Dr. Dan Freed, U Texas (ITP 6/20/01) The Geometry and Topology of p-Form Gauge Fields II

The geometry of p-form gauge fields

X - smooth manifold

j ∈ \Omega^p(X) - current, \; d\mathbf{s} = 0

Q ∈ H^{p+1}(X, \mathbb{Z})

We want j - element in a refined color theory
mapping to (Q, j)

Similarly we had F \in \Omega^2(X) - field strength
\lambda \in H^2(X, \mathbb{Z})
and we want \lambda \mapsto (\lambda), F).

First attempt: Take j, F to be elements in the fiber product

A^q\mathbb{H}(X) \longrightarrow \Omega^q(X)

\lambda \downarrow \downarrow

H^p(X, \mathbb{Z}) \longrightarrow H^q(X, \mathbb{R})

Example: Look at the q = 1 situation from yesterday. There we knew the answer.
\[ \phi \mapsto d\phi \]

\[ \text{Map}(X; \mathbb{R}/2\pi \mathbb{Z}) \to \text{L}_G(X) \]

\[ H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{R}) \]

Clearly, \( \text{Map}(X; \mathbb{R}/2\pi \mathbb{Z}) \) is not \( A^1_H(X) \).

In fact, we have a SES:

\[ 0 \to \frac{H^0(X, \mathbb{R})}{H^0(X, \mathbb{Z})} \to \text{Map}(X, \mathbb{R}/2\pi \mathbb{Z}) \to A^1_H(X) \to 0 \]

locally constant maps from \( X \) to the circle \( \mathbb{R}/2\pi \mathbb{Z} \).

Conclusion: The actual refinement captures more information than the fiber product \( A^1_H(X) \).

- Look at \( q=2 \). Here we also found an answer yesterday.

\[ \text{M}-\text{bundles with connections} \to \text{JCD}(X) \]

\[ \text{Char. class} \]

\[ H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}) \]
Again we have a ses

\[ 0 \to \tilde{H}^1(X, \mathbb{R}) \to \tilde{H}^1(X, \mathbb{C}) \to \tilde{H}^1(X, \mathfrak{g}) \to \tilde{A}^1(X) \to 0 \]

so the actual answer does have more info than \( \tilde{A}^1(X) \).

In the early 70s Cheeger-Simons and Deligne constructed for each \( q \)

\[ \tilde{H}^q(X) \quad \text{Cheeger-Simons diff.} \]

\[ H^q(X) \quad \text{characters = smooth Deligne cohomology} \]

s.t. we have a ses

\[ 0 \to \tilde{H}^q(X, \mathbb{R}) \to \tilde{H}^q(X) \to \tilde{A}^q(X) \to 0 \]

Examples:

\[ \tilde{H}^0(X) = \text{Map}(X, \mathbb{C}) \]

\[ \tilde{H}^1(X) = \text{Map}(X, \mathfrak{g}) \]

\[ \tilde{H}^2(X) = \{ \text{circle bundles with connections} \} \]

So the right answer for the refinement will be \( \tilde{H}^q(X) \).

However: in order to do field theory we need not only \( \tilde{H}^q(X) \) but also a calculus of cochains for elements in \( \tilde{H}^q(X) \), i.e. we want geometric representatives of cohomology classes.
We want something that generalizes the familiar models:

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1st model: A cocycle for an element in $H^q(X)$ is a triple $(c, h, w)$ where

- $c \in C^q(X, \mathbb{Z})$, $\delta c = 0$
- $\omega \in \Omega^q(X)$, $d\omega = 0$
- $h \in C^{q+1}(X, \mathbb{R})$, $\delta h = \omega - c$

Examples:

- $q = 1$  
  $h \in C^0(X, \mathbb{R})$ i.e. $h : X \to \mathbb{R}$
  - function (not necessary continuous)
  
However, the condition $\delta h = \omega - c$ implies $A$ path.

$\Rightarrow h(q) - h(p) = \int_{p}^{q} \omega - \langle c, x \rangle$
and so

\[ e^{ih(p)} = \exp \left( \frac{i}{\hbar} W(p) \right) \]

i.e. \( h \) can be arbitrary but \( \exp \) must be a nice smooth map.

What is the equivalence relation on the triples \((c, \omega, h)\)?

