

Mike Hopkins

07-16-03

Lecture 3

Last time we looked at the Spin^c Dirac Operator and the associated 3d + 5d TFTs.

We saw that index theory implies that certain integrals of characteristic classes are integral, e.g.

$$\underline{3d} \quad \int_{M^4} \frac{c^2}{8} - \frac{p_1}{24} \in \mathbb{Z}$$

$$\underline{5d} \quad \int_{M^6} \frac{c^3 - p_1 \cdot c}{48} \in \mathbb{Z}$$

In questions coming from physics one would like to prove similar statements but in situations where we don't have an obvious index theory interpretation. So we would like to have a purely topological way of getting such results in higher dimensional situations.

For this we will need some machinery. First we need to recall the notion of a spectrum.

A spectrum E is a sequence of spaces

$$\{E_n\}_{n \in \mathbb{Z}}$$

together with specified homotopy equivalences

$$t_n : E_n \xrightarrow{\sim} S^1 E_{n+1}$$

Note: Sometimes one needs to work with spectra in which t_n are actually homeomorphisms.

Note: that if we have a spectrum the set

$$E^n(X) := [X, E_n]$$

is a group and so E_* can be viewed as representing a cohomology theory.

In a sense all cohomology theories come this way.

Example: If E_* is the Eilenberg-MacLane spectrum, i.e.

$$E_n = K(A, n)$$

represents $H^*(X, A)$.

Remark: For technical purposes it is convenient to sometimes rewrite the homotopy equivalences

$$t_n : E_n \xrightarrow{\sim} S^1 E_{n+1}$$

as maps $s_n : S^1 X E_n \rightarrow E_{n+1}$

Cobordism spectra:

Consider \mathbb{E}_n

&

$$BO(n) \hookrightarrow BO(n+1) \hookrightarrow \dots \quad BO$$

$$MO(n) = Thom(BO(n), \mathbb{E}_n)$$

We have $\sum MO(n) \rightarrow MO(n+1)$ which by adjunction corresponds to $MO(n) \rightarrow \mathcal{S} MO(n+1)$

$$MO = \{MO_n\}$$

$$MO_n = \lim_{N \rightarrow \infty} \mathcal{S}^N MO(N+n)$$

If B is any classifying space related to $BO(n)$. (e.g. $B = BSpm$)

\Rightarrow get

$$B(n) \xrightarrow{\text{?}} B$$

& \downarrow

$$BO(n) \rightarrow BO$$

and we can define

$$B^X = \{B_n^X\}$$

$$B_n^X = \lim_{N \rightarrow \infty} \mathcal{S}^{N+n} Thom(B(n), X)$$

$$S \rightarrow MO_n$$

$$\downarrow \quad \quad \quad S^n MO(n+N)$$

$$R^n \times S \rightarrow MO(n+N)$$

$$E \subset R^n \times S$$

$$\text{codim} = N+n$$

$$E \rightarrow S \text{ - relative}$$

dimension in

Differential cohomology groups

let $E = \{E_n\} | s_n : S^1 \times E_n \rightarrow E_{n+1}\}$

be a spectrum

Consider

$$C^*(E_{n+k}) \xrightarrow{\quad} C^*(S^1 \times E_n) \xrightarrow{\quad} C^{*-n}(E_n)$$

$$\text{define } C^*(E) = \lim_{\leftarrow} C^{*-n}(E_n) \quad n \in \mathbb{Z}$$

$$\text{let } i \in \mathbb{Z}^k(E) \hookrightarrow \{i_n \in \mathbb{Z}^{n+k}(E_n)\}$$

$$(x_{(n+k)})_{n \in \mathbb{Z}} \text{ such that } x_{n+k} = \sum_{m=n}^{n+k-1} x_m$$

$$\pi_k E = \pi_{n+k} E_n \leftarrow \text{independent of } n$$

$$k \in \mathbb{Z}$$

Basic fact: $H^0(E; \mathbb{R}) = \text{Hom}(\pi_0(E), \mathbb{R})$

$$= \text{Hom}(\pi_0(E) \otimes \mathbb{R}, \mathbb{R})$$

let $V := \pi_0(E) \otimes \mathbb{R}$

$$H^0(E, V) = \text{Hom}(\pi_0(E), V)$$

$$= \text{Hom}(V, V) \ni \text{id}$$

The id in $\text{Hom}(V, V)$ gives thus a class in $H^0(E, V)$ - the fundamental cohomology class

Now choose a cocycle $i \in Z^0(E, V)$ representing the fundamental class

If M is a smooth manifold, then define the differential cohomology groups of M corresponding to (E, i) by

$$E(n)^k(M) \stackrel{\text{def}}{=} \pi_N \text{filt}_{k+n} (E_{\leq k}, i)^M$$

6.

Note that

$$E^{(n)}_k(M) = \begin{cases} E^k(M) & k \geq n \\ E^{k-n}(M, \mathbb{R}/\mathbb{Z}) & k < n \end{cases}$$

$$E^{n-1}(M) \otimes \mathbb{R}/\mathbb{Z} \hookrightarrow E^{(n)}_n(M) \xrightarrow{\cong} A_{\mathbb{E}}^n(M)$$

Here

$$A_{\mathbb{E}}^n(M) \longrightarrow \mathcal{D}_{cl}(M, V)^n$$



$$E^n(M) \longrightarrow H^n(M, V)$$

uses \mathbb{E}

The last input we need is the natural counterpart of Grothendieck-Serre duality in stable homotopy theory.

This is realized by the Anderson dualizing spectrum.

This spectrum is denoted by $\tilde{\mathbb{F}}$.

