Strongly Nonlinear Discrete Metamaterials: Origin of New Wave Dynamics

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- From discrete system to higher gradients continuum
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- Tensegrity based strongly nonlinear systems – unprecedented tunabilty, interfaces between elastically softening and elastically hardening systems
- Conclusions
- Questions/suggestions for future
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Motivation

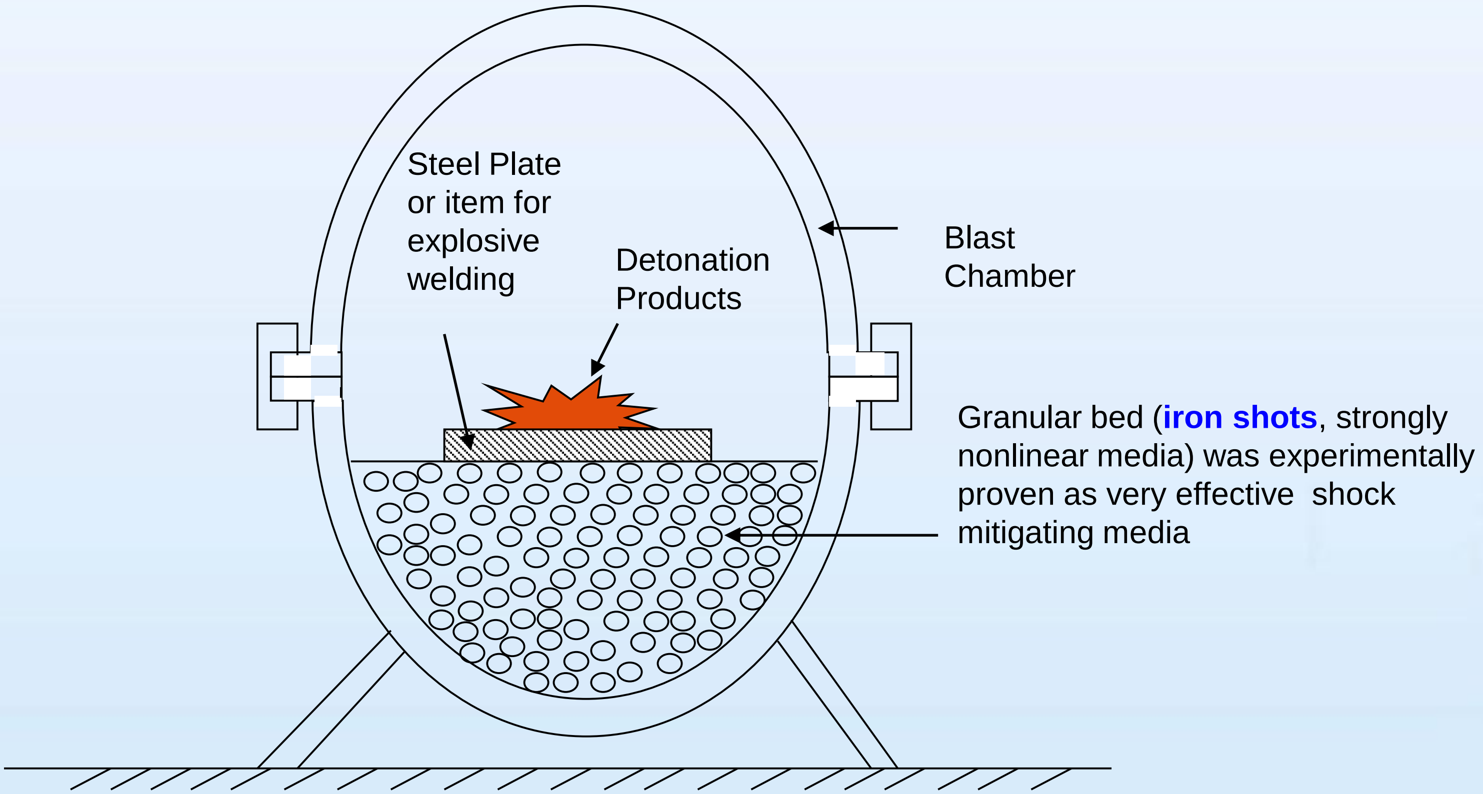
Develop barriers to mitigate shock wave generated by contact explosion in a then new type of devices – blast chambers. “Pure and dirty” engineering problem.

This technical problem outlined major parameters (scales) of future research:

1. Amplitude of pulse was high due to contact explosion;

2. Duration of incoming pulse was very short – 10-100 microseconds;

3. Barrier should have capability to mitigate multiple impacts.



Why *new* wave dynamics?

Discrete strongly nonlinear systems (e.g., granular materials, systems with strongly nonlinear elements (O-rings, tensegrity structures)) represent a new class of media being a natural extension from weakly to strongly nonlinear case:

- *Support new type of solitary wave, compact, space scale independent on amplitude, dictated by mesoscale and interaction law*
- *Highly tunable behavior, sensitive to low amplitude external mechanical field*
- *The only known system that can be tuned from strongly nonlinear regime to weakly and linear regimes*
- *Multiscale systems based on tensegrity concept allows tuning from strongly nonlinear elastically stiffening behaviour to elastically softening regime*
- *Absence of acoustic impedance in case of "sonic vacuum", or its negligible influence at low precompression, new interfacial phenomena, e.g. continuum-discrete-continuum pulse transformation at interface, targeted energy transfer*

General statement

The area of strongly nonlinear dynamic behavior of discrete "soft" condensed matter is an exciting new domain of research, still in infant stage

Research in this area is a logical step forward in general strongly nonlinear wave dynamics with possible similar developments in totally different areas than mechanical systems

The main goal is a design of strongly nonlinear tunable mesostructures with optimal performance in such applications as mitigation of high amplitude pulses, acoustic lensse, delay lines, scrambling devices

Presentation will be mainly restricted to propagating waves and to theoretical and numerical results connected with existing experimental data or with future experiments

This presentation is not a review of this now very broad and active area of theoretical, numerical and experimental research

Examples of Strongly Nonlinear Materials

– Low dimensional and three dimensional granular materials
– Polymer and metal foams
– Metamaterials, like laminates foam/steel plates or steel plates/polymer o-rings
– Tensegrity structures
– “Forest” of carbon nanotubes
– Colloids, magnetorheological slurry

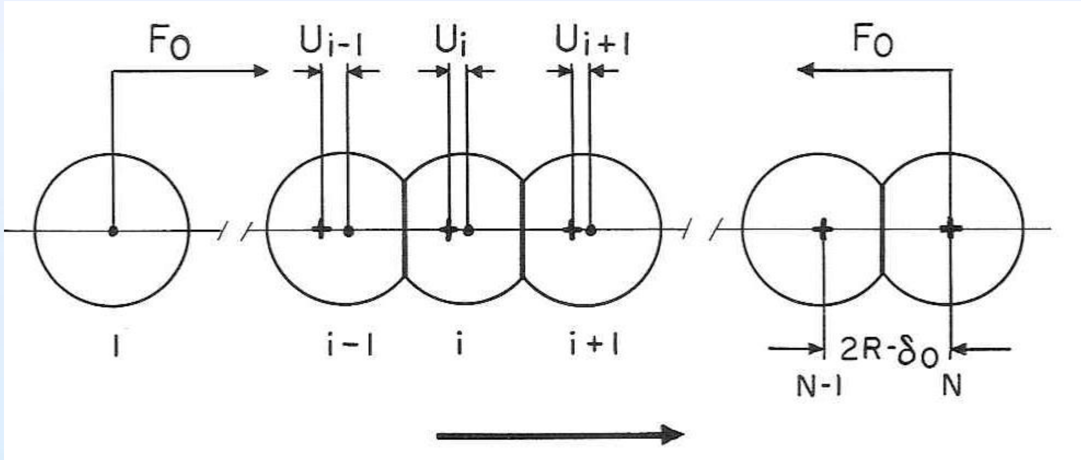
Sources of ***strong*** nonlinearity

- Nonlinear interaction law: linear term is absent or small in compression combined with zero tensile strength; Hertzian interaction between beads, double power law for compression of o-rings between rigid plates
- "Configurational" nonlinearity: structural rearrangements under applied load, ***microscopic***: change of coordination number with compression ***mesoscopic***: rearrangements of force chains
- Elastic instability: (buckling) of walls of cells in foams or carbon nanotubes in "forest" of carbon nanotubes or in three-dimensional lattices
- Nonlinear dissipation, especially in rubber elements like O-rings

1-D granular crystals/Hertzian chains

First attempt to understand performance of granular bed was based on assumption of strongly compressed, ***weakly*** nonlinear ($\Delta\sigma << \sigma_o$) discrete system with Hertzian interaction between spherical particles

$$F_c\left(\delta_{i,i+1}\right)=\frac{2E}{3\left(1-\upsilon^2\right)}\left(\frac{R_iR_{i+1}}{R_i+R_{i+1}}\right)^{1/2}\left(\delta_{i,i+1}\right)^{3/2};\quad \delta_{i,i+1}>0.$$



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1. $\Delta\sigma << \sigma_o$
2. L >> 2R

⇓

Continuum description is based on corresponding two small parameters

Equations of motion for discrete granular chain

$$\ddot{u}_i = A\left(\delta_0 - u_i + u_{i-1}\right)^{3/2} - A\left(\delta_0 - u_{i+1} + u_i\right)^{3/2}, \quad N-1 \geq i \geq 2,$$

$$m = \frac{4}{3}\pi R^3 \rho_0, \quad A = \frac{E(2R)^{1/2}}{3\left(1-\nu^2\right)m},$$

Anharmonic approximation for strongly compressed chain, this case is directly related to FPU problem

$$\frac{\left|u_{i-1} - u_i\right|}{\delta_0} << 1.$$

$$\ddot{u}_i = \alpha\left(u_{i+1} - 2u_i + u_{i-1}\right) + \beta\left(u_{i+1} - 2u_i + u_{i-1}\right)\left(u_{i-1} - u_{i+1}\right), \quad N-1 > i \geq 2,$$

$$\alpha = \frac{3}{2} A\delta_0^{1/2}, \qquad \beta = \frac{3}{8} A\delta_0^{-1/2}.$$

FPU problem, discrete chain equations: weakly nonlinear problem

"..the purpose of our computations was to see how, due to nonlinear forces ***perturbing the periodic linear solution,*** *the string would assume more and more complicated shapes, and, for t tending to infinity, would get into states where all the Fourier modes acquire increasing importance". Below is original part of the text:*

The solution to the corresponding linear problem is a periodic vibration of the string. If the initial position of the string is, say, a single sine wave, the string will oscillate in this mode indefinitely. Starting with the string in a simple configuration, for example in the first mode (or in other problems, starting with a combination of a few low modes), the purpose of our computations was to see how, due to nonlinear forces perturbing the periodic linear solution, the string would assume more and more complicated shapes, and, for t tending to infinity, would get into states where all the Fourier modes acquire increasing importance. In order to see this, the shape of the string,

(1)
$$x_i = (x_{i+1} + x_{i-1} - 2\,x_i) + \alpha\,[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2]$$
$$(i = 1\,, 2\,, \cdots, 64),$$

or

(2)
$$x_i = (x_{i+1} + x_{i-1} - 2\,x_i) + \beta\,[(x_{i+1} - x_i)^3 - (x_i - x_{i-1})^3]$$
$$(i = 1\,, 2\,, \cdots, 64).$$

