Mock modular and quantum modular forms

Modularity in quantum systems, KITP

by

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Modular forms

Example 1.

Modular forms

Example 1.

Definition. For $m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, the m***th power divisor function*** is defined by

$$\sigma_m(n) := \sum_{\substack{d \mid n \\ d>0}} d^m.$$

Modular forms

A generating function:

$$\sum_{n=1}^{\infty} \sigma_m(n) q^n$$

Modular forms

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$$\sum_{n=1}^{\infty} \sigma_m(n) q^n = \sigma_m(1)q + \sigma_m(2)q^2 + \sigma_m(3)q^3 + \sigma_m(4)q^4 + \cdots$$

Modular forms

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$$= q + (1 + 2^m)q^2 + (1 + 3^m)q^3 + (1 + 2^m + 4^m)q^4 + \cdots$$

Modular forms

Let $m = 2k - 1,\ k \in \{2, 3, 4, 5, \ldots\}$.

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Definition. The **weight** $2k$ **Eisenstein series** is defined by

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

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$B_\kappa := \kappa$th Bernoulli number.

Modular forms

- The modular group

$$\Gamma = \mathsf{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

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- Γ is generated by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Modular forms

- Γ acts on $\mathbb{H}$ by

$$\Gamma \cdot \mathbb{H} \longrightarrow \mathbb{H}$$
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Modular forms

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Ex. $T \cdot \tau = \tau + 1, \quad S \cdot \tau = -1/\tau$

- E_{2k} is "symmetric" with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$...

Modular forms

That is, for all $\tau \in \mathbb{H}$,

$$E_{2k}(\tau + 1) = E_{2k}(\tau),$$
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Modular forms

That is, for all $\tau \in \mathbb{H}$,

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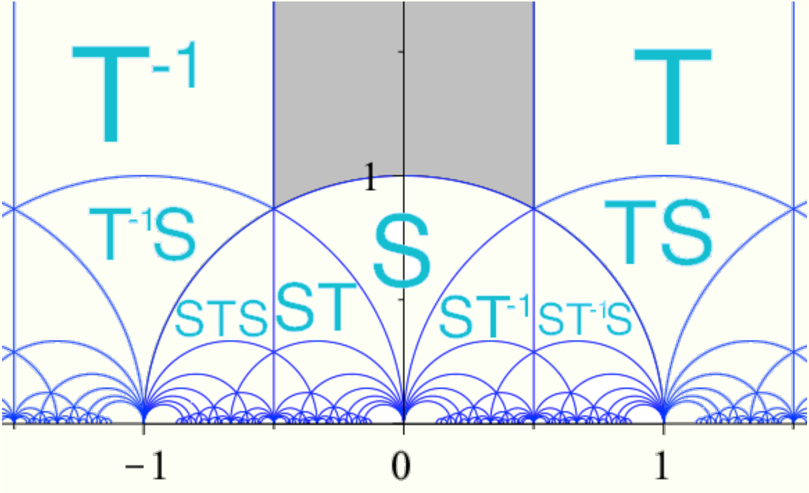
$$E_{2k}(-1/\tau) = \tau^{2k} E_{2k}(\tau),$$

and in general, for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} E_{2k}(\tau).$$

Modular forms



Modular forms

Modular forms

A **weight** $k \in \frac{1}{2}\mathbb{Z}$ **modular form** $f : \mathbb{H} \to \mathbb{C}$ **on** $\Gamma' \subseteq \mathsf{SL}_2(\mathbb{Z})$ satisfies

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- f is holomorphic or meromorphic in the cusps.

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- f is holomorphic or meromorphic in the cusps.
  e.g., f has a Fourier expansion of the shape

$$f(\tau) = \sum_{n=m}^{\infty} a_n q^{\frac{n}{h}},$$

where $q = e^{2\pi i\tau}$, $a_n = a_{f,n} \in \mathbb{C}$, $m = m_f \in \mathbb{Z}$, $h = h_f \in \mathbb{N}$.

Modular forms

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We define the ***partition function*** by

$$p(n) := \#\{\text{partitions of } n\},$$

and we let $p(0) := 1$.

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Ex. $p(4) = 5$, since $4 = 4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$.

