf(\tau), \quad |\varepsilon_\gamma| = 1, \quad \forall \tau \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \)
A **weight** $k \in \frac{1}{2}\mathbb{Z}$ **modular form** $f : \mathbb{H} \to \mathbb{C}$ on $\Gamma' \subseteq \text{SL}_2(\mathbb{Z})$ satisfies

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- $f$ is holomorphic or meromorphic in the cusps.
  e.g., $f$ has a Fourier expansion of the shape

$$f(\tau) = \sum_{n=m}^{\infty} a_n q^n,$$

where $q = e^{2\pi i \tau}$, $a_n = a_{f,n} \in \mathbb{C}$, $m = m_f \in \mathbb{Z}$, $h = h_f \in \mathbb{N}$.
Definition. A **partition** of an integer $n \geq 1$ is a set of positive integers $\{\lambda_1, \ldots, \lambda_r\}$ such that
Example 2.

Definition. A \textit{partition} of an integer \( n \geq 1 \) is a set of positive integers \( \{\lambda_1, \ldots, \lambda_r\} \) such that

\begin{itemize}
  \item \( \lambda_1 + \cdots + \lambda_r = n \),
  \item \( \lambda_1 \geq \cdots \geq \lambda_r \).
\end{itemize}
Definition. A **partition** of an integer $n \geq 1$ is a set of positive integers $\{\lambda_1, \ldots, \lambda_r\}$ such that

1. $\lambda_1 + \cdots + \lambda_r = n,$
2. $\lambda_1 \geq \cdots \geq \lambda_r.$

We define the **partition function** by

\[ p(n) := \#\{\text{partitions of } n\}, \]

and we let $p(0) := 1.$

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$$p(n) := \# \{\text{partitions of } n\},$$

and we let $p(0) := 1.$

Ex. $p(4) = 5,$ since $4 = 4, \ 3 + 1, \ 2 + 2, \ 2 + 1 + 1, \ 1 + 1 + 1 + 1.$
Theorem (Euler). Let $|q| < 1$. The partition generating function

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

satisfies

$$P(q) = \prod_{k=1}^{\infty} \frac{1}{1 - q^k} = \frac{1}{(1 - q)(1 - q^2)(1 - q^3) \cdots}.$$
That is,

\[ q^{-\frac{1}{24}} \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\eta(\tau)}, \]

where the Dedekind \( \eta \)-function

\[ \eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \]

with \( q = e^{2\pi i \tau}, \tau \in \mathbb{H} \), is a modular form of weight \( 1/2 \).
Example 2 (cont.)

A consequence of modularity:

Theorem (Hardy-Ramanujan-Rademacher)

We have the exact formula

\[ p(n) = 2\pi \left( \frac{24}{n-1} \right)^{\frac{3}{2}} \sum_{m=1}^{\infty} \frac{A_m(n)}{m} \]
Example 2 (cont.)

A consequence of modularity:

**Theorem (Hardy-Ramanujan-Rademacher)** We have the exact formula

\[
p(n) = \frac{2\pi}{(24n - 1)^{\frac{3}{4}}} \sum_{m=1}^{\infty} \frac{A_m(n)}{m} I_{\frac{3}{2}} \left( \frac{\pi \sqrt{24n - 1}}{6m} \right).
\]
Modular symmetry

Question.
Question.

What could be gained by perturbing modular symmetry?
Example 1 revisited.

Let $k = 1$. The function $E_2(\tau) := 1 + 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$ is not a (weight 2) modular form. Namely, we have that $E_2(1/\tau) = \tau^2 E_2(\tau) - 6i \pi \{\text{error to modularity}\}$. 
Example 1 revisited.

Let \( k = 1 \). The function

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E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n
\]

is not a (weight 2) modular form.
Example 1 revisited.

Let $k = 1$. The function

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is not a (weight 2) modular form. Namely, we have that

$$E_2(-1/\tau) = \tau^2 E_2(\tau) - \frac{6i\tau}{\pi}. \quad \text{“error to modularity”}$$
Example 1 revisited.

Define the function

$$b_{E_2}(\tau) := E_2(\tau) - \pi \operatorname{Im}(\tau).$$

Then

$$b_{E_2}(\frac{1}{\tau}) = \tau^2 b_{E_2}(\tau).$$

That is, $b_{E_2}$ is an almost holomorphic weight 2 modular form. (Kaneko-Zagier)
Example 1 revisited.

Define the function

\[ \widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi \Im(\tau)}. \]
Example 1 revisited.

Define the function

$$\hat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}.$$

Then

$$\hat{E}_2(-1/\tau) = \tau^2 \hat{E}_2(\tau).$$
Example 1 revisited.