If we have a cochain complex

\[ \cdots \rightarrow C^{q-2} \rightarrow C^{q-1} \rightarrow C^q \rightarrow C^q \rightarrow \cdots \]

The \( q \)-th cohomology group \( H_q \) of this complex can be realized as

\[ T^a (\text{category}) \]

This category will have

**Objects:** \( \mathbb{Z}^q \)

**Morphisms:** \( a \rightarrow a' \) will be \( b \in C^{q-1} \)

s.t. \( a' = a + \delta b \)

In particular, the automorphisms of \( a \) are all \( b \) s.t. \( \delta b = 0 \) i.e. \( \delta^2 = 0 \).

Now to find the equivalences of the cochain \((c, \omega, h)\) we will just define the corresponding category.
A map \((c, w, h) \rightarrow (c', w', h')\) will be a pair

\[
(s, t) : (c, h, w) \rightarrow (c', h', w')
\]

where \(s \in C^{q-1}(X, \mathbb{Z})\), \(t \in C^{q-2}(X, \mathbb{R})\)

\[
\begin{align*}
   c' &= c + \delta s \\
   w' &= w \\
   h' &= h - s - \delta t
\end{align*}
\]

Also we can define equivalence of maps

\[
(s, t) \sim (s + \delta e, t + \delta f)
\]

\(e \in C^{q-2}(X, \mathbb{Z})\), \(f \in C^{q-2}(X, \mathbb{R})\)

and get further a structure of a \(2\)-category etc.

**Variants:**

1. Čech model
2. tame triples: \(c : X \to K(q, \mathbb{Z})\)

\(c, h, w\) was before

\[
\delta h = c_4 - c^2 n
\]

where \(n \in C^4(K(q, \mathbb{Z}), \mathbb{R})\)

is a fixed cocarin.
To get the cochain model we considered above we need to form the fibered product
\[ C^0(q)(x) \rightarrow \mathbb{R}_x^q \]
\[ \downarrow \quad \downarrow \]
\[ C^0(x, \mathbb{R}) \rightarrow C^0(x, \mathbb{R}) \]
and consider the bigraded Deligne theory
\[ H(p)^q(X) := H^p(C^0(q)(x)) \]
Then
\[ H^q(X) := H(q)^q(X) \]

Example: Let us work out the cochain model for \( q = 2 \).

\[ \alpha = (c, b, \omega) \]

What is a trivialization of \( \alpha \)? This is an isomorphism of \( \alpha \) with \( \delta \).

To prescribe such a trivialization we need to solve
\[ \delta b = c \]
\[ db = c \]
\[ d\omega = c \]
How to do this for generalized cohomology theories?

A generalized cohomology theory

\[ X \rightarrow \Gamma^*(X) \]

satisfying the Eilenberg-MacLane axioms

without the normalization axiom.

Normalizations:

1. \( \Gamma = H \mathbb{Z} \):
   \[ H^q(pt) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0 \end{cases} \]

2. \( \Gamma = K \):
   \[ K^q(pt) = \begin{cases} \mathbb{Z} & q \equiv 0 \mod 2 \\ 0 & q \equiv 1 \mod 2 \end{cases} \]

3. \( \Gamma = K0 \):
   \[ K0^q(pt) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2 & q = 1 \\ \mathbb{Z}/4 & q = 2 \\ 0 & q \equiv 3 \mod 4 \end{cases} \]

Note: For every \( \Gamma \)-generalized cohomology theory, there is a natural map

\[ \Gamma^*(X) \rightarrow (H^*(X, \mathbb{K}) \otimes \Gamma^*(pt)) \]

Example: \( \Gamma^*(pt) \otimes \mathbb{K} \cong \mathbb{K} \mathbb{C}^* \mathbb{C}^* \) deg \( u = 2 \)

\( u^{-1} = \text{Hopf bundle over } S^2 \)

The map \( K^*(X) \rightarrow H^*(X, \mathbb{K} \mathbb{C}^* \mathbb{C}^*) \) is just the Chern character.
For example

\[ ch : K^0(X) \to (H(Y, K(C(U, U^{-1}), J)) \times J \to ch_0(x) + ch_1(x)u^{-1} + ch_2(x)u^{-2} + \ldots) \]

A cochain model for \( K^0 \) can be described as follows.

Let \( B \) be a smooth model for the classifying space of \( K \) (Recall: the classifying space of \( K \) is \( \mathbb{Z} \times B \)).