(3)
$$\ddot{x}_i = \delta_1\,(x_{i+1} - x_i) - \delta_2\,(x_i - x_{i-1}) + c$$

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Continuum approximation

Linear wave equation, dispersive and weakly nonlinear Boussinesque wave equations (approach is based on two small parameters)

TRADITIONAL WAVE EQUATION:

$$u_{tt} = c_0^2 u_{xx} \quad , \qquad c_0^2 = \frac{1}{\rho}\frac{\partial\sigma}{\partial\xi};$$

WEAKLY NONLINEAR WAVE EQUATIONS:

$$u_{tt} = c_0^2 u_{xx} + 2 c_0 \gamma u_{xxxx} - \sigma u_x u_{xx},$$

$$u_{tt} = c_0^2 u_{xx} + 2 \gamma u_{ttxx} - \sigma u_x u_{xx},$$

$$c_0^2 = A\delta_0^{1/2} 6R^2, \quad \gamma = \frac{c_0 R^2}{6}, \qquad \sigma = \frac{c_0^2 R}{\delta_0}.$$

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Discrete ***strongly precompressed*** granular chain

(transformation to KdV equation, specifics of granular chain in this case is mainly related to tunability of coefficients in KdV equation by relatively small external forces)

If there are two small parameteres KdV approximation can be obtained:

$$\xi_t + c_0 \xi_x + \gamma \xi_{xxx} + \frac{\varepsilon}{2c_0} \xi \xi_x = 0$$

$$\Delta\xi = \Delta\xi_m \sec^2 \left\{ \left(\frac{\varepsilon \Delta\xi_m}{24 c_0 \gamma} \right)^{1/2} (x - Vt) \right\}$$

$$V = c_0 + \frac{\varepsilon}{6c_0} \Delta\xi_m$$

Properties of KdV soliton in Hertzian chain

Approaching a big problem - failure of KdV solitary wave in case of zero sound speed at zero precompression.

$$C_0^2 = \frac{3C_s^2}{\pi(1-\nu^2)}(\xi_0)^{\frac{1}{2}}\,;\ \gamma = \frac{C_0R^2}{6}\,,\ \varepsilon = \frac{C_0^2R}{\delta_0}\,,\ a = 2R$$

$$\lambda = \left(\frac{24C_0\gamma}{\varepsilon\Delta\xi_m}\right)^{\frac{1}{2}} = a\left(\frac{\xi_0}{\Delta\xi_m}\right)^{\frac{1}{2}},\ V = C_0 + \Delta V\,,\ \Delta V = \frac{C_0}{12}\left(\frac{\Delta\xi_m}{\xi_0}\right)$$

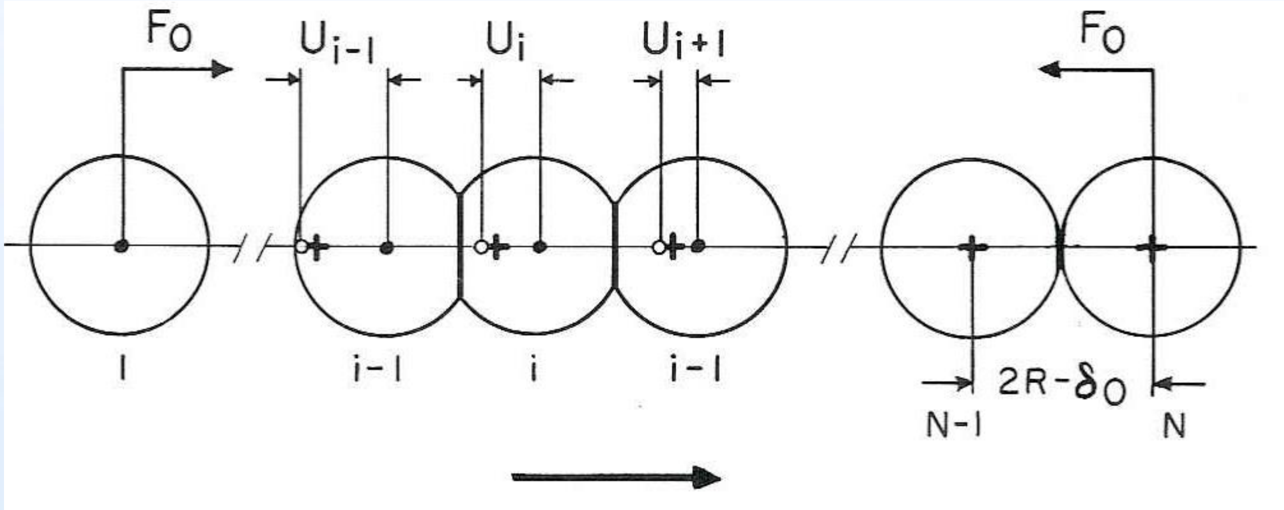
C_0 - The main parameter of medium in linear approximation controls the behaviour of weakly nonlinear solution.

Extreme Case: $\xi_0 \to 0 : C_0 \to 0, \lambda \to 0, V \to \infty$

Apparently new approach is required in this case.

Non-compressed or weakly compressed, ***strongly*** nonlinear ($\Delta\sigma >> \sigma_o$) discrete system

Breaking from KdV (based on two small parameters) and FPU paradigms (weakly nonlinear)



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1. ~~$\Delta\sigma << \sigma_o$~~
2. L >> 2R

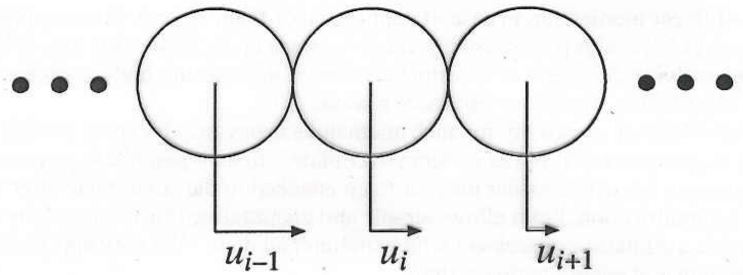
We still may try to use continuum description (not always applicable for granular matter!). No weakly nonlinear approximation possible, no more two small parameters, only one small parameter left. This is very “simple”, but very instructive example of general strongly nonlinear behavior.

Overcoming KdV equation paradigm

(Can we call any signal a sound wave?)

For Hertzian chain sound speed ($c_0 = c_n \sqrt{n}\,\xi_0^{(n-1)/2}$) is zero at zero ξ_0 - "sonic vacuum". Naming is important to identify a new domain of research

$$\sigma = b\xi^n,\ n > 1,\ C_0(\xi_0 \to 0) \to 0$$



$$\ddot{u}_i = A(u_{i-1} - u_i)^{\frac{3}{2}} - A(u_i - u_{i+1})^{\frac{3}{2}}$$

Only One Small Parameter:

$$\frac{a}{L} << 1$$

wrong way

$$c_0^2 \psi_{xx} = \psi_{tt}$$

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Strongly nonlinear wave equation "higher gradients" continuum

$$u_{tt} = -c^2 \left\{ (-u_x)^{3/2} + \frac{a^2}{10}\left[(-u_x)^{1/4} \left((-u_x)^{5/4} \right)_{xx} \right] \right\}_x ,$$

$$u_{tt} = -\left\{ c^2 (-u_x)^{3/2} - \frac{a^2}{12}\left[u_{ttx} + \frac{3}{8} c^2 (-u_x)^{-1/2} u_{xx}^2 \right] \right\}_x ,$$

$$\xi = -u_x$$

For stationary solutions $\xi(x - Vt)$:

$$y_{\eta\eta} = -\frac{\partial}{\partial y} W(y), \qquad W(y) = -\frac{5}{8} y^{8/5} + \frac{1}{2} y^2 + C_3 y^{4/5}.$$

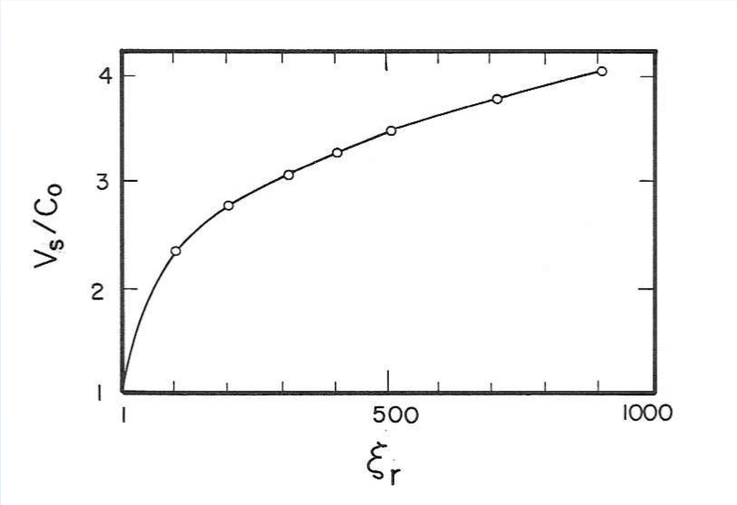
$$\eta = \sqrt{10}\,\frac{x}{a}, \quad y = \xi^{5/4} \left(\frac{c}{V} \right)^5, \quad C_3 = const.$$

General equation for solitary wave

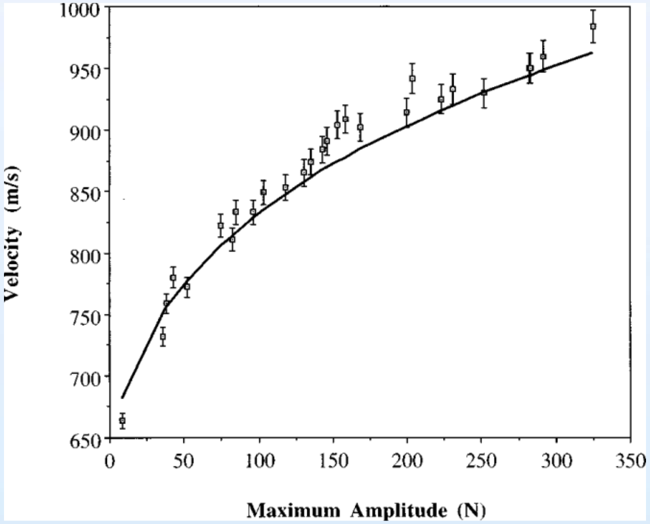
General dependence $V_s(\xi_m)$ for solitary wave

$$V_s = \frac{c}{(\xi_m - \xi_0)} \left\{ \frac{2}{5} \left[3\xi_0^{5/2} + 2\xi_m^{5/2} - 5\xi_0^{3/2}\xi_m \right] \right\}^{1/2}$$

At small dynamic amplitude of $\xi_m - \xi_0 = \xi_d$ this equation gives sound speed and speed of KdV soliton in corresponding approximations



Dependence of solitary wave speed V_s in the initially compressed chain versus the relative amplitude $\xi_r = (\xi_0 + \xi_d)/\xi_0$.