Define the function

$$\hat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}.$$ 

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That is, $\hat{E}_2$ is an \textit{almost holomorphic} weight 2 modular form. (Kaneko-Zagier)
Example 2 revisited.

**Definition (Dyson).** The *rank* of a partition is defined to be its largest part minus the number of its parts.
Example 2 revisited.

Definition (Dyson). The rank of a partition is defined to be its largest part minus the number of its parts. For $n \in \mathbb{N}, m \in \mathbb{Z}$, we define

$$N(m, n) := p(n \mid \text{rank } m),$$

and let $N(m, 0) := \delta_{m,0}$. 


**Example 2 revisited.**

**Definition (Dyson).** The *rank* of a partition is defined to be its largest part minus the number of its parts. For \( n \in \mathbb{N}, m \in \mathbb{Z} \), we define

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N(m, n) := p(n \mid \text{rank } m),
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and let \( N(m, 0) := \delta_{m,0} \).

**Ex.** We have that \( N(m, 4) = \begin{cases} 
1, & m = 0, \pm 1, \pm 3, \\
0, & \text{else}.
\end{cases} \)
Example 2 revisited.

Definition (Dyson). The \textbf{rank} of a partition is defined to be its largest part minus the number of its parts. For \( n \in \mathbb{N}, m \in \mathbb{Z} \), we define

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Ex. We have that \( N(m, 4) = \begin{cases} 1, & m = 0, \pm 1, \pm 3, \\ 0, & \text{else} \end{cases} \).

Note. For fixed \( n \), we have that \( \sum_{m=-\infty}^{\infty} N(m, n) = p(n) \).
Example 2 revisited.

The two-variable partition rank generating function satisfies

$$R(w; q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} R(m, n) w^m q^n = \sum_{n=0}^{\infty} q^n (w^{\frac{1}{2}} q; q^n w^{\frac{1}{2}} q\frac{1}{q}).$$

where for $n \geq 0$, the $q$-Pochhammer symbol is defined by

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3) \ldots (1 - aq^{n-1})$$
Example 2 revisited.

The two variable partition rank generating function satisfies

\[ R(w; q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n}, \]

where for \( n^2 \geq 0 \), the \( q \)-Pochhammer symbol is defined by

\[ (a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}). \]
"Modular" forms

Example 2 revisited.

The two variable partition rank generating function satisfies

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where for \( n \in \mathbb{N}_0 \), the \textit{q-Pochhammer symbol} is defined by

\[ (a; q)_n = (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3) \cdots (1 - aq^{n-1}). \]
Observation. We have that

\[ R(1; q) = \sum_{n=0}^{\infty} p(n)q^n = P(q) \]

is (essentially) a modular form, with \( q = e^{2\pi i \tau}, \tau \in \mathbb{H} \).
**“Modular” forms**

**Observation.** We have that

$$R(1; q) = \sum_{n=0}^{\infty} p(n)q^n = P(q)$$

is (essentially) a modular form, with $q = e^{2\pi i \tau}, \tau \in \mathbb{H}$.

**Question.** Is $R(w; q)$ a modular form for other fixed values of $w$, when viewed as a function of $\tau$, with $q = e^{2\pi i \tau}$?
Let $w = -1$. Then

$$R(-1; q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2} = \sum_{n=0}^{\infty} (N_e(n) - N_o(n))q^n,$$

where $N_e$ (resp. $o$) $(n) := p(n | \text{even (resp. odd) rank})$. 
Mock theta functions

Ramanujan’s mock theta function:

\[ f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots \]
Mock theta functions

\[ f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \ldots \]

\[ \phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^5}{(1+q^2)(1+q^4)} + \ldots \]

\[ \psi(q) = \frac{q}{1-q} + \frac{q^5}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \ldots \]

\[ \chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q+q^2)^2} + \ldots \]

\[ \psi(q) = 1 + q(1+q) + q^6(1+q)(1+q^2) + q^{12}(1+q)(1+q^2)(1+q^2)^2 + \ldots \]

\[ f(q) = 1 + \frac{q}{1+q} + \frac{q^5}{(1+q)(1+q^2)} + \frac{q^{10}}{(1+q)(1+q^2)(1+q^2)^2} + \ldots \]
Mock theta functions

\[
\begin{align*}
I&\text{ have proved that if} \\
f(q) &= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \\
\text{then } f(q) + (1-q)(1-q^3)(1-q^5)\cdots &= \frac{1-2q+2q^4}{1-2q^2+2q^6-2q^9+}\frac{1-2q^2+2q^6}{1-2q^4+2q^8}\frac{1-2q+2q^4}{1-2q^2+2q^6-2q^9+} \\
\text{at all the } &= O(1) \\
\text{points } q &= -1, q^3 = -1, q^5 = -1 \\
\text{and at the same time} \\
f(q) &= (1-q)(1-q^3)(1-q^5)\cdots = O(1) \\
\text{at all the points } q^2 &= -1, q^4 = -1, q^6 = -1, \ldots \\
\text{Also obviously } f(q) &= O(1) \\
\text{at all the points } q = 1, q^2 = 1, q^5 = 1, \ldots
\end{align*}
\]

