A topological umbral moonshine conjecture

KITP, 17 Nov 2020
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These slides: http://categorified.net/KITPslides.pdf. Based on 2006.02422

M_{24} Moonshine [EOT11, GHV12, CD12, GPRV13, ...]:

$$H(\tau) = 2q^{-1/8}\left(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \ldots\right)$$

is mock modular. Shadow $= 24\,\eta(\tau)^3$. Coefs are dims of M_{24} irreps.

In fact, $\forall g,h \in M_{24}$, $[g,h]=1$, can define $H_{g,h}(\tau)$, just as if H = character of some "mock holomorphic VOA" w/ M_{24}-action.

Goal for this talk: Suggest a "topological" description.

Outline: (I) Whence mock modularity?
(II) Examples: geometric and sporadic
(III) Topological modular and wsp forms
(IV) Other umbral groups.

(I.a) What physics creates modular forms?

I always mean "compact" SQFTs.

many answers.

My favourite: Partition fns of 2D $\mathcal{N}=(0,1)$ SQFTs with grav. anomaly $c_L - c_R = w \rightsquigarrow MF_w$.

some natural normalization factors

This is due to the geometry of the moduli stack $\mathcal{M}$ of super tori. "Partition fn" $\in C^\infty(\mathcal{M})$.

this explanation is a math thm of Dan Berwick-Evans.

$$\mathcal{M} = \hom(\mathbb{Z}^2, \mathbb{R}^{2|1}) // \text{super rotations}$$

$$(z,\bar z,\zeta)\cdot(z',\bar z',\zeta') = (z+z', \bar z+\bar z'+\zeta\zeta', \zeta+\zeta').$$

After unpacking: a fn $Z \in C^\infty(\mathcal{M})$ consists of:

- $Z_0(\tau,\bar\tau,\text{volume}) = \langle 1\rangle_{\mathcal{F}}$
- $Z_\Theta(\tau,\bar\tau,\text{volume}) = \langle \bar G_{\bar z}\rangle_{\mathcal{F}}$
- $Z_\alpha(\tau,\bar\tau,\text{volume}) = \langle \bar G_z\rangle_{\mathcal{F}}$

s.t.

details depend on w

(1) $SL(2,\mathbb{Z})$-equivariance

(2) $\dfrac{\partial Z_0}{\partial \bar\tau} = \dfrac{\partial Z_0}{\partial\,\text{vol}} = 0.$

$\bar G = (\bar G_z, \bar G_{\bar z})$ is the supercurrent.

$\bar G_z = 0$ if superconformal.

(I.b) What physics creates (mock) modular forms?

My favorite answer: Partition fns of families of $\mathcal{N}=(0,1)$ SQFTs.

This is due to the geometry of the moduli stack $\mathcal{M}(X)$ of super tori equipped with a map to X

("infinite string tension": only look at maps that probe infinitesimal nbhds of X.)

parameter space.

This explanation is a thm of Dan Berwick Evans.

Unpack: A function $Z \in C^\infty(\mathcal{M}(X))$ consists of

- $Z_0(\tau, \bar\tau, \mathrm{vol}) \in \Omega^\bullet(X)$
- $Z_\theta(\tau, \bar\tau, \mathrm{vol}) \in \Omega^\bullet(X)$
- $Z_\alpha(\tau, \bar\tau, \mathrm{vol}) \in \Omega^\bullet(X)$

s.t.

(1) $SL(2,\mathbb{Z})$-equivariance

(2)
$$\frac{\partial}{\partial\bar\tau} Z_0 = d_X Z_\theta$$
$$\frac{\partial}{\partial \mathrm{vol}} Z_0 = d_X Z_\alpha.$$
$$0 = d_X Z_0$$

up to normalizing factors.

E.g. $Z_0 = \langle 1 \rangle_F + \langle \theta_i \rangle dx^i + \dots$

where θ_i is the op. that deforms F in the ∂_{x^i} direction.

(I.c) Special case: Suppose $[0,1] \to \{\text{SQFTs}\}$, $x \mapsto \mathcal{F}(x)$.