Modular forms

Theorem (Euler). Let $|q| < 1$. The partition generating function

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

satisfies

$$P(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

Modular forms

That is,

$$q^{-\frac{1}{24}} \sum_{n=0}^{\infty} p(n) q^n = \frac{1}{\eta(\tau)},$$

where the Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

with $q = e^{2\pi i \tau}, \tau \in \mathbb{H}$, is a modular form of weight $1/2$.

Modular forms

Example 2 (cont.)

Modular forms

Example 2 (cont.)

A consequence of modularity:

Theorem (Hardy-Ramanujan-Rademacher) We have the exact formula

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{m=1}^{\infty} \frac{A_m(n)}{m} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6m} \right).$$

Modular symmetry

Question.

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What could be gained by perturbing modular symmetry?

“Modular” forms

Example 1 revisited.

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Example 1 revisited.

Let $k = 1$. The function

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is not a (weight 2) modular form.

“Modular” forms

Example 1 revisited.

Let $k = 1$. The function

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is not a (weight 2) modular form. Namely, we have that

$$E_2(-1/\tau) = \tau^2 E_2(\tau) \underbrace{- \frac{6i\tau}{\pi}}_{\text{"error to modularity"}}.$$

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Define the function

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}.$$

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Then

$$\widehat{E}_2(-1/\tau) = \tau^2\, \widehat{E}_2(\tau).$$

That is, $\widehat{E}_2$ is an *almost holomorphic* weight 2 modular form. (Kaneko-Zagier)

“Modular” forms

Example 2 revisited.

Definition (Dyson). The ***rank*** of a partition is defined to be its largest part minus the number of its parts.

“Modular” forms

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Definition (Dyson). The ***rank*** of a partition is defined to be its largest part minus the number of its parts. For $n \in \mathbb{N}, m \in \mathbb{Z}$, we define

$$N(m, n) := p(n \mid \text{rank } m),$$

and let $N(m, 0) := \delta_{m,0}$.

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Ex. We have that $N(m, 4) = \begin{cases} 1, & m = 0, \pm 1, \pm 3, \\ 0 & \text{else} . \end{cases}$

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Note. For fixed n, we have that $\displaystyle\sum_{m=-\infty}^{\infty} N(m,n) = p(n)$.

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Example 2 revisited.

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Example 2 revisited.

The two variable partition rank generating function satisfies

$$R(w;q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n},$$

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where for $n \in \mathbb{N}_0$, the *q-Pochhammer symbol* is defined by

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots(1-aq^{n-1}).$$

“Modular” forms

Observation. We have that

$$R(1;q) = \sum_{n=0}^{\infty} p(n)q^n = P(q)$$

is (essentially) a modular form, with $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$.

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Question. Is $R(w;q)$ a modular form for other fixed values of w, when viewed as a function of τ, with $q = e^{2\pi i\tau}$?

“Modular” forms

Let $w = -1$. Then

$$R(-1;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = \sum_{n=0}^{\infty} (N_e(n) - N_o(n))q^n,$$

where $N_{e\,(\text{resp. } o)}(n) := p(n \mid \text{even (resp. odd) rank})$.

Mock theta functions

Ramanujan's mock theta function:

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

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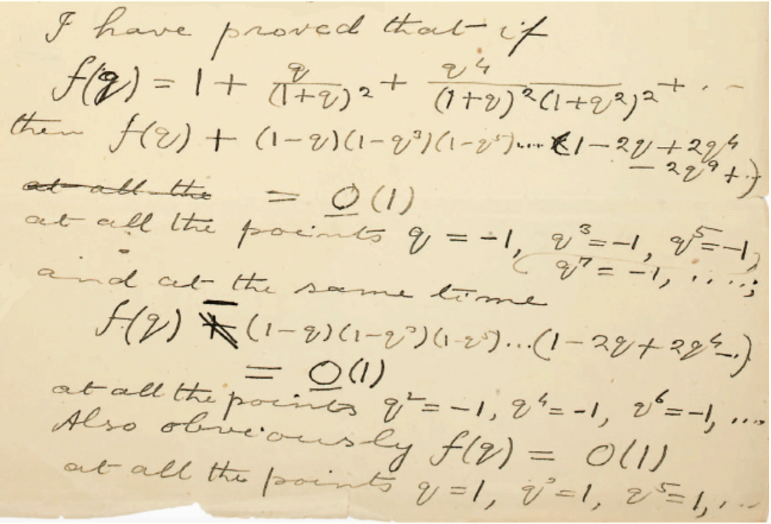
$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots$$