-S. Ramanujan to G.H. Hardy, 1920
Mock theta functions

Ramanujan’s observations:

- There is an explicit modular form $b(q)$ that “cuts out” the exponential singularities of $f(q)$. 

  | $f(q)$ approaches any even order $2k$ root of unity singularity of $f(q)$, then $f(q) \sim 1 \cdot b(q) = O(1)$ asymptotically toward singularities, mock theta modular form = bounded.
Ramanujan’s observations:

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$$f(q) - (-1)^k b(q) = O(1)$$
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  $$f(q) - (-1)^k b(q) = O(1)$$

- That is, asymptotically, towards singularities,

  $$\text{mock theta } \pm \text{ modular form } = \text{ bounded}$$
Mock theta functions

Theorem (Watson). Let $q = e^{-\alpha}$, $\beta = \pi^2/\alpha$, $q_1 = e^{-\beta}$, where $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$. Then

$$q^{-\frac{1}{24}} f(q) = 2 \sqrt{\frac{2\pi}{\alpha}} q_1^{\frac{4}{3}} \omega(q_1^2) + 4 \sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty \frac{\sinh(\alpha t)}{\sinh \left( \frac{3\alpha t}{2} \right)} e^{-\frac{3\alpha t^2}{2}} dt.$$
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**Remark.** This may be interpreted as a transformation under $\tau \mapsto -1/(2\tau)$. 
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"error to modularity"
Mock theta functions

“Theorem” (S. Zwegers, 2002). Ramanujan’s mock theta functions are not modular forms,
“Theorem” (S. Zwegers, 2002). Ramanujan’s mock theta functions are not modular forms, but they can be completed to form nonholomorphic modular forms.
Zwegers’ completion:

\[ f(q) \]
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\[ q^{-\frac{1}{24}} f(q) \]
Mock theta functions

Zwegers’ completion:

\[ q^{-\frac{1}{24}} f(q) \]

\[ \uparrow \]

not modular
Zwegers’ completion:

\[ q^{-\frac{1}{24}} f(q) + f^{-}(q) \]

\[ \uparrow \]

\[ \text{not modular} \]
Mock theta functions

Zwegers’ completion:

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\[ \uparrow \]

not modular \hspace{1cm} not holomorphic
Zwegers’ completion:

\[ \hat{f}(\tau) := q^{-\frac{1}{24}} f(q) + f^-(q) \]

not modular

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- modular
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Mock theta functions

Zwegers’ completion:

\[ f(q) := 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]

\[ \hat{f}(\tau) := q^{-\frac{1}{24}} f(q) + 2i\sqrt{3} \int_{-\tau}^{i\infty} \frac{g(z) \, dz}{\sqrt{-i(\tau + z)}} \]
Mock theta functions

Zwegers’ completion:

\[ f(q) := 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]

\[ \hat{f}(\tau) := q^{-\frac{1}{24}} f(q) + 2i\sqrt{3} \int_{-\tau}^{i\infty} \frac{g(z) \, dz}{\sqrt{-i(\tau + z)}} \]

where \( g(\tau) := - \sum_{n=-\infty}^{\infty} (n + \frac{1}{6}) q^{\frac{3}{2}(n + \frac{1}{6})^2} \) is a weight 3/2 modular form.
Harmonic Maass forms

Definition (Bruinier-Funke). A weight $k$ ($k \in \frac{1}{2} \mathbb{Z}$) harmonic Maass form on $\Gamma' = \Gamma_0(N)$, where $4|N$ if $k \in \frac{1}{2} + \mathbb{Z}$, is a smooth $M: \mathbb{H} \to \mathbb{C}$ satisfying

i) For all $a b \quad c d \in \Gamma$ and all $\tau \in \mathbb{H}$, we have
$$M(a\tau + b c\tau + d) = \left( \begin{array}{cc} c\tau + d & \quad k \end{array} \right) M(\tau) \quad \text{if} \quad k \in \frac{1}{2} \mathbb{Z},$$
$$c d \quad \varepsilon - 2 k d (c\tau + d) \quad k \quad M(\tau) \quad \text{if} \quad k \in \frac{1}{2} + \mathbb{Z}.$$