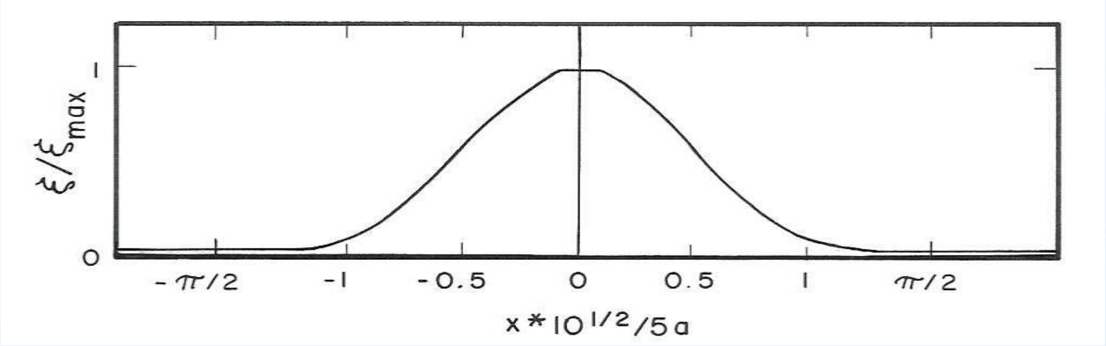
Dependence of solitary wave speed V_s in the initially compressed chain on wave amplitude and the theoretical prediction. From: C. Coste, E. Falcon, S. Fauve, PRE, 56, 1997.

Strongly nonlinear soliton in “sonic vacuum”

$$\xi_0 = 0 \text{ (or } \xi_0 << \xi_m)$$

$$\upsilon = V_p \left(\frac{5V_p^2}{4c^2}\right)^2 \cos^4\left(\frac{\sqrt{10}}{5a}x\right) \qquad \xi = \left(\frac{5V_p^2}{4c^2}\right)^2 \cos^4\left(\frac{\sqrt{10}}{5a}x\right)$$

$$L_s = \left(\frac{5a}{\sqrt{10}}\right)\pi \approx 5a \qquad V_s = \frac{2}{\sqrt{5}} c\,\xi_m^{1/4} = \left(\frac{16}{25}\right)^{1/5} c^{4/5} \upsilon_m^{1/5}$$



1. Width does not depend on amplitude. A strong nonlinearity resulted in a solitary wave width equal only to 5 particle sizes with compact support raising question about possible failure of continuum approach. Number 5 comes from force exponent 3/2.
2. For very small initial compression $V_s/c_0 \rightarrow \infty$. The amplitude of the soliton can be infinitely large in comparison with initial strain
3. $V_s \sim U^{1/10}$, (qualitatively similar to suggested in Einstein’s letter to Born).
4. Soliton can be considered as “quasiparticle” with effective mass ~1.4m

Main features of “sonic vacuum”, qualitative difference with FPU problem, connection with FPU problem in case of strong precompression

1. No phonons, which are direct consequence of linear elastic interaction between particles, exist in “sonic vacuum”, unlike in FPU system
2. Basic excitations are strongly nonlinear solitary waves and not phonons, they are of the same nature even at infinitesimal amplitude
3. No characteristic speed like sound speed independent of signal amplitude exist in “sonic vacuum”
4. Speed of solitary wave can be infinitely larger than sound speed and strongly depends on its energy
5. Unlike width of solitary wave in weakly nonlinear case soliton in “sonic vacuum” has width independent on amplitude
6. “Sonic vacuum” has a zero tensile strength though this requirement main be relaxed at least for compression waves
7. Static precompression reduces “sonic vacuum” to FPU problem, if pulse amplitude is small in comparison with initial strain

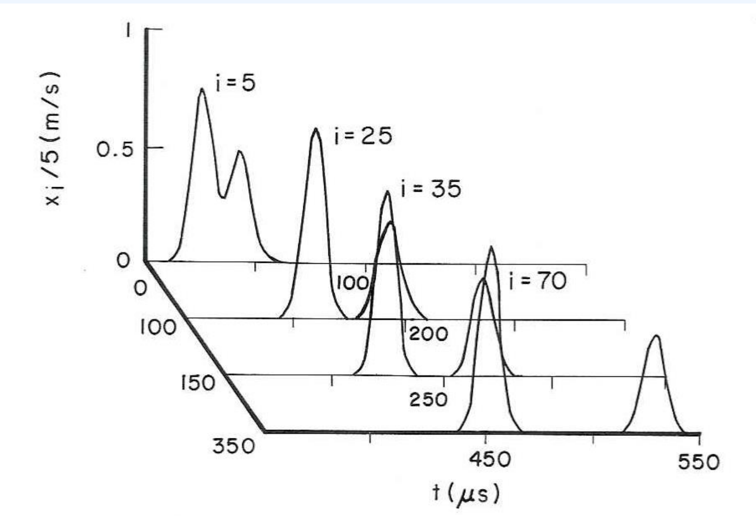
Hierarchy of waves in continuum

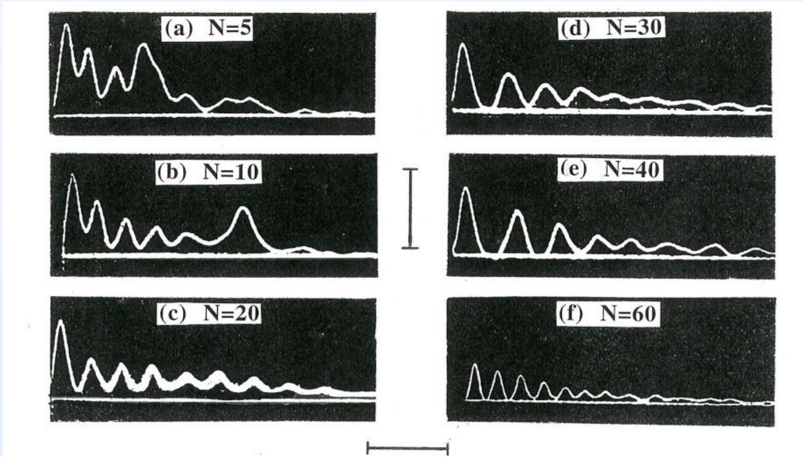
I	$$c_0^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$ 	C_o	Linear (Structure independent, discrete system with linear interaction law)
II	$$c_0^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial^4 u}{\partial x^4}$$ 	$C_o(\lambda)$	Linear dispersive (Structure sensitive, discrete system with linear interaction law)
III	$$c_0^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial^4 u}{\partial x^4} - \sigma \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$ 	$V_s=c_0+\beta u$	Weakly nonlinear dispersive (KdV solitons, weakly nonlinear FPU problem in discrete case)
IV	 $$\rho\xi_{tt} = \left[f + \frac{a^2}{24}\left(2f'\xi_{xx} + f''\xi_x^2\right)\right]_{xx}$$	$V_s=\psi(u)$	"Sonic vacuum"(c_0=0), strongly nonlinear solitary, shock and periodic waves ($V>>c_0$); strongly nonlinear discrete systems, nonanalytical interaction law. Limit of continuum approach, active area of research

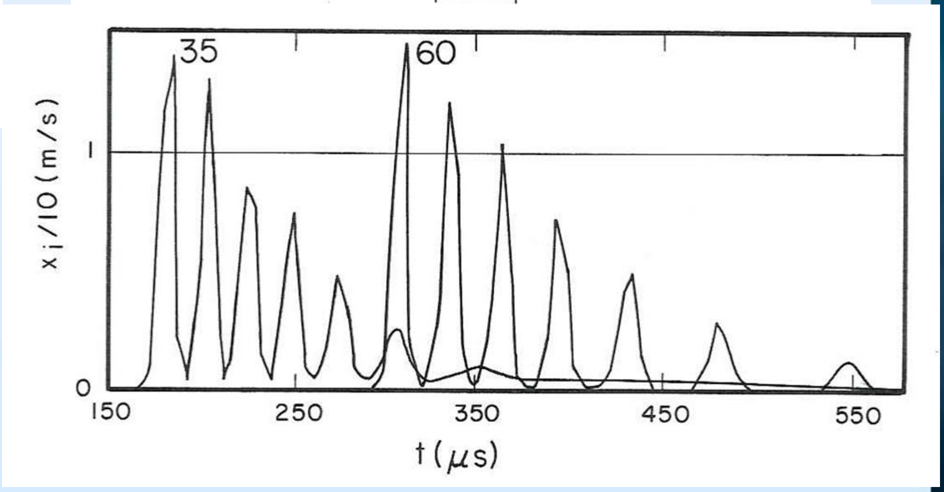
What else is unique for strongly nonlinear materials:

Fast processing of initial pulses: generation of train of solitary waves from initial impact (was very important for their experimental observation because dissipation could be ignored in a first approximation).

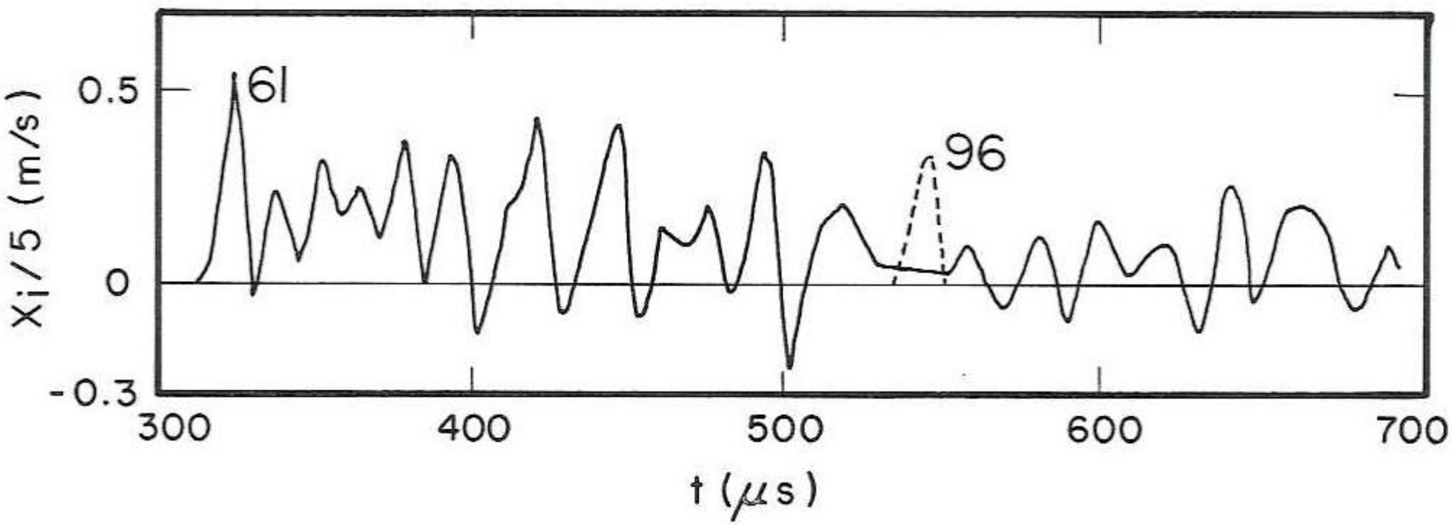
Modification of initial pulses on ***short*** distances from entrance





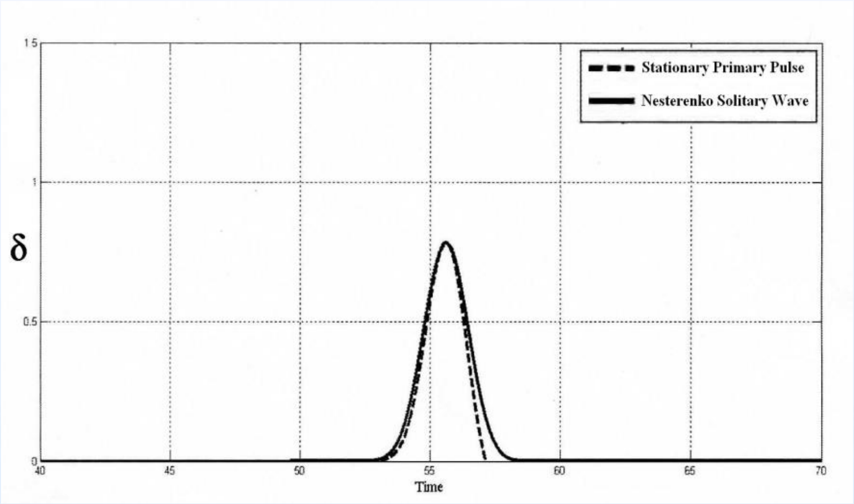


“Solitary” waves in random chain



Decay of the quasisolitary impulse in the random chain: particle velocity for the 61th particle and first oscillation (broken line) for the 96th particle. Random chain was impacted by two particles with a velocity of 5 m/s.