"Total partition function" $\ni$

$$\overbrace{z_0 = z_0^0 + z_0^1\,dx, \quad z_\theta = z_\theta^0 + z_\theta^1\,dx, \quad z_\alpha = z_\alpha^0 + z_\alpha^1\,dx,}^{\text{each} \,\in\, \Omega^\bullet([0,1];\, C^\infty(\tau, \bar\tau, \mathrm{vol}))}$$

solving cocycle relations

Witten index of $\mathcal{F}(x)$.

$dz_0 = 0 \Rightarrow z_0^0$ is constant in x.

$dz_\theta = \frac{\partial}{\partial \bar\tau} z_0 \Rightarrow \frac{\partial z_0^0}{\partial \bar\tau} = 0$, and $\frac{\partial z_0^1}{\partial \bar\tau} = \frac{\partial z_\theta^0}{\partial x}$.

$dz_\alpha = \frac{\partial}{\partial \mathrm{vol}} z_0 \Rightarrow \frac{\partial z_0^0}{\partial \mathrm{vol}} = 0$, and $\frac{\partial z_0^1}{\partial \mathrm{vol}} = \frac{\partial z_\alpha^0}{\partial x}$.

Suppose $\mathcal{F}(0) = \emptyset^*$ and $\mathcal{F}(1)$ is superconformal.

* $\emptyset$ = IR thy w/ spontaneous susy breaking.

$\mathcal{F}$ is a nullhomotopy of $\mathcal{F}(1)$

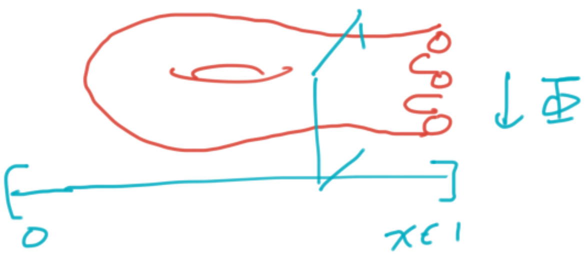
$$\frac{\partial}{\partial \bar\tau} \int_0^1 z_0^1\, dx = \langle \bar{G}_{\bar z} \rangle \text{ in } \mathcal{F}(1)$$

up to normalization.

$$\frac{\partial}{\partial \mathrm{vol}} \int_0^1 z_0^1\, dx = \langle \bar{G}_z \rangle \text{ in } \mathcal{F}(1) = 0.$$

$\int_0^1 z_0$ is the partition fn of a "noncompact SQFT", namely "$\int_0^1 \mathcal{F}(x)\, Dx$".

(II.a) A geometric example. Start w/ $\mathcal{N}=(0,1)$ σ-model with target



K3 − 24 pts. B field w/ H-flux = 1 thru each puncture.

Add λ chiral free fermion. Superpotential

$W = \lambda \cdot (\Phi - x)$. λ acts like a Lagrange mult.

$\rightsquigarrow$ thy $\mathcal{F}(x)$.

$\mathcal{F}(0) = \emptyset$. $\mathcal{F}(1) = \oplus^{24} \underbrace{\overline{\mathrm{Fer}(3)}}_{\uparrow\, S^3 = SU(2)}$. $\bar{G} = :\bar{\Psi}_1 \bar{\Psi}_2 \bar{\Psi}_3:$.

This is the level $(0,2)$ $\mathcal{N}=(0,1)$ $SU(2)$ WZW model.

$\Rightarrow$ mock modular form $H(\tau)$ with

$\overline{\text{shadow}} = \langle \bar{G} \rangle_{24\,\mathrm{Fer}(3)} = 24 \langle \bar{\Psi}_1 \bar{\Psi}_2 \bar{\Psi}_3 \rangle = 24\, \eta(\bar{\tau})^3$.

So this "explains" how K3 $\rightsquigarrow$ $H(\tau)$.

(II.b) All we needed was that the (antiholomorphic!) SCFT $J := 24\,\overline{\mathrm{Fer}(3)}$ was nullhomotopic in $\{(0,1)$ SQFTs$\}$.

<u>Guess:</u> $M_{24} \circlearrowright J$ by permuting the ground states.

Maybe J is M_{24}-equivariantly null-homotopic?

If so, would get MMFs $H_{g,h}$ for "generalized M_{24} moonshine".