ii) We have that
$$\Delta_k(M) = 0 \quad \text{where} \quad (\tau = x + iy)$$
$$\Delta_k := y^2 \frac{\partial^2}{\partial x^2} + y^2 \frac{\partial^2}{\partial y^2} + iky \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}.$$

iii) There exists a polynomial $P_M(\tau) \in \mathbb{C}[q^{−1}]$ such that
$$M(\tau) P_M(\tau) = O(e^{-\epsilon y}) \quad \text{as} \quad y \to \infty$$
for some $\epsilon > 0$. 


Definition (Bruinier-Funke). A **weight** $k$ ($k \in \frac{1}{2} \mathbb{Z}$) **harmonic Maass form on** $\Gamma' = \Gamma_0(N)$, where $4|N$ if $k \in \frac{1}{2} + \mathbb{Z}$, is a smooth $M: \mathbb{H} \to \mathbb{C}$ satisfying

i) For all $\left( \frac{a}{c} \frac{b}{d} \right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have

$$M \left( \frac{a \tau + b}{c \tau + d} \right) = \begin{cases} (c \tau + d)^k M(\tau) & \text{if } k \in \mathbb{Z}, \\ (\frac{c}{d}) \varepsilon_d^{-2k} (c \tau + d)^k M(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$
Harmonic Maass forms

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i) For all \((a \ b \ c \ d) \in \Gamma \) and all \( \tau \in \mathbb{H} \), we have

\[
M \left( \frac{a \tau + b}{c \tau + d} \right) = \begin{cases} 
\frac{(c \tau + d)^k M(\tau)}{c d} & \text{if } k \in \mathbb{Z}, \\
\frac{e^{-2k}}{c d} (c \tau + d)^k M(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}.
\end{cases}
\]

ii) We have that \( \Delta_k(M) = 0 \), where (if \( \tau = x + iy \))

\[
\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i ky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
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Definition (Bruinier-Funke). A **weight** $k$ ($k \in \frac{1}{2} \mathbb{Z}$) **harmonic Maass form on** $\Gamma' = \Gamma_0(N)$, where $4|N$ if $k \in \frac{1}{2} + \mathbb{Z}$, is a smooth $M : \mathbb{H} \to \mathbb{C}$ satisfying

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iii) There exists a polynomial $P_M(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$M(\tau) - P_M(\tau) = O \left( e^{-\varepsilon y} \right)$$

as $y \to \infty$ for some $\varepsilon > 0$. 
Lemma. Let $k \in \frac{1}{2} \mathbb{Z} \setminus \{1\}$ and $\Gamma \in \{\Gamma_0(N), \Gamma_1(N)\}$. If $M$ is a HMF, then $M$ has Fourier expansion

$$M(\tau) = \sum_{n \gg -\infty} c_M^+(n) q^n + \sum_{n < 0} c_M^-(n) \Gamma(1 - k, -4\pi ny) q^n.$$
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\]

The \textit{incomplete gamma function} is defined by

\[
\Gamma(s, z) := \int_z^{\infty} e^{-t} t^{s-1} dt.
\]
Harmonic Maass forms

That is,

\[ M(\tau) = \sum_{n \gg -\infty} c_M^+(n) q^n + \sum_{n < 0} c_M^-(n) \Gamma(1 - k, -4\pi n y) q^n \]

\[ \begin{align*}
& \text{“holomorphic part”} \\
& \text{“nonholomorphic part”}
\end{align*} \]
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Definition (Zagier).
Harmonic Maass forms

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\[ \begin{align*}
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    &\text{"nonholomorphic part"}
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Definition (Zagier). A **mock modular form** (of weight \( k \)) is the holomorphic part of a harmonic Maass form (of weight \( k \)).
That is, 

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“holomorphic part”

“nonholomorphic part”

**Definition (Zagier).** A *mock modular form* (of weight \( k \)) is the holomorphic part of a harmonic Maass form (of weight \( k \))* (*for which the NHP is nontrivial).*
Theorem (Zwegers). Ramanujan’s mock theta functions are weight $1/2$ mock modular forms.
Theorem (Zwegers). Ramanujan’s mock theta functions are* weight 1/2 mock modular forms. That is, if $F$ is one of Ramanujan’s mtf’s, then for some $\alpha_F \in \mathbb{Q}$ and $c_F \in \mathbb{C}$,

$$F(\tau) = q^{\alpha_F} G_F^+(\tau) + c_F,$$

where $G_F^+$ is the holomorphic part of a weight 1/2 HMF.
Theorem (Zwegers). Ramanujan’s mock theta functions are weight 1/2 mock modular forms. That is, if $F$ is one of Ramanujan’s mtf’s, then for some $\alpha_F \in \mathbb{Q}$ and $c_F \in \mathbb{C}$,

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where $G_F^+$ is the holomorphic part of a weight 1/2 HMF.