It is hard to define $\tilde{\mathbb{F}}$ geometrically but we can describe maps to $\tilde{\mathbb{F}}$.

7.

Recall that for a spectrum $\gamma = \{\gamma_n\}$
we can define its k -th suspension

$$\Sigma^k \gamma = \{\Sigma^k \gamma_n\} = \{\gamma_{n+k}\}$$

Now if E -spectrum $[E, \Sigma^\infty \tilde{I}]$
fits into a sequence

$$\text{Ext}(\pi_{k+1} E, \mathbb{Z}) \rightarrow [E, \Sigma^k \tilde{I}] \rightarrow \text{Hom}(\pi_k E, \mathbb{Z})$$

Note on the construction of \tilde{I}

$$X \mapsto \text{Hom}(\pi_0^{st} X, \mathbb{Q}) = [X, I_\mathbb{Q}]$$

$$X \mapsto \text{Hom}(\pi_0^{st} X, \mathbb{Q}/\mathbb{Z}) = [X, I_{\mathbb{Q}/\mathbb{Z}}]$$

$$\begin{array}{c} \tilde{I} \rightarrow I_\mathbb{Q} \\ \downarrow \\ I_{\mathbb{Q}/\mathbb{Z}} \end{array}$$

<u>Remark:</u>	k	$\pi_k \tilde{I}$
	≥ 0	0
	0	\mathbb{Z}
	-1	0
	-2	$\mathbb{Z}/2$
	-3	$\mathbb{Z}/2$
	-4	$\mathbb{Z}/4$

8.

For a manifold M we have

$$\tilde{I}(1)^1(M) = \text{smooth } (M, U(1))$$

$\tilde{I}(2)^2(M)$ = iso classes of graded
 $U(1)$ bundles with
 connections on M

$\tilde{I}(3)^3(M)$ = various interpretations
 via gerbes

$\tilde{I}(4)^4(M)$ = should have interpretations
 as 2-gerbes

General Story

let $M \langle G \rangle$ be some cobordism theory classifying
 some structures (G)

Suppose we are given an integer invariant
 of d -dimensional (G) -manifolds. This is
 encoded in a homomorphism

$$\pi_d : M \langle G \rangle \rightarrow \mathbb{Z}$$

refine this homomorphism to a map

$$M \langle G \rangle \rightarrow \sum^d \tilde{I}$$

(If $T_{d-1}M \langle G \rangle = 0$, then this is free)

If M^{d-1}

ξ + metric along the fibers

\Rightarrow get differential function

$$\xi \rightarrow (M \langle G \rangle_{-(d-1)}, \cdot)$$

$$\begin{array}{ccc} & & \downarrow \\ \nearrow & & \\ \tau & \rightarrow & (\tilde{I}, \cdot) \end{array}$$

function

from ξ to $U(1)$

(d-2) ... get a graded $U(1)$ bundle
with connection on ξ .

Going back to the example: $M^4 - S^1 \times C$

$$K(c) = \frac{c^2}{8} - \frac{p_1}{24}$$

$$q(x) := \frac{x^2 - xc}{2} = K(c+dx) - K(c)$$

Note that

$$q(x+y) - q(x) - q(y) = xy$$

i.e. q - quadratic refinement of the "intersection"

pairing.

Let $L = H^2(M, \mathbb{Z}) / \text{torsion}$

\langle , \rangle - intersection pairing

The map ~~char~~ $L \rightarrow \mathbb{Z}/2$

$$x \rightarrow \langle x, x \rangle$$

is linear \Rightarrow

$$\langle x, x \rangle = \langle x, \bar{c} \rangle, \quad \bar{c} \in L \otimes (\mathbb{Z}/2)$$

Let

$$c \in L \text{ be c.t. } c \rightarrow \bar{c}$$

c is called - characteristic element

$$c^2 - \text{sign}(L) \equiv 0 \pmod{8}$$

$$\frac{x^2 - xc}{2} \quad \text{- quadratic}$$

If M^{4k} - oriented then the characteristic element v_{2k} is known as the $2k$ -th Wu class.

It is characterized by $x^2 - xv_{2k} \in H^{4k}(M, \mathbb{Z}/2)$

Equip M with an integer lift of v_{2k} .

$$\begin{array}{ccc} \text{BSO}(V_{2k}) & \longrightarrow & K(\mathbb{Z}, 2k) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad} & \text{B}\mathbb{S}O \xrightarrow{\quad} K(\mathbb{Z}_{2^k}, 2k) \end{array}$$

$$\pi_{4k} M\mathbb{S}O(V_{2k}) \rightarrow \mathbb{Z} \quad (1)$$

$$M\mathbb{S}O(V_{2k}) \rightarrow \sum^{4k} \tilde{I} \quad (2)$$

The extra data needed to define (2)
 (as opposed as (1)) was defined by
 Milgram and Morgan-Sullivan in the 70's

In the example of the 5 brane

$$B\mathbb{S}pin \times K(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}, 4)$$

$$\begin{array}{ccc} \downarrow & & \\ B\mathbb{S}pin & \longrightarrow & K(\mathbb{Z}_{2^k}, 4) \\ w_4 & & \end{array}$$

$$\pi_7(M\mathbb{S}pin \wedge K(\mathbb{Z}, 4))_+ = 0 \Rightarrow \text{no extra data is needed to define (1)}$$

Remark: There is an analogue of this algebraic story (Hopkins-Freed) for cubic forms.

For M-theory action again one needs to look at

$$MSpin^A K(\mathbb{Z}, 4)_+$$

Again there is a result (due to Stoen?) saying that

$$\pi_{21} MSpin^A K(\mathbb{Z}, 4)_+ = 0.$$