Other solitary waves similar to observed in 1-D chain



Solitary wave in *on site* disturbed chain, comparison to Hertizan chain
(Figure from paper by Y. Starosvetsky, PRE, 85, 051306, 2012)

Is 1-D solitary wave in granular chain related to 2-D or 3-D packings?
From: A. Leonard · F. Fraternali · C. Daraio, Experimental Mechanics, 2011:
"….deviations from the granular crystal's ideal contact lattice were not significant enough to prevent the formation and propagation of solitary waves, with no or minor energy lost in the excitation of adjacent chains of particles. The observed solitary waves appeared to have comparable properties to the extensively studied solitary waves traveling in an uncompressed, one-dimensional chain of spheres.

History lessons: Change of paradigm of nonlinearity

D. Bernoulli, 1741
(30 years before Boussinesq's papers 1871,1872)

"For the elongation will not be proportional to the extending force…and everything must be irregular".

In other words, Bernoulli accepted/introduced paradigm that nonlinearity results in irregular behavior. It survived for very long time and it was one of motivation for FPU problem – expected thermalization due to weak nonlinearity.

Einstein's interest in *strongly* nonlinear systems (original letter)

4 December, 1926

Dear Born

Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the 'old one'. I, at any rate, am convinced that *He* is not playing at dice. Waves in 3-dimensional space, whose velocity is regulated by potential energy (for example, rubber bands). . . I am working very hard at deducing the equations of motion of material points regarded as singularities, given the differential equation of general relativity.

With best wishes

Yours

A. Einstein

Albert Einstein to Max Born (4 December 1926):

Waves in 3-dimensional space whose velocity is regulated by potential energy (for example, rubber bands)…

Comment: In linear case velocity of sound c_0 (or similar pulse in another media) is not regulated by potential energy, it is simply constant. *Rubber bands* are mentioned probably as example of strongly nonlinear system experiencing large strains. Einstein did not mention metals or ceramics which represent a linear system at small amplitudes of wave.

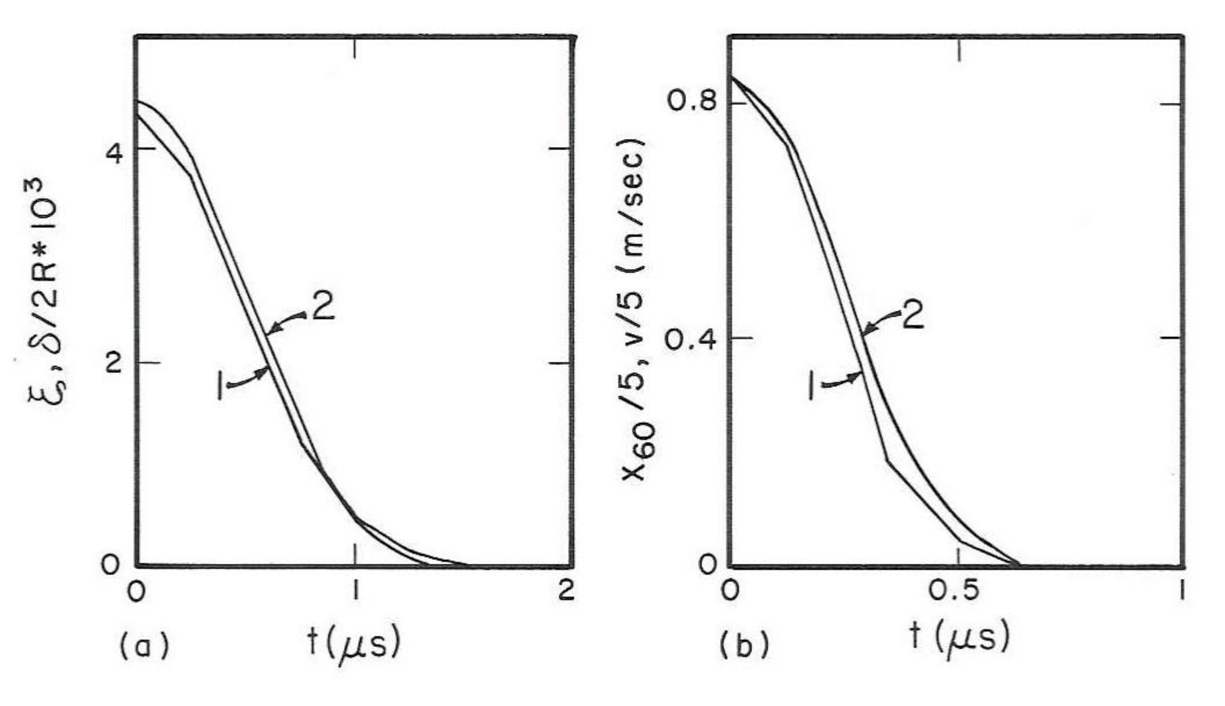
We saw that *in strongly nonlinear* case solitary wave velocity V_s is strongly regulated by potential energy U. For example, it is equal zero when potential energy is equal zero and for specific case of strongly nonlinear Hertzian chain $V_s \sim U^{1/10}$.

Einstein’ pessimism about continuum approximation and discrete versus continuum approach in strongly nonlinear granular chains

- “I consider it quite possible that physics cannot be based on the field concept, i. e., on continuous structures. In that case *nothing* remains of my entire castle in the air, gravitation theory included, [and of] the rest of modern physics”.
  -- Einstein in a 1954 letter to Besso, quoted from: "Subtle is the Lord", Abraham Pais**.**
- Interesting that Einstein did not connect requirement of strong nonlinearity (which in “sonic vacuum” resulted in a solitary wave width equal 5 particle sizes) with possible failure of continuum approach (or I failed to find it).

Strongly nonlinear solitary wave in discrete system and in continuum approximation

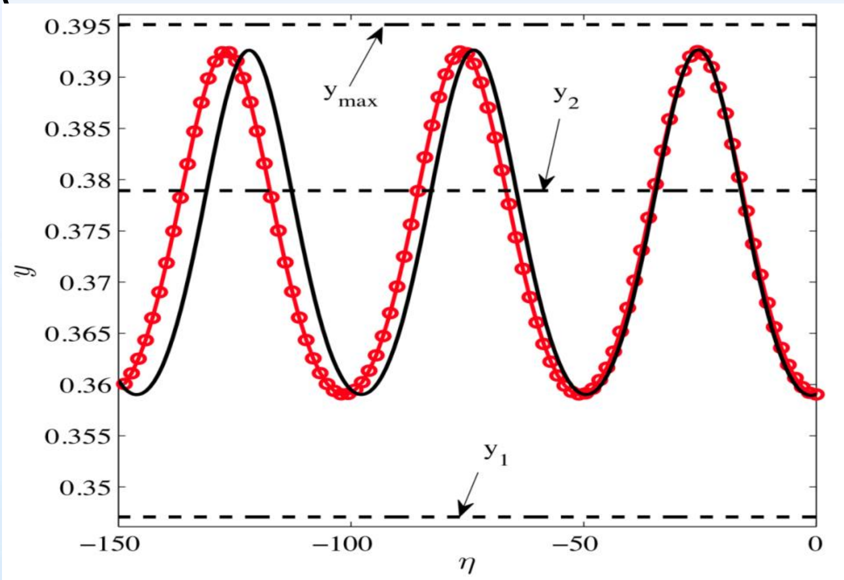
Comparison of strains (left) and particle velocities (right) profiles for a solitary wave (width about 5 particles) in discrete system (1) and continuum (2), generally satisfactory relations between discrete and continuum approaches. Compact support in continuum approach and double exponential decay in tails in numerical calculations (Chatterjee, A. *Phys. Rev. E,* 59,5912, 1999)



Periodic waves, discrete versus continuum

From Nesterenko, Herbold, Physics Procedia, 2010

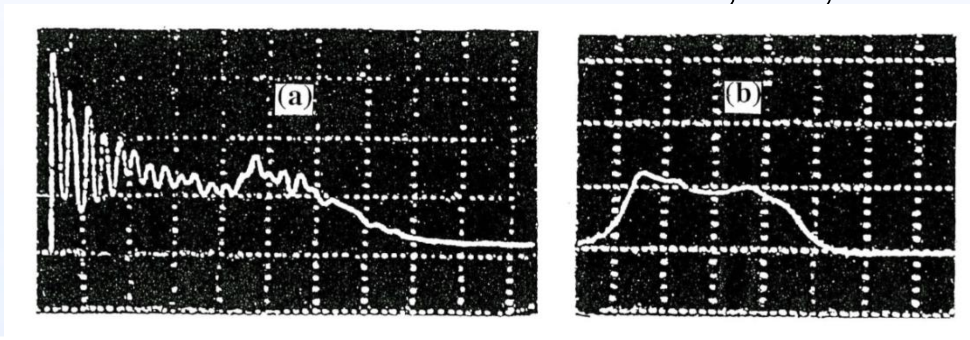
$$V_{\mathrm{p}} = c\left\{\frac{2}{5\left(\xi_{\max}-\xi_{\min}\right)}\left[\frac{2\left(\xi_{\max}^{5/2}-\xi_{\min}^{5/2}\right)-5\xi_{2}^{3/2}\left(\xi_{\max}-\xi_{\min}\right)}{\xi_{\max}+\xi_{\min}-2\xi_{2}}\right]\right\}^{1/2},$$

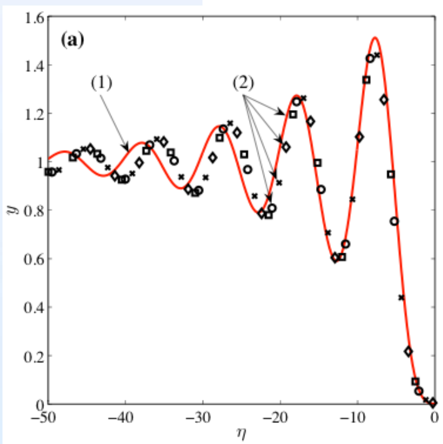


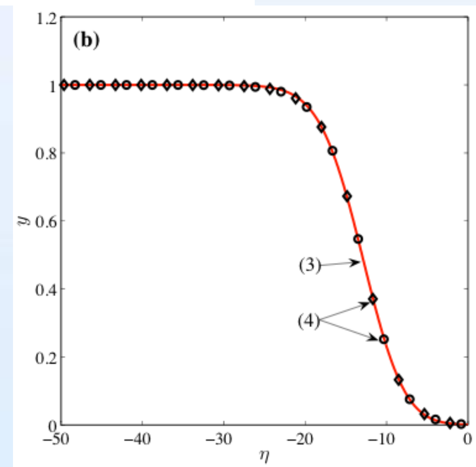
Comparison of the numerical data for discrte chain with a stationary solution of strongly nonlinear continuum wave equation.