(*up to multiplication by a power of $q$ and addition of a constant)
“Modular” forms

Example 2 (revisited.)

A consequence of mock modularity:

Theorem (Bringmann-Ono), Conjectured by Andrews-Dragonette.

We have the exact formula

\[ N_e(n) = N_0(n) = \pi(24n^2 + 1) \times \sum_{m=1}^{\infty} \left( \frac{1}{b_m+1} \right)^{1/4} \left( \frac{1}{c_{A^2m}} \frac{n^m}{m} (1 + \frac{1}{m}) \right) \frac{\pi}{p^{24n^2 + 1}}. \]
**Example 2 (revisited.)**

A consequence of mock modularity:

**Theorem (Bringmann-Ono), Conjectured by Andrews-Dragonette.**

We have the exact formula

\[
N_e(n) - N_o(n) = \frac{\pi}{(24n - 1)^{1/4}} \sum_{m=1}^{\infty} (-1)^{\left\lfloor \frac{m+1}{2} \right\rfloor} A_{2m} \left( n - \frac{m(1+(-1)^m)}{4} \right) \frac{1}{m} \int \frac{1}{2} \left( \frac{\pi \sqrt{24n - 1}}{12m} \right). 
\]
Example 2 (revisited). Let $\zeta_N := e^{2\pi i/N}$.

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$$q^{-\ell_b/24}R(\zeta^a_b; q^\ell_b) + i(3^{-1}\ell_b)^{1/2}\sin\left(\frac{\pi a}{b}\right)\int_{-\tau}^{i\infty} \frac{\Theta\left(\frac{a}{b}; \ell_b z\right)}{\sqrt{-i(z+\tau)}} \, dz$$

is a harmonic Maass form of weight $1/2$ and level $144$. 
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Here, $\Theta$ is a weight 3/2 modular form, and $\ell_b \in \mathbb{N}$. 
Example 2 revisited. We have the weight 1/2 HMF

\[ q^{-\ell_b/24} R(\zeta_b^a; q^{\ell_b}) + i(3^{-1} \ell_b)^{1/2} \sin\left(\frac{\pi a}{b}\right) \int_{-\tau}^{i\infty} \frac{\Theta\left(\frac{a}{b}; \ell_b z\right)}{\sqrt{-i(z + \tau)}} \, dz. \]
**Example 2 revisited.** We have the weight 1/2 HMF

\[
q^{-\ell_b/24} R(\zeta_b^a; q^\ell_b) + i(3^{-1}\ell_b)^{1/2} \sin\left(\frac{\pi a}{b}\right) \int_{-\tau}^{i\infty} \frac{\Theta\left(\frac{a}{b}; \ell_b z\right)}{\sqrt{-i(z + \tau)}} dz.
\]

The **shadow** of the mock modular form \( q^{-\ell_b/24} R(\zeta_b^a; q^\ell_b) \) is (up to a constant multiple) the theta function \( \Theta\left(\frac{a}{b}; \ell_b z\right) \).
Mock theta functions

Question.
Question.

Ramanujan’s “definition” of a mock theta function?
“Definition” (Ramanujan). A mock theta function $F$ satisfies:

1. infinitely many roots of unity are exponential singularities,
2. for every root of unity $\zeta$ there is a modular form $\vartheta^\zeta(q)$ such that the difference $F(q) - q^{\alpha} \vartheta^\zeta(q)$ is bounded as $q \to \zeta$ radially,
3. there does not exist a single modular form $\vartheta(q)$ such that $F(q) - q^{\alpha} \vartheta(q)$ is bounded as $q \to$ any root of unity radially.
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3. there does not exist a single modular form $\vartheta(q)$ such that $F(q) - q^\alpha \vartheta(q)$ is bounded as $q$ approaches any root of unity radially.
Mock theta functions

I have proved that if

\[ f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots \]

then \[ f(q) + (1-q)(1-q^2)(1-q^3) \cdots \approx 1 - 2q + 2q^2 - 2q^3 + \cdots \]

at all the \( q = 0 \) \( = O(1) \)

at all the points \( q = -1, q^3 = -1, q^5 = -1 \)

and at the same time

\[ f(q) \approx (1-q)(1-q^2)(1-q^3) \cdots (1-2q+2q^3-\cdots) \]

