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Lecture 4

$(M, J : T \oplus T^* \circlearrowright)$ – generalized complex manifold

Examples: $J_1 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ – symplectic

$J_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ – complex

Recall: A Kahler manifold is a manifold M together with:

- complex structure I and a symplectic structure ω
- ω is of type $(1,1)$ w.r.t. I
- the metric $\omega(\cdot, I\cdot)$ is positive definite

Note: The condition that ω is of type $(1,1)$ w.r.t. I can be expressed in terms of the corresponding generalized complex structures is: $[J_1, J_2] = 0$.

Indeed $J_1 J_2 - J_2 J_1 = \begin{pmatrix} 0 & \omega^{-1}I + I\omega^{-1} \\ \omega I + I\omega & 0 \end{pmatrix} = 0$

$\Leftrightarrow \quad \omega(X, IY) + \omega(IX, Y) = 0$

$\Leftrightarrow \quad \omega$ is of type $(1,1)$

This motivates the following

2.

Def: A *generalized Kähler structure* on M is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures so that

$$(\mathcal{J}_1\mathcal{J}_2(X+\xi), (X+\xi))$$

is positive definite on $T \oplus T^*$.

In the same spirit:

Def: A *generalized Calabi-Yau metric* is a pair of generalized CY structures on a manifold M^{2n} given by pure spinors ψ_1, ψ_2 such that the corresponding $\mathcal{J}_1, \mathcal{J}_2$ describe a generalized Kähler structure and such that

$$\langle \psi_1, \bar{\psi}_1 \rangle = c \langle \psi_2, \bar{\psi}_2 \rangle$$

c - constant.

Remark: This is a generalization of the ordinary Ricci flatness conditions:

$(M, \mathbb{I}, \omega)$ is a CY if $\exists$ Ω - holomorphic n-form on M s.t. $i\,\Omega \wedge \bar{\Omega} = c\,\omega^n$

Now ω and Ω specify two generalized CY structures on M which satisfy the condition in the previous definition

2.

Thm (M. Gualtieri) A generalized Kähler structure on M is the same thing as:

- a metric g on M
- two integrable complex structures I_+ and I_-
- a 2-form b

so that

(i) g is hermitian with respect to I_+ and I_-

(ii) $\frac{1}{2}(\nabla_+ + \nabla_-) = \nabla$ – Levi-Civita connection of g on T

(iii) $\frac{1}{2}(\nabla_+ - \nabla_-) = db$ $\quad (\Lambda^3 T^* \subset \Lambda^2 T^* \otimes T^*)$

Remarks:

- ∇_I – the unique connection on T compatible with I_I and preserving g (the Chern connection)
- Sometimes one may want to allow (iii) with a rhs = closed 3-form which is not necessarily exact. For instance the Hopf surface is not generalized Kähler but does satisfy (i) – (iii) with the rhs of (iii) being a closed non-exact 3-form
- An ordinary Kähler manifold (M, I, ω) has such a structure: $I_+ = I_- = I$ and b – closed.

9.

- Acting by a closed B-field on the above data shifts b by the B-field $\Rightarrow$ preserves generalized Kähler structures.

Sketch of proof: Suppose (M, J_1, J_2) - generalized Kähler.

Then $J_1 J_2 = J_2 J_1$

$\Rightarrow$ $(J_1 J_2)^2 = 1$

Let V = $+1$ eigenspace of $J_1 J_2$

$V^\perp$ = -1 eigenspace of $J_1 J_2$

$\Rightarrow$ $T \oplus T^* = V \oplus V^\perp$ - orthogonal decomposition w.r.t. the inner product

$$\left((X+\xi), (X+\eta) \right) = 2\langle \xi, X \rangle$$

Also V - positive definite and $V^\perp$ - negative definite w.r.t. this product:

$$(J_1 J_2 (v+w), v+w) = \langle v, v \rangle - (w, w).$$

V - positive definite $\Rightarrow$ $V \cap T = V \cap T^* = 0$

(T, T^* - isotropic)

Hence V is a graph of a map $T \to T^*$ i.e. we get an element

$$g + b \in T^* \otimes T^*.$$

$g \in S^2 T^*$, $b \in \Lambda^2 T^*$

5.

We can identify T with V and V^+ via the maps

$$X \longmapsto X + i_X(g+b)$$

$$X \longmapsto X + \bar{i}_X(-g+b)$$

respectively.

Note that $(X + i_X(g+b), X + i_X(g+b)) = g(X,X) + b(X,X)$ (the $b(X,X)$ term $\to 0$)

Hence we get a metric on T, induced from g.