The critical viscosity for transition from oscillatory to monotonous shock for power law materials and for Hertzian interaction

From: E*. B. Herbold and V. F. Nesterenko, PRE,2007*







$$p_{c,sv} = V_{sh} / a\sqrt{n(n-1)/3}$$

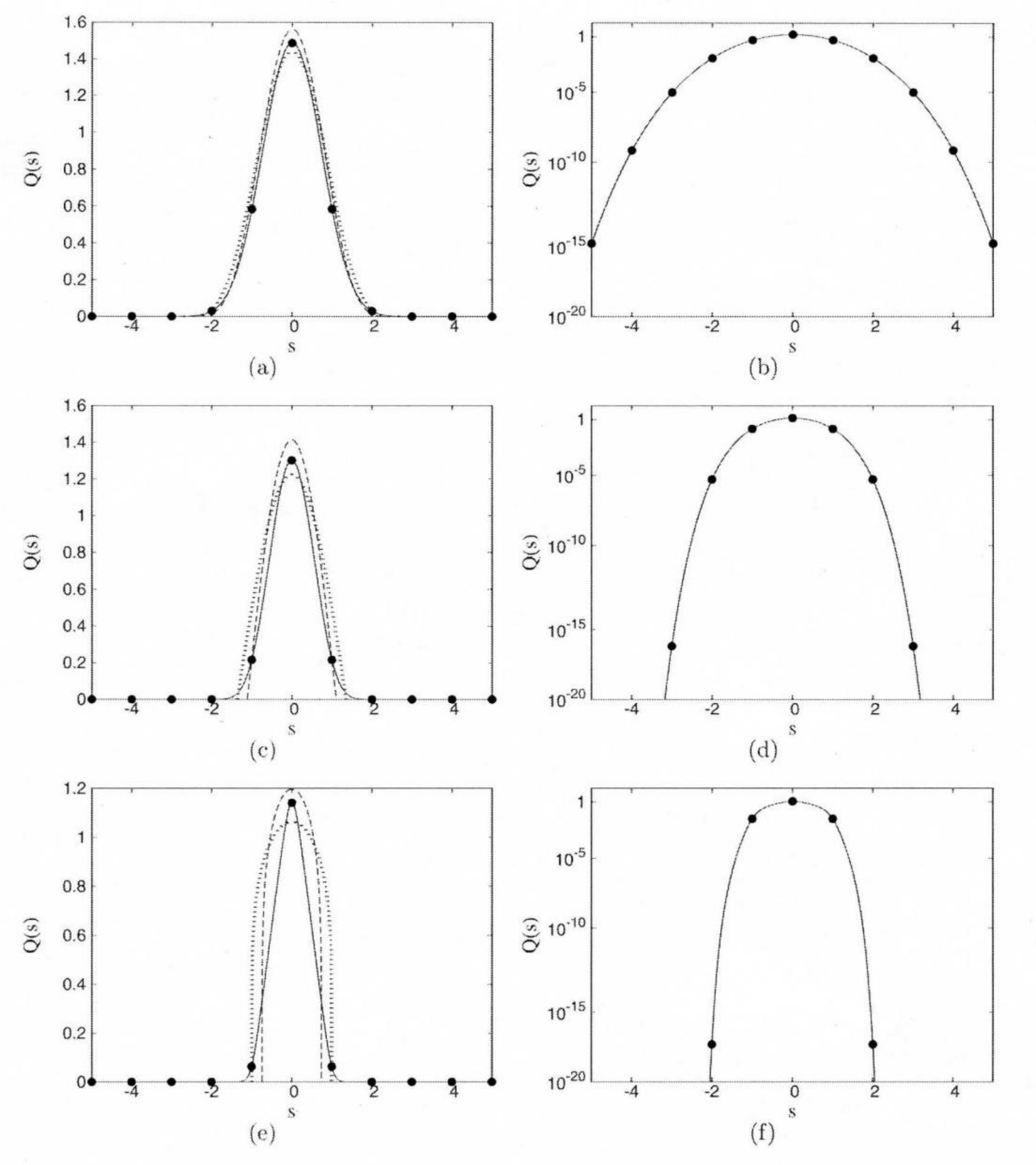
$$p_{c,sv} = V_{sh} / 2a$$

$$V_{sh} = c^{4/5} \upsilon_c^{1/5}.$$

Continuum versus discrete system, satisfactory relations between discrete and continuum, minimal shock front width is about 7 particles for Hertzian chain

Continuum versus discrete system

(From Ahnert and Pikovsky, PRE, 026209, 2009)



•Markers show the wave on the lattice
•dashed lines show the corresponding solutions of the continuum approximation based on expansion of displacements (Nesterenko, 1983, 2001)
• dotted lines show the corresponding solutions of the continuum approximation based on expansion of differences (Rosenau, Hyman 1993)
• **n=3/2:** (a, normal scale) and (b, logarithmic scale);
•**n=3** (c, normal scale) and (d, logarithmic scale);
•**n=11** (e, normal scale) and (f, logarithmic scale);

Probably Einstein was too pessimistic about continuum approximation?

- Question about limits of continuum approach is still an open question. How many grains are enough to consider them in continuum limit?
- Frieske and Watt's theorem for *discrete lattice* (Friesecke, Wattis, Comm. Math. Phys, **161**, 391, 1994) and proof of existence of compressive strongly solitary wave in ***continuum approximation*** for general interaction law (Nesterenko, 2001) require the same property of interaction law – "normal" elastic stiffening under compression

Two wave structure in dissipative strongly nonlinear system excited by δ-function pulse

(A. Rosas, A.H. Romero, V.F. Nesterenko and K. Lindenberg, PRL, 2007)

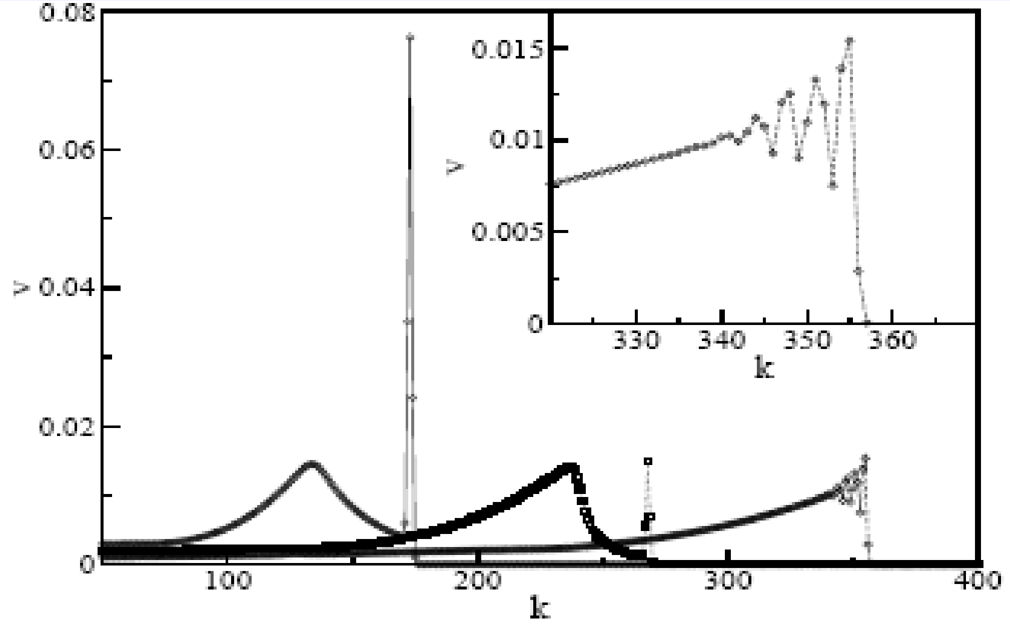
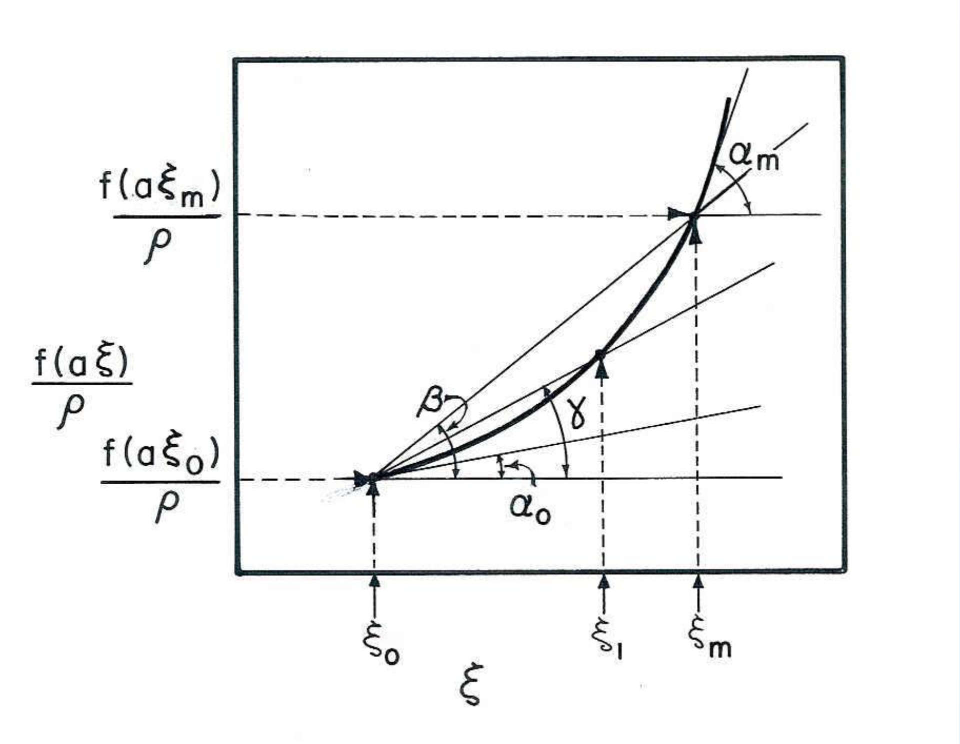


FIG. 3: Snapshots of the velocity profile for small viscosity ($\gamma = 0.02$) at different times whose progression is easily recognizable as both pulses move forward, the secondary pulse steepens, and the primary pulse disappears. The times are 500, 900, and 1400 and $n = 5/2$. Inset: detailed view of the crest of the velocity profile at time 1400.