\( = O(1) \)

at all the points \( q^2 = -1, q^4 = -1, q^6 = -1, \ldots \)

Also obviously \( f(q) = O(1) \)

at all the points \( q = 1, q^2 = 1, q^3 = 1, \ldots \)

-S. Ramanujan to G.H. Hardy, 1920
Mock theta functions

**Ramanujan's mock theta function**

\[ f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \frac{q^9}{(1+q)^2(1+q^2)^2(1+q^3)^2} + \cdots \]

has singularities when
Mock theta functions

**Ramanujan’s mock theta function**

\[ f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \frac{q^9}{(1+q^2)^2(1+q^3)^2} + \cdots \]

has singularities when \( q^n = -1 \)  \( (n \in \mathbb{N}) \).
Mock theta functions

...roots of unity.
Mock theta functions

Ramanujan’s observations:

- There is a(n explicit) modular form $b(q)$ that “cuts out” the exponential singularities of $f(q)$. 
Mock theta functions

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- There is an explicit modular form \( b(q) \) that “cuts out” the exponential singularities of \( f(q) \).

- That is, as \( q \) approaches any even order \( 2k \) root of unity singularity of \( f(q) \), then

\[
\lim_{q \to \text{root of unity}} f(q) = O(1)
\]
Ramanujan’s observations:

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\[ f(q) - (-1)^k b(q) = O(1) \]
Mock theta functions

Ramanujan’s observations:

- There is an explicit modular form $b(q)$ that “cuts out” the exponential singularities of $f(q)$.

- That is, as $q$ approaches any even order $2k$ root of unity singularity of $f(q)$, then

  $$f(q) - (-1)^k b(q) = O(1)$$

- That is, asymptotically, towards singularities,

  mock theta $\pm$ modular form $=$ bounded
\[ b(q) := q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)} \]

(joint work with K. Ono, R.C. Rhoades)
$b(q) := q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)}$

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As $q \to -1$, we computed (with help of R. Lemke Oliver)
Ramanujan revisited

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\[ f(-0.994) \sim -1 \cdot 10^{31}, \]
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(joint work with K. Ono, R.C. Rhoades)

As $q \to -1$, we computed (with help of R. Lemke Oliver)

$f(-0.994) \sim -1 \cdot 10^{31}$, $f(-0.996) \sim -1 \cdot 10^{46}$,
Ramanujan revisited

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(joint work with K. Ono, R.C. Rhoades)

As \( q \to -1 \), we computed (with help of R. Lemke Oliver)

\[ f(-0.994) \sim -1 \cdot 10^{31}, \quad f(-0.996) \sim -1 \cdot 10^{46}, \quad f(-0.998) \sim -6 \cdot 10^{90} \ldots \]
Ramanujan’s observation gives:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$-0.990$</th>
<th>$-0.992$</th>
<th>$-0.994$</th>
<th>$-0.996$</th>
<th>$-0.998$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(q) + b(q)$</td>
<td>3.961...</td>
<td>3.969...</td>
<td>3.976...</td>
<td>3.984...</td>
<td>3.992...</td>
</tr>
</tbody>
</table>
Ramanujan’s observation gives:

\[
\begin{array}{c|ccccc}
q & -0.990 & -0.992 & -0.994 & -0.996 & -0.998 \\
\hline
f(q) + b(q) & 3.961 \ldots & 3.969 \ldots & 3.976 \ldots & 3.984 \ldots & 3.992 \ldots \\
\end{array}
\]

This suggests that

\[
\lim_{q \to -1} (f(q) + b(q)) = 4.
\]
This suggests that \( \lim_{q \to i} (f(q) - b(q)) = 4.46 \).
This suggests that
\[
\lim_{{q \to i}} (f(q) - b(q)) = 4i.
\]
i) What are the $O(1)$ constants in

$$\lim_{q \to \zeta}(f(q) - (-1)^k b(q)) = O(1)?$$
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$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = O(1)?$$

ii) How do they arise?
Theorem (F-Ono-Rhoades)

If $\zeta$ is an even $2k$ order root of unity, then

$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = -4 \sum_{n=0}^{k-1} (1 + \zeta)^2 (1 + \zeta^2)^2 \cdots (1 + \zeta^n)^2 \zeta^{n+1}.$$
Theorem (F-Ono-Rhoades)