Next observe $[J_1, (J_1 J_2)] = 0$

$J_1 : V \to V \qquad J_1^2 = -1$

$\Rightarrow$ gives an almost complex structure on V

Identifying T with V using the map above we get an almost complex structure I_+ on T

For integrability: $V \otimes \mathbb{C} = \underset{(1,0)}{V_+} \oplus \underset{(0,1)}{\bar{V}_+}$

$$\underbrace{V_+ \oplus \bar{V}_+}_{V} \oplus \underbrace{V_- \oplus \bar{V}_-}_{V^\perp}$$

$$V_+ = \ker(J_1 - i) \cap \ker(J_2 + i)$$

and since we have integrability for J_1, J_2

$\Rightarrow$ V_+ is closed under the Courant bracket

6.

Sections of $V_+ = X + i_X(g+b)$

where

$$I_+ X = iX \qquad (\text{by the def. of } I_+)$$

$\Rightarrow$

$$[X + i_X(g+b)_1 Y + i_Y(g+b)] = [X,Y] + \cdots$$

$\Rightarrow$ X, Y of type $(1,0)$ implies $[X,Y]$ is of type $[1,0]$.

Hence I_+ - integrable by Newlander-Nirenberg.

Similarly one gets I_- - integrable.

Suppose $z_1, \ldots, z_m$ are local holomorphic coordinates (for I_+)

V_+ is spanned by $\dfrac{\partial}{\partial z_i} + i\left(\dfrac{\partial}{\partial z_i}\right)(g+b)$

$$\| \qquad \dfrac{\partial}{\partial z_i} + i\left(\dfrac{\partial}{\partial z_i}\right)(-i\omega + b)$$

$$\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right] = 0 \Rightarrow$$

$$\left[\frac{\partial}{\partial z_i} + i\left(\frac{\partial}{\partial z_i}\right)(-i\omega+b), \frac{\partial}{\partial z_j} + i\left(\frac{\partial}{\partial z_j}\right)(-i\omega+b)\right] = 0$$

Note $$\frac{\partial}{\partial z_i} + i\left(\frac{\partial}{\partial z_i}\right)(-i\omega+b) = A\left(\frac{\partial}{\partial z_i}\right)$$

where

$$A : X+\xi \longmapsto X+\xi + i_X B$$

is given by the B-field $B = -i\omega + b$

7.

Note : B is not closed in general
Hence A does not preserve the Courant bracket.

In fact

$$A[X+\xi, Y+\eta] = [A(X+\xi), B(Y+\eta)]$$
$$= i_X i_Y (dB)$$

$$\Rightarrow \quad i\left(\frac{\partial}{\partial z_i}\right) i\left(\frac{\partial}{\partial \bar{z}_j}\right) d(-i\omega + b) = 0$$

$$\Rightarrow \quad -i\partial\omega + \partial b^{1,1} + \bar{\partial} b^{2,0} = 0$$
$$\partial b^{2,0} = 0$$

$$\left.\begin{aligned} -i\partial\omega + \partial b^{1,1} + \bar{\partial} b^{2,0} &= 0 \\ i\bar{\partial}\omega + \bar{\partial} b^{1,1} + \partial b^{0,2} &= 0 \end{aligned}\right\} \Rightarrow i(\partial - \bar{\partial})\omega = db$$

The Chern connection now is

$$\nabla + \frac{1}{2} g^{-1} i(\partial - \bar{\partial})\omega$$

both for I_+ and I_-.

This proves one direction of the theorem. We will not discuss the proof in the other direction. □.

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Remark: Apostolov - Gauduchon - Grantcharov have studied similar structures in dimension 4 : bihermitian structures.

$V_+ \oplus \bar{V}_+ \oplus V_- \oplus \bar{V}_-$ odd ∩ odd, even ∩ even — even dim

odd ∩ even — odd dim

$\dim V_+ = \dim_{\mathbb{C}} M$

Get types of J_1, J_2 labeled by parity:

$\dim_{\mathbb{C}} M$		
odd	even J_1	odd J_2
even	even J_1	even J_2
	odd J_1	odd J_2

Let's look at $\dim_{\mathbb{R}} M = 4$.

Spin (4, 4) - triality

Spinors = 8 dim space + quadratic form

pure spinors ⟺ null spinors

$\varphi_0 + \varphi_2 + \varphi_4$ is pure iff $\langle \varphi_1, \varphi \rangle = 0$

iff $\varphi_2^2 = 2\varphi_0\varphi_4$

9.

Aside: Generalized CY structures in 4 dimensions: φ - spinor $\in \Lambda^{ev} T^*$ s.t.

$$\langle \varphi, \varphi \rangle = 0$$
$$\langle \varphi, \bar{\varphi} \rangle \neq 0$$
$$d\varphi = 0$$

$d\varphi = 0 \Rightarrow \varphi_0 = \text{const.}$ If $\varphi_0 \neq 0 \Rightarrow$ can renormalize to get $\varphi_0 = 1$

$\Rightarrow$ must have $\varphi = 1 + \varphi_2 + \varphi_4$ with

$$\varphi_2^2 = 2\varphi_4$$

$\Rightarrow$ $\varphi = e^{B+i\omega}$ $\qquad \varphi_2 = B + i\omega$

$$0 \neq \langle e^{\varphi_2}, e^{\bar{\varphi}_2} \rangle = \langle e^{\varphi_2 - \bar{\varphi}_2}, 1 \rangle = \frac{(\varphi_2 - \bar{\varphi}_2)^2}{2} \neq 0$$