Leading solitary wave and shock-like wave travel with different speeds and different rates of attenuation

General “normal” interaction law

Solitary wave speed with strain in maximum ξ_m is equal to square root of $\tan\gamma$, angle γ is determined by position of minimum of effective potential energy *W*, $\tan\beta$ is related to speed of shock wave with final strain ξ_m



$$\rho\xi_{tt} = \left[f + \frac{a^2}{24}\left(2 f' \xi_{xx} + f'' \xi_x^2\right)\right]_{xx}$$

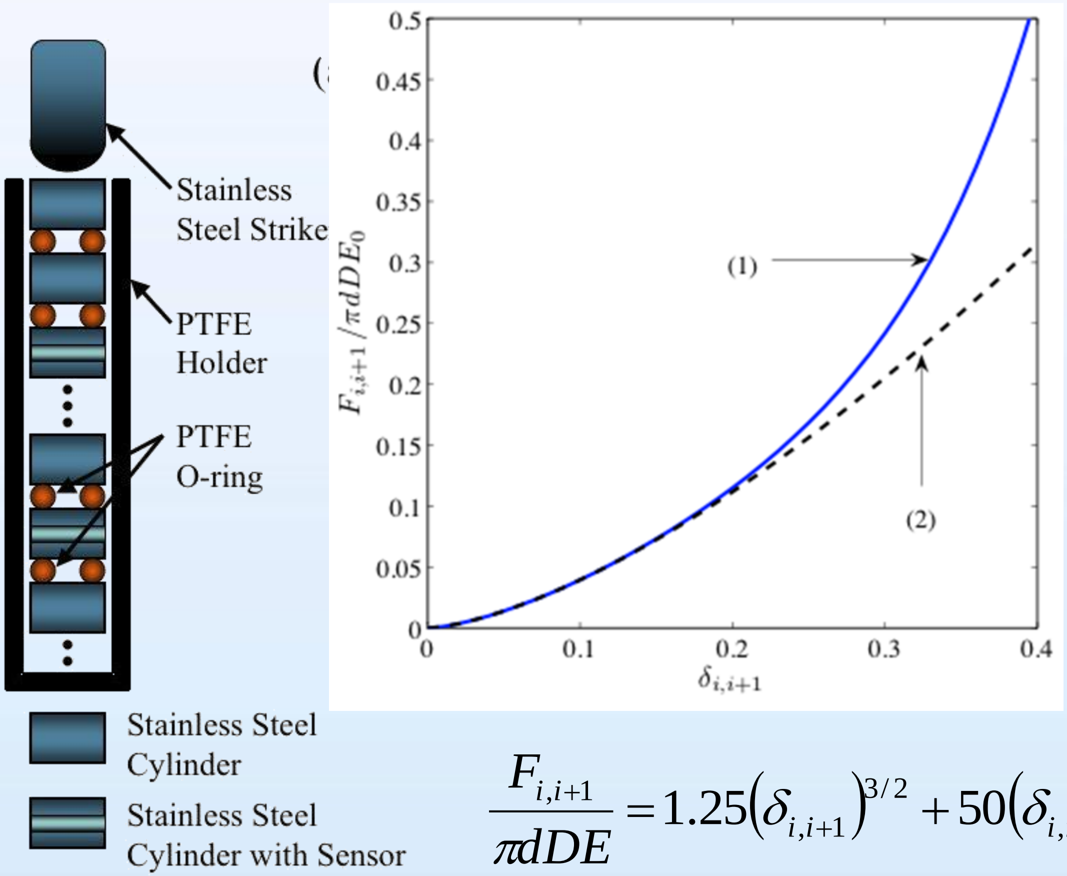
Power law materials, compression solitary wave

$$V = c_n \left(\frac{2}{n+1}\right)^{1/2} \left(\xi_{\max}\right)^{\frac{n-1}{2}}; \quad L_n = \frac{\pi a}{n-1}\sqrt{\frac{n(n+1)}{6}}.$$

1. Width of strongly nonlinear compression solitary wave is scaled with diameter of particle, but also strongly depends on force exponent (for hertzian interaction n=3/2 and $L_n \approx 5a$.
2. When n is approaching 1, the width L_n is approaching infinity, at large n, L_n is close to a.
3. The width does not depend on the amplitude

Another examples of strongly nonlinear discrete systems

(double power law materials if O-rings are used as strongly nonlinear elements. This approach allows design of strongly nonlinear metamaterials with practically any interaction forces between elements)

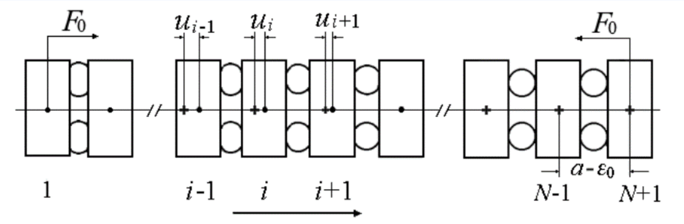


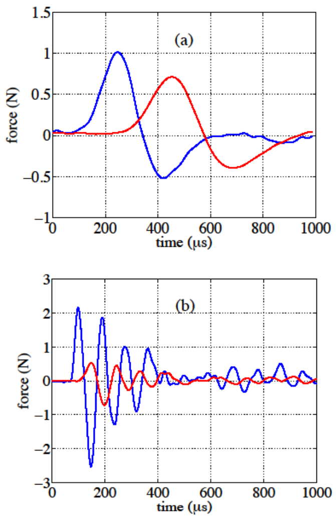
$$\frac{F_{i,i+1}}{\pi dDE} = 1.25\left(\delta_{i,i+1}\right)^{3/2} + 50\left(\delta_{i,i+1}\right)^{6}$$

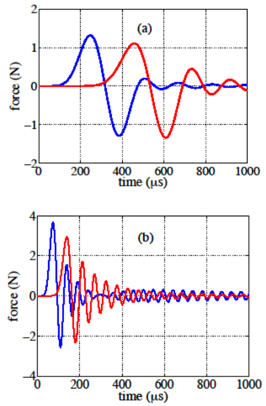
From: *E.B. Herbold, V.F. Nesterenko, Applied Physics Letters, 2007*

Short stress pulses in a chain of nitrile O-rings and stainless steel cylinders generated by a steel striker (0.455 g) with an initial velocity of 2.62 m/s

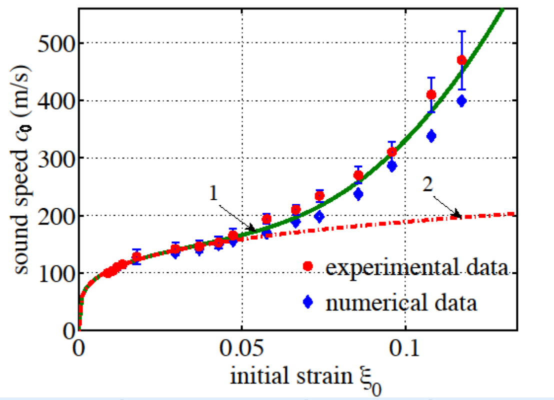
(From: Yi. Xu and Vitali F. Nesterenko, *Phil. Trans. R. Soc. A,* vol. 372, 20130186, 2014)







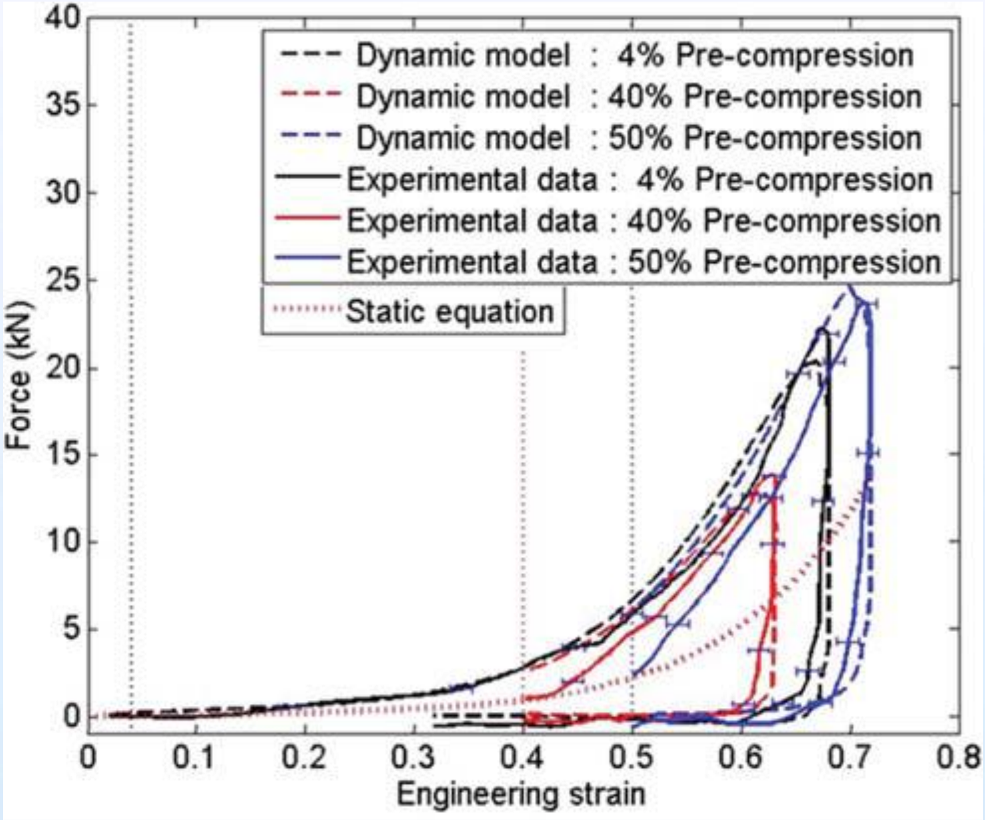
Experimental (left) and numerical results (right) at static precompression force (a) 10N and (b) 193 N.



Experimental and numerical of pulse speed dependence on initial strain. Green curve represents the long-wave sound speed at E_0 =105 MPa. Dashed red curve represents sound speeds corresponding to the Hertzian part of the interaction.

Experiments with O-ring and in the model (viscoelastic, strongly nonlinear)

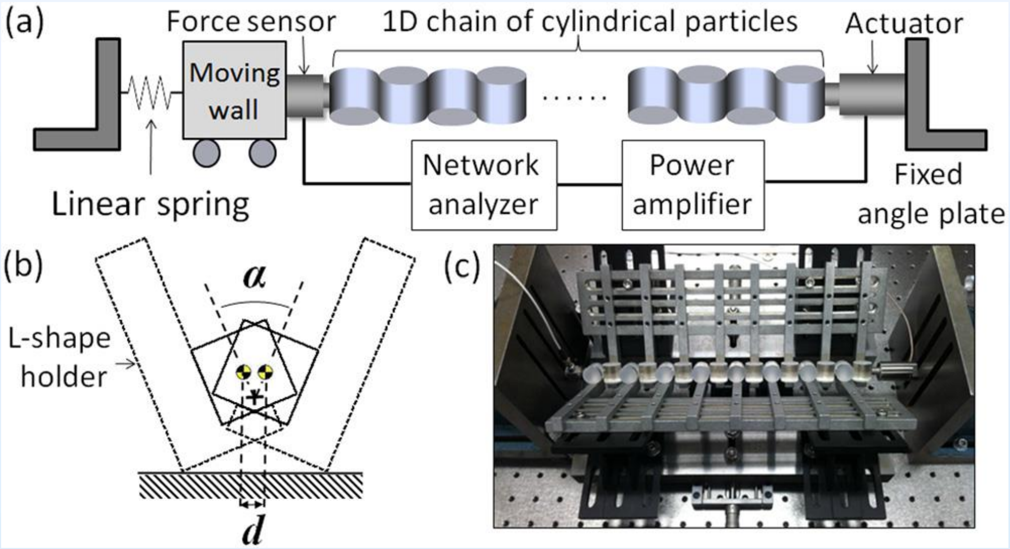
From: C-W Lee and V. F. Nesterenko, Journal of Applied Physics, vol. 116, 083512 (2014)



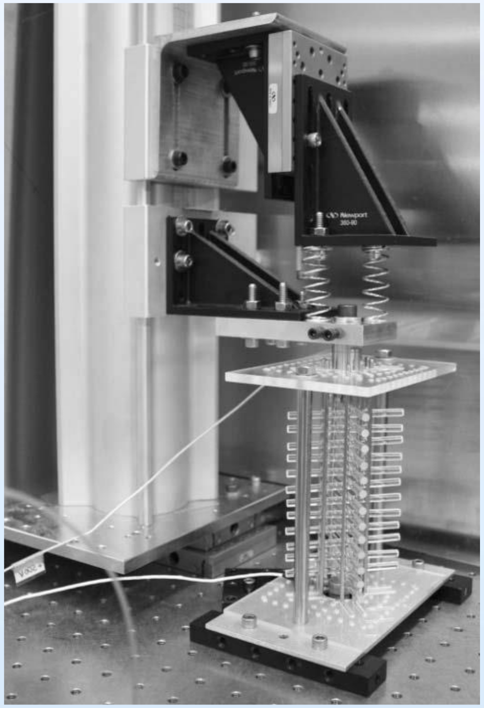
The pre-compression strains are indicated by corresponding vertical lines: 4%, 40%, and 50%, which also correspond to the beginning of curves for dynamic forces in experiments (at corresponding points on the static curve) and in the model (at points elevated vertically from the static curve). A static equation is shown by a dotted line.