If $\zeta$ is an even $2k$ order root of unity, then

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Remark. We prove this as a special case of a more general theorem involving:
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\[
R(w; q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq; q)_n(w^{-1}q; q)_n} \quad \text{(Dyson’s rank)}
\]

\[
C(w; q) := \frac{(q; q)_\infty}{(wq; q)_\infty(w^{-1}q; q)_\infty} \quad \text{(Andrews-Garvan crank)}
\]

\[
U(w; q) := \sum_{n=0}^{\infty} (wq; q)_n(w^{-1}q; q)_nq^{n+1} \quad \text{(Unimodal rank)}
\]
Let

\[ N(m, n) := \#\{\text{partitions } \lambda \text{ of } n \mid \text{rank}(\lambda) = m\}, \]

\[ M(m, n) := \#\{\text{partitions } \lambda \text{ of } n \mid \text{crank}(\lambda) = m\}, \]

\[ u(m, n) := \#\{\text{size } n \text{ strongly unimodal sequences with rank } m\}. \]
A sequence \( \{a_j\}_{j=1}^{s} \) of integers is called *strongly unimodal* of size \( n \) if

\[
\begin{align*}
& a_1 + a_2 + \cdots + a_s = n, \\
& 0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > a_s > 0
\end{align*}
\]
A sequence \( \{a_j\}_{j=1}^s \) of integers is called **strongly unimodal** of size \( n \) if

- \( a_1 + a_2 + \cdots + a_s = n \),
- \( 0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots a_s > 0 \) for some \( r \).
A sequence \( \{a_j\}_{j=1}^s \) of integers is called \textit{strongly unimodal} of size \( n \) if

- \( a_1 + a_2 + \cdots + a_s = n \),
- \( 0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots a_s > 0 \) for some \( r \).

The \textit{rank} equals \( s - 2r + 1 \) (difference between \# terms after and before the “peak”).
Combinatorial “modular” forms

\[
R(w; q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n},
\]

mock modular [Bringmann-Ono]

\[
C(w; q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) w^m q^n = \frac{(q; q)_\infty}{(wq; q)_\infty (w^{-1}q; q)_\infty},
\]

modular

\[
U(w; q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} u(m, n) (-w)^m q^n = \sum_{n=0}^{\infty} (wq; q)_n (w^{-1}q; q)_n q^{n+1}.
\]
Theorem (F-Ono-Rhoades)

If $\zeta_b = e^{\frac{2\pi i}{b}}$ and $1 \leq a < b$, then for every suitable root of unity $\zeta$ there is an explicit integer $c$ for which

$$\lim_{q \to \zeta} \left( R(\zeta_b^a; q) - \zeta_b^c C(\zeta_b^a; q) \right) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) U(\zeta_b^a; \zeta).$$
Remark

The first theorem is the special case $a = 1, b = 2$, using that

\[ R(-1; q) = f(q) \quad \text{and} \quad C(-1; q) = b(q). \]
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*The first theorem is the special case* $a = 1$, $b = 2$, *using that*

\[ R(-1; q) = f(q) \text{ and } C(-1; q) = b(q). \]

Remark

*Specializations of* $R(w; q)$ *give rise to other mock theta functions.*
\[
\lim_{q \to \zeta} \left( f(q) - (-1)^k b(q) \right) = -4U(-1; \zeta).
\]
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\]
\[
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\]

mock modular \quad modular
\[
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\]

mock modular \hspace{2cm} \textit{modular} \hspace{2cm} ?
Quantum modular forms

- D. Zagier, 2010
Quantum modular forms

“...we want to discuss...another type of modular object which...we call quantum modular forms.

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“...we want to discuss...another type of modular object which...we call quantum modular forms. These are objects which live at the boundary of the space..., ...and have a transformation behavior of a quite different type...”

-D. Zagier, 2010
Quantum modular forms are defined in $\mathbb{Q}$,
Quantum modular forms are defined in $\mathbb{Q}$, and take values in $\mathbb{C}$. 
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They exhibit modular symmetry in \( \mathbb{Q} \)...

...up to the addition of smooth error functions in \( \mathbb{R} \).
Quantum modular forms

Let

\[ F : \mathbb{H} \to \mathbb{C}, \quad \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \text{SL}_2(\mathbb{Z}), \quad \tau \in \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \]
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Modular transformation:
\[
F(\tau) - \epsilon^{-1}(\gamma)(c\tau + d)^{-k}F\left(\frac{a\tau + b}{c\tau + d}\right) = 0
\]
Let $F : \mathbb{Q} \to \mathbb{C}$, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \text{SL}_2(\mathbb{Z})$, $x \in \mathbb{Q}$.