<u>Sadness:</u> $J \neq \emptyset$ in $\{M_{24}$-equiv. SQFTs$\}$.

<u>Pf:</u> Gauge $M_{23} \subseteq M_{24}$. $J/\!/M_{23} = 17\,\overline{\mathrm{Fer}(3)} \neq \emptyset$. $\square$

Possible because $H^3(M_{23}; U(1)) = 0$.

In hindsight, this cannot be the answer. The shadows would be twisted-twined indices of the permutation rep, and these are not the shadows of [GPRV13].

(II.c) A nongeometric example

Duncan's $V^{f\natural}$ is a holomorphic SCFT, $c=12$, with an (anomalous) Co_1 action.

$$\mathrm{Aut}\big(\underbrace{\overline{V}^{f\natural} \times \overline{\mathrm{Fer}(3)}}_{\text{new } \mathfrak{Z}\text{. It is an antiholomorphic SCFT.}}\big) = Co_1 \times SO(3) \supseteq M_{24}$$ with correct anomaly.

Witten Index of $\overline{V}^{f\natural} = 24 \Rightarrow \langle \overline{G} \rangle_{\mathfrak{Z}} = 24\, \overline{\eta(\tau)}^{3}$.

Conjecture: $\overline{V}^{f\natural} \times \overline{\mathrm{Fer}(3)}$ is M_{24}-equivariantly null-homotopic.

Reality check: This thy does produce the correct shadows for generalized M_{24} moonshine, i.e. they match the shadows in [GPRV13].

(III.a) When is an $\mathcal{N}=(0,1)$ SQFT null homotopic?

Conjecture (Stolz–Teichner, ...): For the appropriate topology, $\{\text{SQFTs}\} = TMF^{\bullet}$, the spectrum of Topological Modular Forms.

Weaker: There is a topological Witten index $\{\text{SQFTs}\} \to TMF^{\bullet}$.
This is all I really need, because there are maps $TMF \to MF_{\mathbb{C}}$ and $TMF \to KU$, and together these "know about" moonshine.

Examples: $\overline{\text{Fer}(3)}$ represents the class $\nu \in TMF^{-3}(pt)$
$\overline{V}^{f\natural}$ represents the class $\{24\Delta\} \in TMF^{-24}(pt)$.

N.B.: Δ itself does not exist.

Since $Co_1 \circlearrowright \overline{V}^{f\natural}$ with anomaly $\alpha = \frac{p_1}{2}(\text{Leech} \otimes \mathbb{R})$,

Conjecture: $\{24\Delta\}$ refines to a class in $\underbrace{TMF_{\alpha}^{-24}(BCo_1)}_{\text{twisted equiv. coh.}}$ "$pt // Co_1$".

Topological M_{24} moonshine Conjecture:
$\{24\Delta\}\nu \simeq 0$ in $TMF_{\alpha}^{-27}(BM_{24})$.

(III.5) Supporting evidence:

There is a map $TMF(\mathbb{B}G) \to TMF(BG)$

$\mathbb{B}G$ = classifying stack

$BG = |\mathbb{B}G|$ = classifying space.

$TMF(\mathbb{B}G)$ = genuinely equiv. coh.

$TMF(BG)$ = Borel equiv. coh.

Compare: $KU(\mathbb{B}G) = R(G)$ representation ring.

$KU(BG) = \widehat{RG}$ is its completion at augmentation ideal.

Similarly, for finite gps, expect $TMF(\mathbb{B}G) \to TMF(BG)$ to be a completion.

TMF = weakly holomorphic.

Tmf = holomorphic at cusps.

$tmf = Tmf\langle 0\rangle$. Does not have equiv. refinement.

Thm: $tmf_{\alpha}^{-27}(BM_{24})[\frac{1}{2}] = 0$.

In other words the conjecture holds perturbatively at odd primes.

$H^\bullet(BM_{24})_{(2)}$ unknown, so I couldn't do the 2-local comp.

Pf: Fun AHSS calculation.

(III.c) Optimal growth

There is a spectrum Tcf of "Topological cusp forms".
It has not been well studied.

$Tcf := \ker(Tmf \to KO)$.

Nontopologically, $cf = mf \cdot \Delta$.
But $\Delta \notin Tmf$, and $Tcf \not\simeq Tmf \cdot$ anything.