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)}$$

$$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3).$$

$$f(q) = 1 + \frac{q^2}{1+q} + \frac{q^6}{(1+q)(1+q^2)} + \frac{q^{12}}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

Mock theta functions


I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$

~~at all the~~ $= O(1)$

at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, $\ldots$;

and at the same time

$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots)$

$= O(1)$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, $\ldots$

Also obviously $f(q) = O(1)$

at all the points $q=1$, $q^3=1$, $q^5=1$, $\ldots$

-S. Ramanujan to G.H. Hardy, 1920

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Ramanujan's observations:

- There is a(n explicit) modular form $b(q)$ that "cuts out" the exponential singularities of $f(q)$.

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- That is, asymptotically, towards singularities,

$$\boxed{\text{mock theta} \pm \text{modular form} = \text{bounded}}$$

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Theorem (Watson). Let $q = e^{-\alpha}, \beta = \pi^2/\alpha, q_1 = e^{-\beta}$, where $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$. Then

$$q^{-\frac{1}{24}} f(q) = 2\sqrt{\frac{2\pi}{\alpha}} q_1^{\frac{4}{3}} \omega(q_1^2) + 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty \frac{\sinh(\alpha t)}{\sinh\left(\frac{3\alpha t}{2}\right)} e^{-\frac{3\alpha t^2}{2}} dt.$$

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Remark. This may be interpreted as a transformation under $\tau \mapsto -1/(2\tau)$.

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Mock theta functions

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Mock theta functions

> "Theorem" (S. Zwegers, 2002). Ramanujan's mock theta functions are not modular forms, but they can be completed to form nonholomorphic modular forms.

Mock theta functions

Zwegers' completion:

$$f(q)$$

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$\nearrow$ modular — $\uparrow$ not modular — $\nwarrow$ not holomorphic

Mock theta functions

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$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

$$\downarrow$$

$$\widehat{f}(\tau) := q^{-\frac{1}{24}} f(q) + 2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{g(z)\, dz}{\sqrt{-i(\tau+z)}}$$

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where $g(\tau) := -\sum_{n=-\infty}^{\infty} (n+\frac{1}{6}) q^{\frac{3}{2}(n+\frac{1}{6})^2}$ is a weight $3/2$ modular form.

Harmonic Maass forms

Definition (Bruinier-Funke). A ***weight*** k ***(***$k \in \frac{1}{2}\mathbb{Z}$***) harmonic Maass form on*** $\Gamma' = \Gamma_0(N)$, where $4|N$ if $k \in \frac{1}{2} + \mathbb{Z}$, is a smooth $M\colon \mathbb{H} \to \mathbb{C}$ satisfying

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i) For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have

$$M\left(\frac{a\tau + b}{c\tau + d}\right) = \begin{cases} (c\tau + d)^k M(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-2k} (c\tau + d)^k M(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

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$$M\left(\frac{a\tau + b}{c\tau + d}\right) = \begin{cases} (c\tau + d)^k M(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-2k} (c\tau + d)^k M(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

ii) We have that $\Delta_k(M) = 0$, where (if $\tau = x + iy$)
$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$

Harmonic Maass forms

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iii) There exists a polynomial $P_M(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$M(\tau) - P_M(\tau) = O\left(e^{-\varepsilon y}\right)$$

as $y \to \infty$ for some $\varepsilon > 0$.

Harmonic Maass forms

Lemma. Let $k \in \frac{1}{2}\mathbb{Z}\backslash\{1\}$ and $\Gamma \in \{\Gamma_0(N), \Gamma_1(N)\}$.
If M is a HMF, then M has Fourier expansion

$$M(\tau) = \sum_{n \gg -\infty} c_M^+(n) q^n + \sum_{n<0} c_M^-(n) \Gamma(1-k, -4\pi n y) q^n.$$

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The *incomplete gamma function* is defined by

$$\Gamma(s, z) := \int_z^\infty e^{-t} t^s \frac{dt}{t}.$$

Harmonic Maass forms

That is,

$$M(\tau) = \underbrace{\sum_{n \gg -\infty} c_M^+(n) q^n}_{\text{“holomorphic part”}} + \underbrace{\sum_{n<0} c_M^-(n) \Gamma(1-k, -4\pi n y) q^n}_{\text{“nonholomorphic part”}}$$

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Definition (Zagier).