**Modular transformation:**

$$F(x) - \varepsilon^{-1}(\gamma)(cx + d)^{-k}F\left(\frac{ax + b}{cx + d}\right) = ?$$
Definition (Zagier ’10)

A quantum modular form of weight $k$ ($k \in \frac{1}{2}\mathbb{Z}$) is function $F : \mathbb{Q} \to \mathbb{C}$, such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the functions

$$h_\gamma(x) = h_{F,\gamma}(x) := F(x) - \epsilon^{-1}(\gamma)(cx + d)^{-k}F\left(\frac{ax + b}{cx + d}\right)$$

extend to suitably continuous or analytic functions in $\mathbb{R}$. 
Quantum modular forms

Zagier’s examples arise from areas such as:

➤ theta series associated to indefinite quadratic forms
➤ quantum invariants of 3-manifolds
➤ Jones polynomials for knots
Quantum modular forms

The real part of a quantum modular form

\[ g(x) \]

Image Credit: D. Zagier, 2010
Quantum modular forms

The real part of a quantum modular form

The real and imaginary parts of its error to symmetry

\( g(x) \)

Image Credit: D. Zagier, 2010
Quantum modular forms

The real part of a quantum modular form

```
g(x) - g(-1/x)
```

Image Credit: D. Zagier, 2010
Theorem (F-Ono-Rhoades)

If $\zeta$ is an even $2k$ order root of unity, then

$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = -4U(-1; \zeta) = -4 \sum_{n=0}^{k-1} (1 + \zeta)^2 (1 + \zeta^2)^2 \cdots (1 + \zeta^n)^2 \zeta^{n+1}$$

This can be realized as the value of a function on $Q_{\zeta}$, which is a quantum modular form.
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This can be realized as the value of a function on $\mathbb{Q}$... ...which is a quantum modular form.

(Bryson-Ono-Pittman-Rhoades, F-Ki-Truong Vu)
Let $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$, and $q = e^{2\pi i \tau}$. 

Rational number $r/s$ 

Root of unity $e^{2\pi i \tau}$
\[
\lim_{q \to \zeta} \left( f(q) - (-1)^k b(q) \right) = -4U(-1; \zeta).
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mock modular
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Proof ingredients

- Ramanujan’s identity

\[
\sum_{n=0}^{\infty} \frac{(\alpha \beta)^n q^{n^2}}{(\alpha q; q)_n (\beta q; q)_n} + \sum_{n=1}^{\infty} q^n (\alpha^{-1}; q)_n (\beta^{-1}; q)_n = iq^{1/8} (1 - \alpha) (\beta \alpha^{-1})^{1/2} (q \alpha^{-1}; q)_\infty (\beta^{-1}; q)_\infty \mu(u, v; \tau).
\]

\(q = e^{2\pi i \tau}, \alpha = e^{2\pi i u}, \beta = e^{2\pi i v}\)
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\((q = e^{2\pi i \tau}, \alpha = e^{2\pi i u}, \beta = e^{2\pi i v})\)

- Transformation theory (Zwegers) of the mock Jacobi form

\[
\mu(u, v; \tau) := \frac{e^{\pi i u}}{\vartheta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}.
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Proof ingredients

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- Explicit asymptotic calculations
Further results

Bringmann-Rolen/Jang-Lobrich: more general “radial limit” theorems for the Gordon-McIntosh universal mock $\vartheta$s:

\[
g_2(w; q) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{\frac{n(n+1)}{2}}}{(w; q)_{n+1} (w^{-1}q; q)_{n+1}}, \quad g_3(w; q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(w; q)_n (w^{-1}q; q)_n}.
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\]

Theorem (Bringmann-Rolen)

There is a linear combination of theta functions \( \vartheta_{a,b,A,B,\zeta}(q) \) such that the radial limit difference \( \lim_{q \to \zeta} (g_2(\zeta^a q^A; q^B) - \vartheta_{a,b,A,B,\zeta}(q)) \) is bounded, and is the special value of a quantum modular form.
Further results

“...[no one has] proved that any of Ramanujan’s mock theta functions are really mock theta functions according to his definition.”

-B.C. Berndt, *Ramanujan, his lost notebook, its importance.*
Further results

By incorporating the theory of harmonic Maass forms,
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Theorem (Griffin-Ono-Rolen)
Ramanujan’s mock theta functions satisfy his definition.
Further results

Theorem (Choi-Lim-Rhoades)

Let $F$ be a mock modular form and $\zeta$ a root of unity. There is a weakly holomorphic modular form $q^{\alpha} \vartheta_{\zeta}(q)$ such that the radial limit $\lim_{q \to \zeta} (F(q) - q^{\alpha} \vartheta_{\zeta}(q))$ is the special value of a quantum modular form.
Further results

- Mathematical physics
- Moonshine, Representation theory
- Combinatorics
- Topology
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Thank you