Other strongly nonlinear discrete systems with behavior similar to Hertzian chain

(experimental setup composed of a one-dimensional chain of alternating cylindrical particles or rods)



From: F. Li, D. Ngo, J. Yang, and Chiara Daraio, Applied Physics Letters 101, 171903 (2012)



E. Kim and J. Yang, arXiv:1404.6972

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October 23, 2014

Anomalous, strongly nonlinear discrete systems

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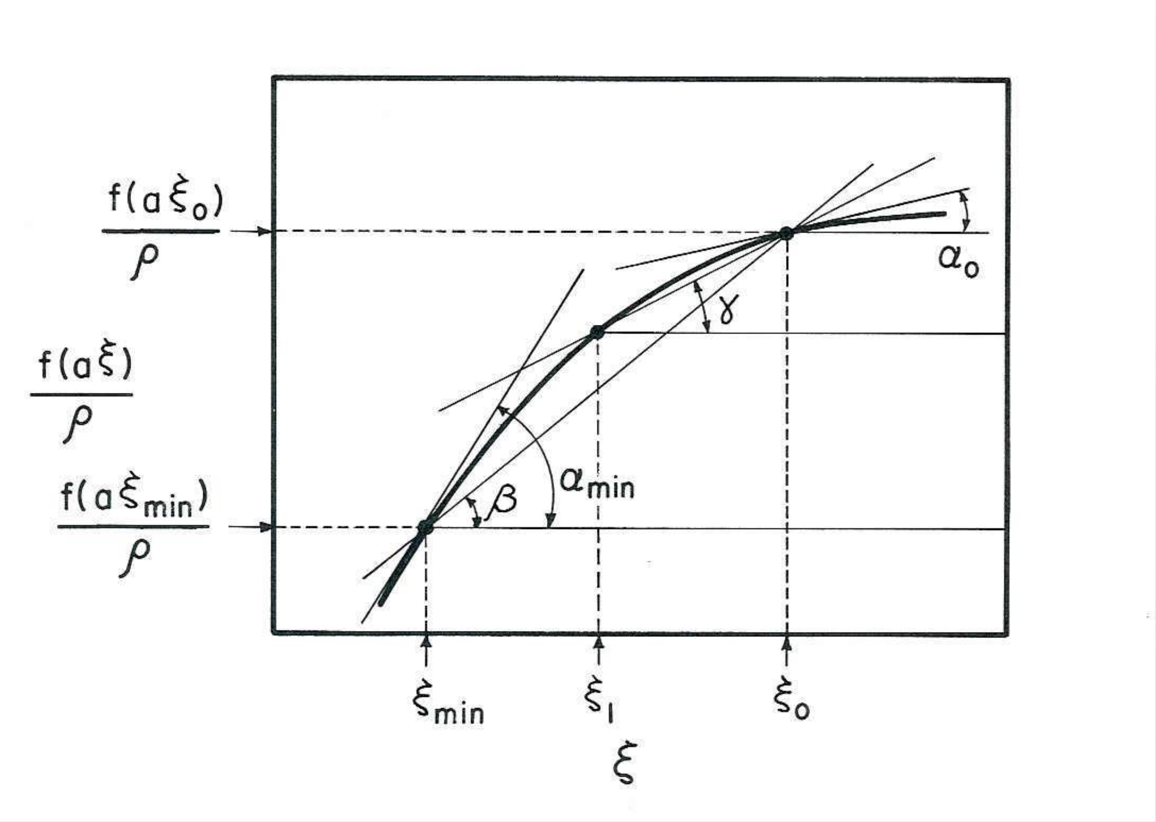
October 23, 2014

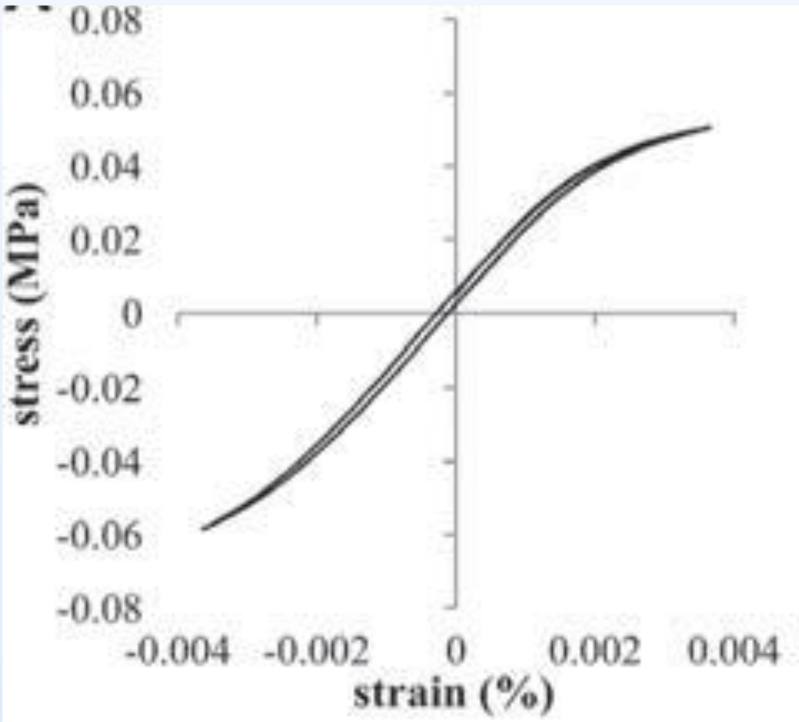
“Abnormal” Materials

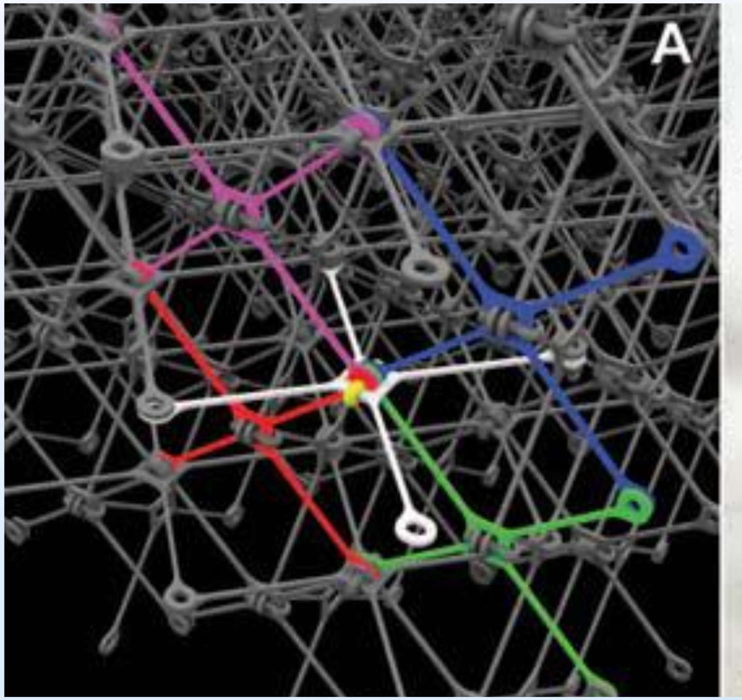
Graphical illustration of the relative speeds of sound, rarefaction solitary wave, and rarefaction shock for abnormal materials



Linear elastic response followed by a nonlinear response due to coordinated buckling

(Figures from: K. C. Cheung and N. Gershenfeld, *Science* 341, 1219 (2013))

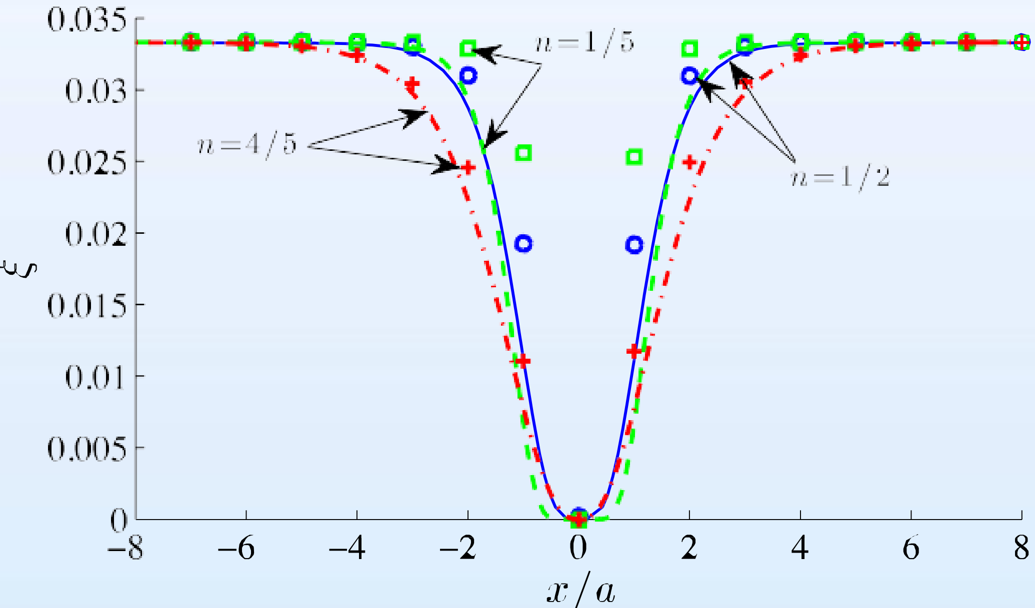




Elastic response of composite materials (left) made by reversibly assembling a three-dimensional lattice (right) of mass-produced carbon fiber–reinforced polymer composite parts with integrated mechanical interlocking connections.