Harmonic Maass forms

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> Definition (Zagier). A ***mock modular form*** (of weight k) is the holomorphic part of a harmonic Maass form (of weight k)*
> (*for which the NHP is nontrivial).

Harmonic Maass forms

Theorem (Zwegers). Ramanujan's mock theta functions are* weight 1/2 mock modular forms.

Harmonic Maass forms

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$$F(\tau) = q^{\alpha_F} G_F^+(\tau) + c_F,$$

where G_F^+ is the holomorphic part of a weight 1/2 HMF.

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(*up to multiplication by a power of q and addition of a constant)

“Modular” forms

Example 2 (revisited.)

“Modular” forms

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A consequence of mock modularity:

Theorem (Bringmann-Ono), Conjectured by Andrews-Dragonette.
We have the exact formula

$$N_e(n) - N_o(n)$$
$$= \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{m=1}^{\infty} (-1)^{\lfloor \frac{m+1}{2} \rfloor} \frac{A_{2m}\left(n - \frac{m(1+(-1)^m)}{4}\right)}{m} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12m}\right).$$

Harmonic Maass forms

Example 2 (revisited). Let $\zeta_N := e^{2\pi i/N}$.

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Theorem (Bringmann-Ono). Fix $w = \zeta_b^a \neq 1$. Then $R(\zeta_b^a; q)$ (with $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$) is a weight $1/2$ mock modular form.

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$$q^{-\ell_b/24} R(\zeta_b^a; q^{\ell_b}) + i(3^{-1}\ell_b)^{\frac{1}{2}} \sin(\tfrac{\pi a}{b}) \int_{-\overline{\tau}}^{i\infty} \frac{\Theta(\frac{a}{b}; \ell_b z)}{\sqrt{-i(z+\tau)}} dz$$

is a harmonic Maass form of weight $1/2$ and level 144.

Harmonic Maass forms

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Here, Θ is a weight $3/2$ modular form, and $\ell_b \in \mathbb{N}$.

Harmonic Maass forms

Example 2 revisited. We have the weight $1/2$ HMF

$$q^{-\ell_b/24}R(\zeta_b^a;q^{\ell_b}) + i(3^{-1}\ell_b)^{\frac{1}{2}}\sin(\tfrac{\pi a}{b})\int_{-\overline{\tau}}^{i\infty}\frac{\Theta(\frac{a}{b};\ell_b z)}{\sqrt{-i(z+\tau)}}dz.$$

Harmonic Maass forms

Example 2 revisited. We have the weight $1/2$ HMF

$$q^{-\ell_b/24}R(\zeta_b^a;q^{\ell_b})+i(3^{-1}\ell_b)^{\frac{1}{2}}\sin(\tfrac{\pi a}{b})\int_{-\overline{\tau}}^{i\infty}\frac{\Theta(\frac{a}{b};\ell_b z)}{\sqrt{-i(z+\tau)}}dz.$$

The **shadow** of the mock modular form $q^{-\ell_b/24}R(\zeta_b^a;q^{\ell_b})$ is (up to a constant multiple) the theta function $\Theta(\frac{a}{b};\ell_b z)$.

Mock theta functions

Question.

Mock theta functions

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Ramanujan’s “definition” of a mock theta function?