We can take

\[ B = \text{Fred}(T^*) \quad \text{for separable (or Hilbert)} \]

Fix

\[ x \in (H(B, K(C(U, U^{-1}) \to )) \]

e.g. as the Chern character of the universal connection \( \nabla_{\text{uni}} \).

Then a class in \( K^0(X) \) is a triple \((c, h, w)\) where

\[ c : X \to B \quad \text{and} \quad h : (C(X, K(C(U, U^{-1}) \to )) \to B \]

\[ w \in (T^2(X, \ldots)) \quad \text{with} \quad dw = 0 \]

Alternatively, one may dance \((E, \nabla)\) - vector bundle with connection as a cocycle representative.

(Left)
Remark: \( \tilde{\Gamma} \) has a multiplicative structure,
\[ \tilde{\Gamma} \otimes \tilde{\Gamma} \rightarrow \tilde{\Gamma} \]
This can also be lifted to cochains!

- \( \tilde{\Gamma} \) has evaluation maps (integration).

For example, \( \iota : W \rightarrow X \) - inclusion \( \pi : X \rightarrow T \) - fiber bundle.

If \( \tilde{\alpha} \in \tilde{\Gamma}(X) \) has curvature \( \omega \), then

\[
\text{curvature} \left( \tilde{\alpha} \right) = \int_{X/T} \hat{A}_p \left( \frac{X}{T} \right) \wedge \omega
\]

where \( \hat{A}_p \left( \frac{X}{T} \right) \) is a Todd-like class, depending on \( \tilde{\Gamma} \).

- \( \tilde{\Gamma} = H \mathbb{Z} \Rightarrow \hat{A}_p = 1 \)
- \( \tilde{\Gamma} = KO \Rightarrow \hat{A}_p = \hat{A} \)
- \( \tilde{\Gamma} = K \Rightarrow \hat{A}_p = \hat{A} \cdot e \)

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How to recast the discussion of higher gauge fields from yesterday in terms of generalized differential cohomology theory?
Data:
(1) degree \( d = (d_1, d_2, \ldots, d_k) \)

(2) coupling constant \( e = (e_1, e_2, \ldots, e_k) \)

(3) \( \Gamma \) - cohomology theory

(4) normalization

\( \omega^x = (\omega^x_1, \ldots, \omega^x_k) \) degree 0

These are differential forms that depend functorially on \( X \).

Example:
If \( \Gamma = \mathbb{H}^2 \Rightarrow \omega^x = 2\pi \)

If \( \Gamma = \mathbb{K} \Rightarrow \omega^x = 2\pi \sqrt{A(x)} \)

(5) homomorphisms \( \Gamma' \to \mathbb{H}^2 \)

Example:
If \( \Gamma = \mathbb{K} \Rightarrow \mathbb{K}' \to \mathbb{H}^2 \)

\( f : \mathbb{K} \to \Omega(\mathbb{K}) \) def \( f \)

Ingredients:
\( j_x \in \mathbb{Z}_{p+1}(X) \) currents

\( j_x \in \mathbb{Z}_p \)

\( \gamma \to (\Omega, \frac{j_x}{\gamma}) \).
gauge field \( A \in \mathfrak{g}^d(X) \) - non-flat

\[ \text{covariant derivative } = \frac{F_A}{\omega_X} \]

let \( F_A \) = equivalence class of \( A \).

Action for a family of manifolds:

\[
\begin{align*}
\mathcal{X} & \supset \mathcal{X}_t \\
\uparrow & \quad \uparrow \\
\mathcal{T} & \supset \mathcal{T}_t
\end{align*}
\]

\[
\mathcal{L}(A) = \exp \left( -\frac{1}{2e^2} \left( \sum_{X/T} F_A \wedge \ast F_A \right) \right)
\]

\[
\exp \left( -\frac{i}{e} \int_{X/T} \mathcal{F}_X \cdot \mathcal{F}_X \right)
\]

trivialization or \( \exp \left( -\frac{i}{e} \int_{X/T} \mathcal{F}_X \cdot \mathcal{F}_X \right) \)

which can be interpreted as an element in \( \mathfrak{g}/\mathfrak{t} \) via the map (6) i.e.

\[
\exp \left( -\frac{i}{e} \int_{X/T} \mathcal{F}_X \cdot \mathcal{F}_X \right) \text{ is a trivialization of a circle bundle with connection}
\]