$\Rightarrow$ φ is a B-field transform of a symplectic generalized complex structure, i.e. ω - symplectic and

$$\varphi = e^B \cdot e^{i\omega}.$$

If $\varphi_0 = 0 \Rightarrow \langle \varphi, \varphi \rangle = 0 \Rightarrow \varphi_2^2 = 0$

i.e. φ_2 is locally decomposable

$$\varphi_2 = \theta_1 \wedge \theta_2$$

$\langle \varphi, \bar{\varphi} \rangle = \varphi_2 \wedge \bar{\varphi}_2 \neq 0 \Rightarrow \varphi_2$ is an almost complex structure.

Also $d\varrho_2 = 0 \Rightarrow \varrho_2$ is integrable.

Hence (M, ϱ_2) is a CY manifold

i.e. $M = K3$ or T^4.

What about ϱ_4?

- ϱ_4 - exact $\Rightarrow$ $\varrho_4 = \partial\bar\partial\beta^{1,1} = -\bar\partial\partial\beta^{1,1}$

  $\partial\beta^{1,1} \in \Omega^{2,1}$

  $\| $

  $\varrho_2 \wedge \theta^{0,1}$

$B = -d(\theta^{0,1} + \theta^{1,0})$ - closed 2 form

$B \wedge \varrho_2 = \bar\partial\theta^{0,1} \wedge \varrho_2 = \varrho_4$

- ϱ_4 - not exact $\Rightarrow$ $\varrho_4 = \lambda\varrho_2 \wedge \bar\varrho_2 +$ exact

<u>Conclusion</u>: If $\dim_{\mathbb{R}} M = 4 \Rightarrow$ a generalized CY structure on M is either the B-field transform of a symplectic structure or the B-field transform of an ordinary CY structure.

How about generalized Kähler structures in four dimensions?

If (M^4, J_1, J_2) - generalized Kähler

then

$J_1 J_2 = J_2 J_1$, $\ker(J_1 \pm i) \cap \ker(J_2 \pm i)$

$\|$

2-dim

11.

$\Rightarrow \quad \langle \varphi, \psi \rangle = 0 \quad$ iff $\quad \dim (E_\varphi \cap E_\psi) > 0$

even $\cap$ even $\Rightarrow$ even dimensional

Hence $J_1 J_2 = J_2 J_1$ if for (local) pure spinors φ_1, φ_2 we have

$$\langle \varphi_1, \varphi_2 \rangle = 0$$
$$\langle \varphi_1, \bar{\varphi}_2 \rangle = 0$$

and

$$\langle \varphi_1, \varphi_1 \rangle = \langle \varphi_2, \varphi_2 \rangle = 0$$

Generic case:

$$\langle e^{\pm i\omega_1}, e^{B+i\omega_2} \rangle = 0$$

or

$$\langle e^{B+i\omega_2 \pm i\omega_1}, 1 \rangle = 0$$

$\Rightarrow \quad (B + i(\omega_2 \pm \omega_1))^2 = 0$ — algebraic conditions

We get

$$B\omega_1 = B\omega_2 = \omega_1\omega_2 = 0 = \omega_1^2 + \omega_2^2 - B^2$$

If we can solve these algebraic equations we get a generalized Kähler metric.

12.

Example:

(1) M - hyperkähler

ν_1, ν_2, ν_3 - hk forms

$\nu_1^2 = \nu_2^2 = \nu_3^2 \qquad \nu_1\nu_2 = \nu_2\nu_3 = \nu_3\nu_1 = 0$

Take $B = \nu_1$, $\omega_1 = \frac{1}{2}(\nu_2 - \nu_3)$ $\omega_2 = \frac{1}{2}(\nu_2 + \nu_3)$

(2) (Joyce) In the previous setting choose f - function on M and put X for the Hamiltonian vector field (for ν_1)

Let F_t be the symplectomorphisms corresponding to X.

Take

$B = \nu_1$

$\omega_1 = \frac{1}{2}(\nu_2 - F_t^* \nu_3)$

$\omega_2 = \frac{1}{2}(\nu_2 + F_t^* \nu_3)$ $\qquad \nu_1 F_t^* \nu_3 = F_t^*(\nu_1 \nu_3) = 0$

Compact 4-dimensional case:

- $I_+ \neq \pm I_-$ $\Rightarrow$ the underlying complex manifold is either $K3$, T^4 or a Hopf surface

13.

- $I_+ = \pm I_-$

I_+, I_- - sections in $\mathbb{P}^1$ bundle over M

Apostolov et al show that these sections must coincide an anticanonical divisor in M (anticanonical for both I_+ and I_-)

In particular they get these are Kähler.