Mock theta functions

“Definition” (Ramanujan). A **mock theta function** F satisfies

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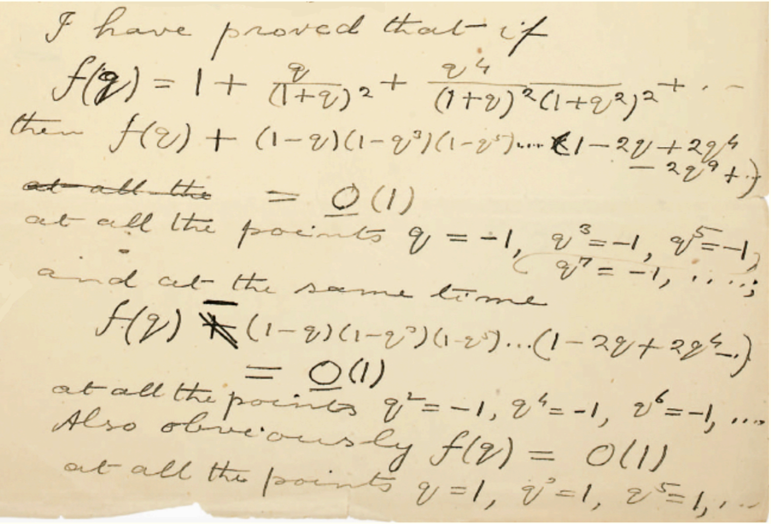
Mock theta functions

"Definition" (Ramanujan). A **mock theta function** F satisfies

1. infinitely many roots of unity are exponential singularities,
2. for every root of unity ζ there is a modular form $\vartheta_\zeta(q)$ such that the difference $F(q) - q^\alpha \vartheta_\zeta(q)$ is bounded as $q \to \zeta$ radially,
3. there does not exist a single modular form $\vartheta(q)$ such that $F(q) - q^\alpha \vartheta(q)$ is bounded as q approaches any root of unity radially.

Mock theta functions



I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$

~~at all the~~ $= O(1)$

at all the points $q=-1,\ q^3=-1,\ q^5=-1,\ q^7=-1, \ldots;$

and at the same time

$$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots) = O(1)$$

at all the points $q^2=-1,\ q^4=-1,\ q^6=-1, \ldots$

Also obviously $f(q) = O(1)$

at all the points $q=1,\ q^3=1,\ q^5=1, \ldots$

-S. Ramanujan to G.H. Hardy, 1920

Mock theta functions

Ramanujan's mock theta function

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \frac{q^9}{(1+q)^2(1+q^2)^2(1+q^3)^2} + \cdots$$

has singularities when

Mock theta functions

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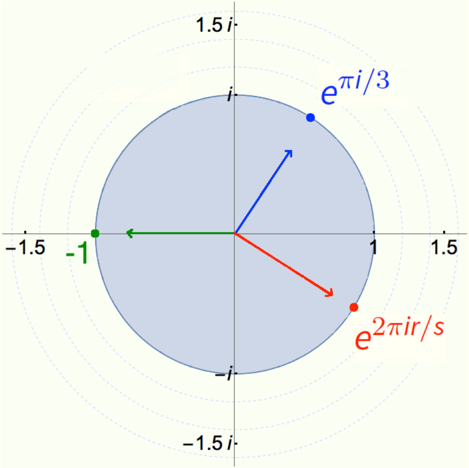
has singularities when $q^n = -1 \quad (n \in \mathbb{N})$.

Mock theta functions



...roots of unity

Mock theta functions

Ramanujan's observations:

- There is a(n explicit) modular form $b(q)$ that "cuts out" the exponential singularities of $f(q)$.

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- That is, as q approaches any even order $2k$ root of unity singularity of $f(q)$, then

$$\boxed{f(q) - (-1)^k b(q) = O(1)}$$

- That is, asymptotically, towards singularities,

$$\boxed{\text{mock theta} \pm \text{modular form} = \text{bounded}}$$

Ramanujan revisited

$b(q) := q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)}$

(joint work with K. Ono, R.C. Rhoades)

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As $q \to -1$, we computed (with help of R. Lemke Oliver)

$f(-0.994) \sim -1{\cdot}10^{31}$, $f(-0.996) \sim -1{\cdot}10^{46}$, $f(-0.998) \sim -6{\cdot}10^{90} \ldots$

Ramanujan revisited

Ramanujan's observation gives:

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\ldots$	$3.969\ldots$	$3.976\ldots$	$3.984\ldots$	$3.992\ldots$

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Ramanujan revisited

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.