$\{24\Delta\} \in Tcf(pt)$.

Conjecture: It has a twisted Co_1-equivariant refinement, and $\{24\Delta\}\nu \simeq 0$ in $Tcf_{\alpha}^{-27}(\mathbb{B}M_{24})$.

After you include all normalization factors, I think that this would provide the "optimal growth condition". But I didn't do all computations.

Wide open question: What is the physics of $Tcf^{\bullet}$?
What physics leads to cusp forms?

(IV. a) Umbral group $2M_{12}$ (Niemeier lattice A_2^{12})

Write $\widetilde{SU(2)}_k$ for the $\mathcal{N}=(0,1)$ WZW model with bosonic levels $(k-1, k+1)$.

E.g.: $\widetilde{SU(2)}_1 = \overline{Fer(3)}$.

$\widetilde{SU(2)}_k$ represents the class $k \cdot \nu \in TMF^{-3}(pt)$.

Take $\overline{V}^{f\natural} \times \widetilde{SU(2)}_k$. $\quad Aut = Co_1 \times SO(4)_{LR}$.

$= \dfrac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$.

Gauge the order-2 sym that acts by $-1 \in SO(4)$ and with frame shape 2^{12} in Co_1.

These are each separately anomalous, but the anomalies cancel.

$(\overline{V}^{f\natural} \times \widetilde{SU(2)}_k) // _{\text{this } \mathbb{Z}_2}$ represents $\{24\Delta\} \cdot \frac{2\nu}{2} = \{24\Delta\}\nu$ (I think).

Choice of element with frame shape 2^{12} breaks $Co_1 \to \mathbb{Z}_2^{10} \rtimes M_{12}$, and the M_{12} subgp extends to $2M_{12}$ on the twisted sector.

Conjecture $(\overline{V}^{f\natural} \times \widetilde{SU(2)}_k) // \mathbb{Z}_2$ is $2M_{12} \times SU(2)_L$ - equivariantly null homotopic. In fact, is null in Tcf.

(IV.b) Umbral group $2AGL_3(2) \simeq 2^4.L_3(2)$ (Niemeier lattice A_3^8).

Start with $\bar{V}^{f\natural} \times \widetilde{SU}(2)_3$. Gauge the $\mathbb{Z}_2$ subgroup acting as (Frame shape $1^8 2^8$, central elt) $\in Co_1 \times SO(4)_{LR}$.

$\bar{V}^{f\natural} \times \widetilde{SU}(2)_3$ represents $\{24\Delta\}\cdot 3\nu$. I think the quotient $\rightsquigarrow \{24\Delta\}\cdot 2\nu$.

This choice breaks $Co_1 \rightsquigarrow \mathbb{Z}_2^8 . O_8^+(2)$, but there is also an "outer" $\mathbb{Z}_2$ sym of the quotient that does not lift to Co_1. Gauge it.

Resulting thy has chiral alg $SU(2)_2$ and antichiral alg an extension of $Spin(8)_1^3 \times SU(2)_4 \times Fer(3)$.

It has an action by $\approx SU(2)_L \times AGL_4(2) \times SU(2)_R$.

Don't quote me on this!

$$\begin{array}{c} AGL_4(2) \\ \shortparallel \\ 2^4.L_4(2) = 2^4.A_8 \subseteq 2^8.S_8 \subseteq 2^8.O_8^+(2). \\ \cup\!\shortmid \\ 2^4.L_3(2). \end{array}$$

Conjecture: Is $SU(2)_L \times 2^4.L_4(2)$-equivariantly nullhomotopic in Tmf.

(IV.c) Caveats:

These are really just guesses. I have not done even the most basic reality checks.

I do not know a general method, so I do not have guesses for the other Niemeier lattices.

What I want is a "quantum hol(e)y construction":

Niemeier lattice of Coxeter # $= h$ $\longmapsto$ interesting "orbifold" of $\overline{V}^{f\natural} \times \widetilde{SU}(2)_{h-1}$.

by a fusion cat of generalized symmetries.

e.g. maybe it would use the ADE classification of modular invariants for $SU(2)$?

And of course the question of constructing a nullhomotopy remains...