This suggests that

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

Ramanujan revisited

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$2 \cdot 10^6 - 4.6 \cdot 10^6 i$	$2 \cdot 10^8 - 4 \cdot 10^8 i$	$1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.05 + 3.85i$	$\sim 0.04 + 3.89i$	$\sim 0.03 + 3.92i$

Ramanujan revisited

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$f(q) - b(q)$	$\sim 0.05 + 3.85i$	$\sim 0.04 + 3.89i$	$\sim 0.03 + 3.92i$

This suggests that

$$\lim_{q \to i} (f(q) - b(q)) = 4i.$$

Ramanujan revisited

i) What are the $O(1)$ constants in

$$\lim_{q\to\zeta}(f(q) - (-1)^k b(q)) = O(1)?$$

Ramanujan revisited

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$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = O(1)?$$

ii) How do they arise?

Ramanujan revisited

Theorem (F-Ono-Rhoades)

If ζ is an even $2k$ order root of unity, then

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

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Ramanujan revisited

Remark. We prove this as a special case of a more general theorem involving:

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$$R(w;q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n} \qquad \text{(Dyson's rank)}$$

$$C(w;q) := \frac{(q;q)_\infty}{(wq;q)_\infty (w^{-1}q;q)_\infty} \qquad \text{(Andrews-Garvan crank)}$$

$$U(w;q) := \sum_{n=0}^{\infty} (wq;q)_n (w^{-1}q;q)_n q^{n+1} \qquad \text{(Unimodal rank)}$$

Combinatorial “modular” forms

Let

$$N(m,n) := \#\{\text{partitions } \lambda \text{ of } n \mid \text{rank}(\lambda) = m\},$$

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$$u(m,n) := \#\{\text{size } n \text{ strongly unimodal sequences with rank } m\}.$$

Combinatorial “modular” forms

A sequence $\{a_j\}_{j=1}^{s}$ of integers is called *strongly unimodal* of size n if

- $a_1 + a_2 + \cdots + a_s = n$,

Combinatorial "modular" forms

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- $0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots a_s > 0$ for some r.

The *rank* equals $s - 2r + 1$ (difference between # terms after and before the "peak").

Combinatorial “modular” forms

$$R(w;q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n},$$

mock modular [Bringmann-Ono]

$$C(w;q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) w^m q^n = \frac{(q;q)_\infty}{(wq;q)_\infty (w^{-1}q;q)_\infty},$$

modular

$$U(w;q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} u(m,n) (-w)^m q^n = \sum_{n=0}^{\infty} (wq;q)_n (w^{-1}q;q)_n q^{n+1}.$$

Ramanujan revisited

Theorem (F-Ono-Rhoades)

If $\zeta_b = e^{\frac{2\pi i}{b}}$ and $1 \leq a < b$, then for every suitable root of unity ζ there is an explicit integer c for which

$$\lim_{q \to \zeta} \left(R(\zeta_b^a; q) - \zeta_{b^2}^c C(\zeta_b^a; q) \right) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) U(\zeta_b^a; \zeta).$$

Ramanujan revisited

Remark

The first theorem is the special case $a = 1$, $b = 2$, using that

$$R(-1; q) = f(q) \quad \text{and} \quad C(-1; q) = b(q).$$

Ramanujan revisited

Remark

The first theorem is the special case $a = 1$, $b = 2$, *using that*

$$R(-1;q) = f(q) \text{ and } C(-1;q) = b(q).$$

Remark

Specializations of $R(w;q)$ *give rise to other mock theta functions.*

Ramanujan revisited

$$\lim_{q\to\zeta}\Big(f(q)-(-1)^k b(q)\Big) = -4U(-1;\zeta).$$

Ramanujan revisited

$$\lim_{q \to \zeta} \Big(f(q) - (-1)^k b(q) \Big) = -4U(-1;\zeta).$$

↗

mock modular

Ramanujan revisited

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$\nearrow$ mock modular

$\uparrow$ modular

Ramanujan revisited

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$\uparrow$ modular

$\nwarrow$?

Quantum modular forms

Quantum modular forms

-D. Zagier, 2010

Quantum modular forms

Quantum modular forms

"...we want to discuss...another type of modular object which...we call **quantum modular forms***.*

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"...we want to discuss...another type of modular object which...we call **quantum modular forms***.*

These are objects which live at the boundary of the space...,

...and have a transformation behavior of a quite different type..."

-D. Zagier, 2010

Quantum modular forms

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Quantum modular forms

- Quantum modular forms are defined in $\mathbb{Q}$, and take values in $\mathbb{C}$.
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  ...up to the addition of smooth error functions in $\mathbb{R}$.

Quantum modular forms

Let

$F : \mathbb{H} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathsf{SL}_2(\mathbb{Z}),\ \tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathsf{Im}(\tau) > 0\}$

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Modular transformation:

$$F(\tau) - \epsilon^{-1}(\gamma)(c\tau + d)^{-k} F\left(\frac{a\tau + b}{c\tau + d}\right) = 0$$

Quantum modular forms

Let $F : \mathbb{Q} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathsf{SL}_2(\mathbb{Z}), \quad x \in \mathbb{Q}.$

Modular transformation:

$$F(x) - \epsilon^{-1}(\gamma)(cx+d)^{-k} F\left(\frac{ax+b}{cx+d}\right) = ?$$

Quantum modular forms

Definition (Zagier '10)

A **quantum modular form of weight** k ($k \in \frac{1}{2}\mathbb{Z}$) is function $F : \mathbb{Q} \to \mathbb{C}$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the functions

$$h_\gamma(x) = h_{F,\gamma}(x) := F(x) - \epsilon^{-1}(\gamma)(cx+d)^{-k} F\left(\frac{ax+b}{cx+d}\right)$$

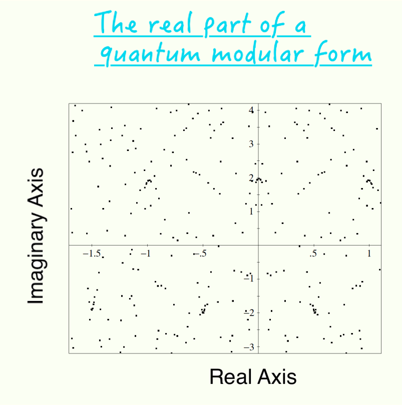
extend to suitably continuous or analytic functions in $\mathbb{R}$.

Quantum modular forms

Zagier's examples arise from areas such as:

- theta series associated to indefinite quadratic forms
- quantum invariants of 3-manifolds
- Jones polynomials for knots

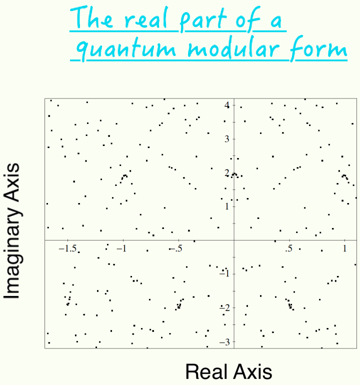
Quantum modular forms

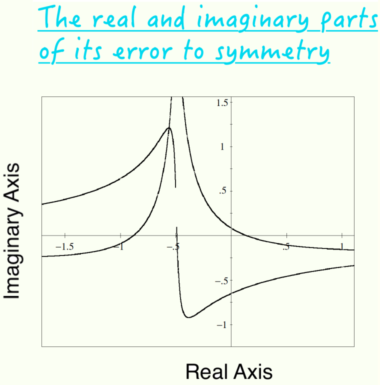


$g(x)$

Image Credit: D. Zagier, 2010

Quantum modular forms



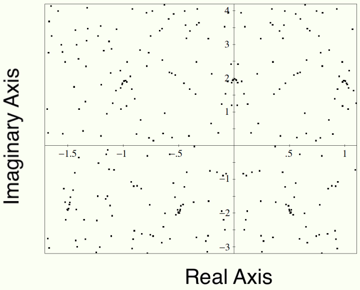


$g(x)$

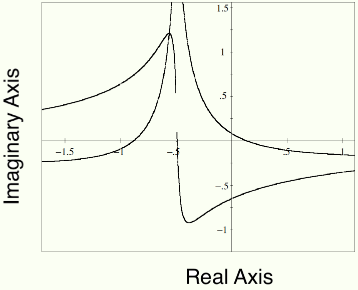
Image Credit: D. Zagier, 2010

Quantum modular forms

The real part of a quantum modular form



The real and imaginary parts of its error to symmetry



$g(x)$

"$g(x) - g(-1/x)$"

Image Credit: D. Zagier, 2010

Ramanujan revisited

Theorem (F-Ono-Rhoades)

If ζ is an even $2k$ order root of unity, then

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4U(-1;\zeta) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}$$

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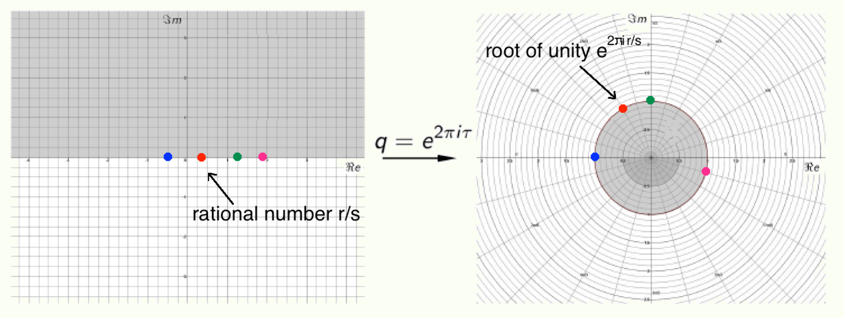
  (Bryson-Ono-Pittman-Rhoades, F-Ki-Truong Vu)

Ramanujan revisited

Let $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, and $q = e^{2\pi i\tau}$.



Ramanujan revisited

$$\lim_{q \to \zeta} \left(f(q) - (-1)^k b(q) \right) = -4U(-1; \zeta).$$

Ramanujan revisited

$$\lim_{q\to\zeta}\Big(f(q)-(-1)^k b(q)\Big) = -4U(-1;\zeta).$$

↗

mock modular

Ramanujan revisited

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$\nearrow$ mock modular

$\uparrow$ modular

Ramanujan revisited

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↗ ↑ ↖

mock modular   modular   quantum modular

Proof ingredients

- Ramanujan's identity

$$\sum_{n=0}^{\infty} \frac{(\alpha\beta)^n q^{n^2}}{(\alpha q;q)_n(\beta q;q)_n} + \sum_{n=1}^{\infty} q^n(\alpha^{-1};q)_n(\beta^{-1};q)_n = iq^{\frac{1}{8}}(1-\alpha)(\beta\alpha^{-1})^{\frac{1}{2}}\left(q\alpha^{-1};q\right)_{\infty}\left(\beta^{-1};q\right)_{\infty}\mu(u,v;\tau).$$

($q = e^{2\pi i\tau}, \alpha = e^{2\pi iu}, \beta = e^{2\pi iv}$)

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- Transformation theory (Zwegers) of the mock Jacobi form

$$\mu(u,v;\tau) := \frac{e^{\pi iu}}{\vartheta(v;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}} e^{2\pi inv}}{1-e^{2\pi iu}q^n}.$$

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- Explicit asymptotic calculations

Further results

Bringmann-Rolen/Jang-Lobrich: more general "radial limit" theorems for the Gordon-McIntosh universal mock ϑs:

$$g_2(w;q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\frac{n(n+1)}{2}}}{(w;q)_{n+1}(w^{-1}q;q)_{n+1}}, \; g_3(w;q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(w;q)_n(w^{-1}q;q)_n}.$$

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Theorem (Bringmann-Rolen)

There is a linear combination of theta functions $\vartheta_{a,b,A,B,\zeta}(q)$ such that the radial limit difference $\lim_{q\to\zeta}(g_2(\zeta_b^a q^A; q^B) - \vartheta_{a,b,A,B,\zeta}(q))$ is bounded, and is the special value of a quantum modular form.

Further results

“...[no one has] proved that any of Ramanujan’s mock theta functions are really mock theta functions according to his definition.”

-B.C. Berndt, *Ramanujan, his lost notebook, its importance.*

Further results

By incorporating the theory of harmonic Maass forms,

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Theorem (Griffin-Ono-Rolen)

Ramanujan's mock theta functions satisfy his definition.

Further results

Theorem (Choi-Lim-Rhoades)

Let F be a mock modular form and ζ *a root of unity. There is a weakly holomorphic modular form* $q^{\alpha}\vartheta_{\zeta}(q)$ *such that the radial limit* $\lim_{q\to\zeta}(F(q) - q^{\alpha}\vartheta_{\zeta}(q))$ *is the special value of a quantum modular form.*

Further results

- Mathematical physics
- Moonshine, Representation theory
- Combinatorics
- Topology

⋮

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⋮

Thank you