Solitary waves in discrete chain versus continuum for three different values of power exponent *n*

$$y = \left(\frac{2n}{n+1}\right)^{(1+n)/2(1-n)} \left|\tanh^{(1+n)/n}\left(\frac{\eta}{\sqrt{3(n+1)/(n(1-n))}}\right)\right|.$$



Strain from discrete simulations for n = 1/5 (blue), 1/2 (green), and 4/5 (red) is compared to equation above which slightly overestimates the width (FWHM) of the solitary rarefaction wave

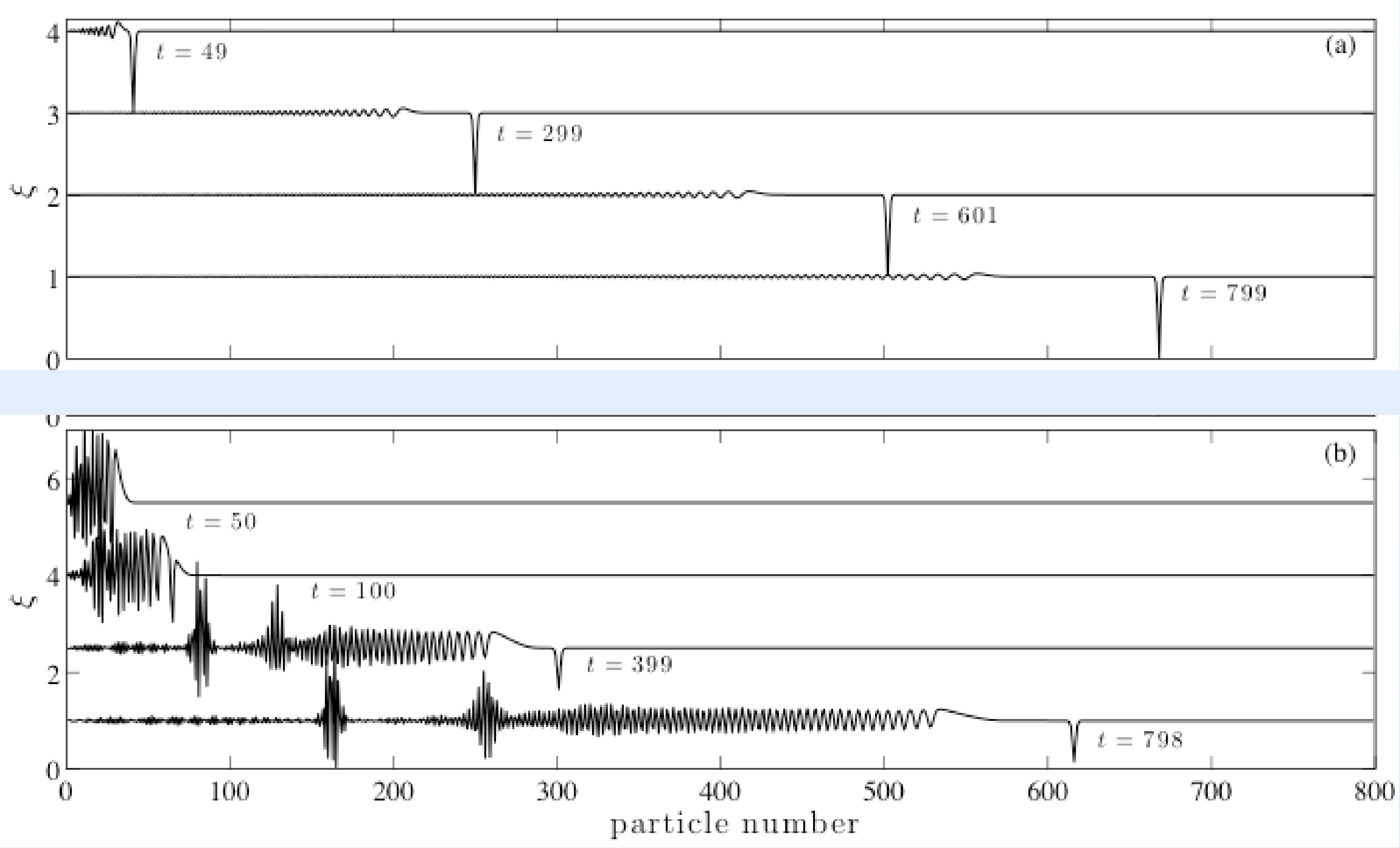
(From; E.B. Herbold and V.F. Nesterenko, Phys. Rev. Lett., **110**, 144101, 2013)

Width of strongly nonlinear rarefaction solitary wave is scaled with diameter of particle and weakly depends on force exponent (unlike the case with compression wave).
The width does not depend on the amplitude

Pulse processing by anomalous material (n=1/2), static force 1 N

(a) initial velocity of first particle **−1.373 m/s** results in rarefaction wave with a tail;
(b) impact with velocity **5 m/s** initially generates nonsteady compression wave which disintegrates into leading *rarefaction wave* and oscillatory tail with decaying amplitude.
Ultimate protection shield without dissipation!

(From; E.B. Herbold and V.F. Nesterenko, “Propagation of Rarefaction Pulses in Discrete Materials with Strain-Softening Behavior”, Phys. Rev. Lett., vol. 110, 144101, 2013)



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Interaction of pulse with interfaces between strongly nonlinear discrete materials

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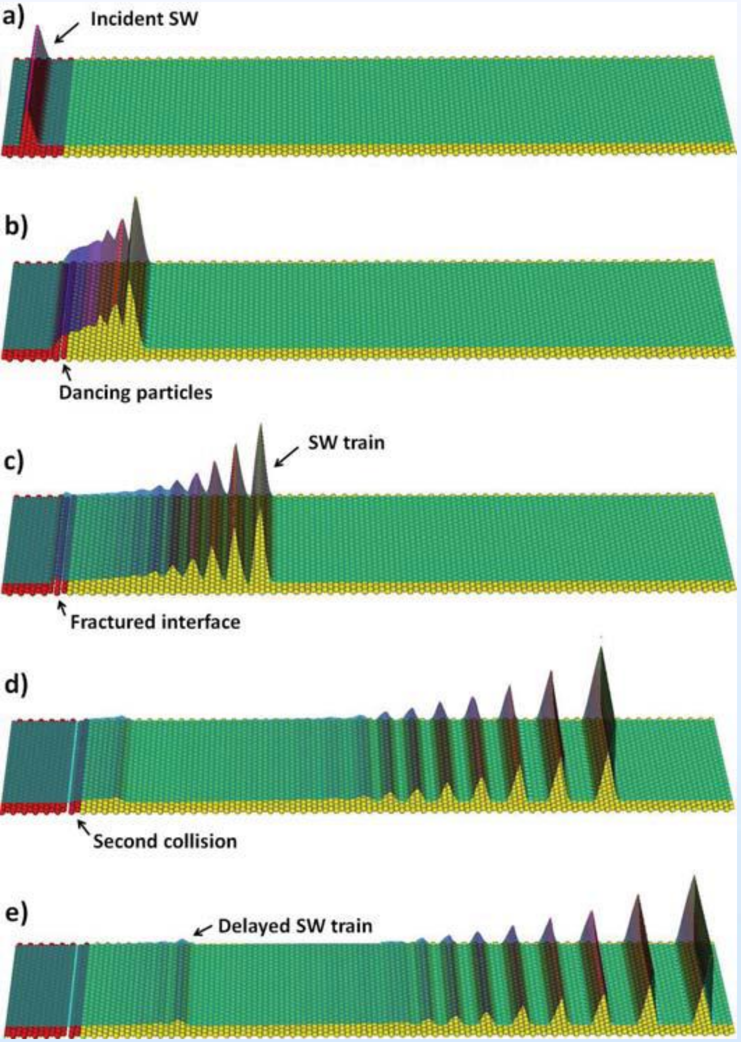
October 23, 2014

Generation of a solitary wave train by incident solitary wave

From: A. M. Tichler, L. R. Gomez, N. Upadhyaya, X. Campman, V. F. Nesterenko, and V. Vitelli, Physical Review Letters, vol. 111, 048001 (2013)



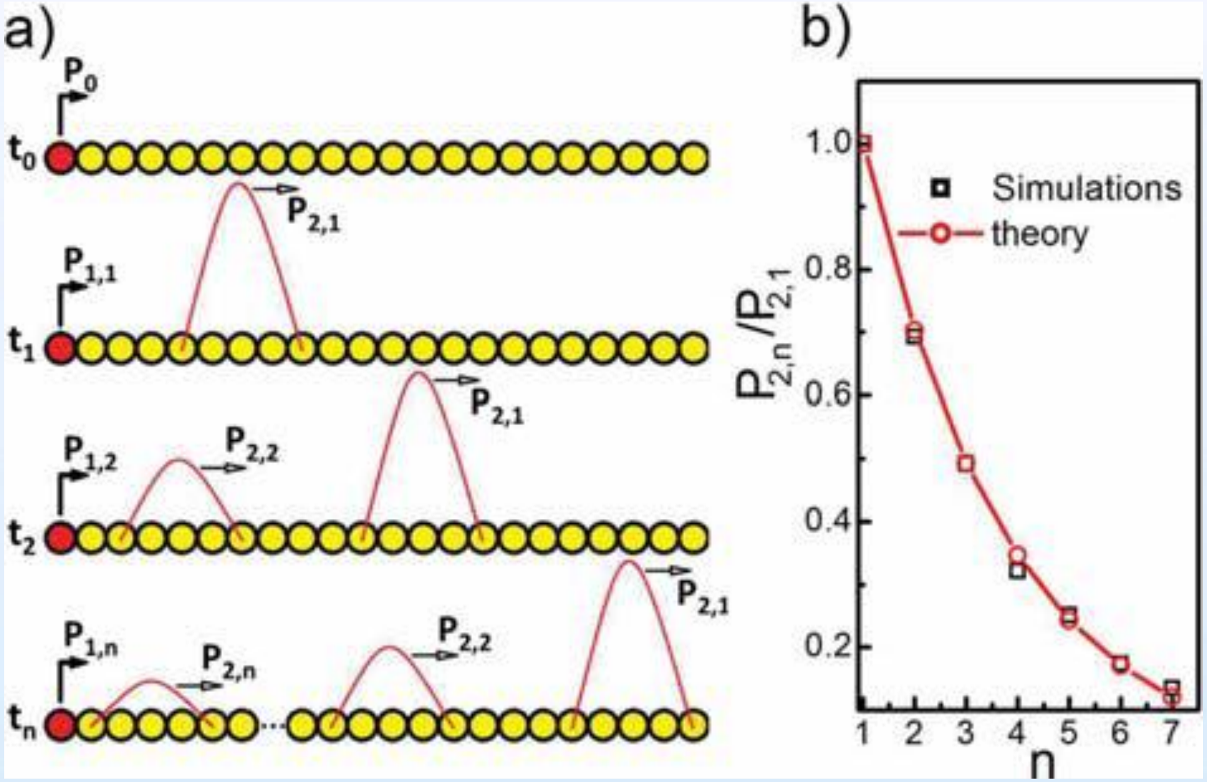
Time sequence leading to the generation of a solitary wave train in simulations. The (red) beads on the left of the interface constitute the heavier medium with mass m_1 and the (yellow) beads on the right of the interface constitute the lighter medium with mass m_2. The mass ratio $A \equiv m_2/m_1 = 0.125$. Practically the whole pulse was transmitted into the system with small particle masses.

Completely different behavior was observed when incident solitary wave approached interface from the other side.

Model for the formation of a solitary-wave train

From: A. M. Tichler, L. R. Gomez, N. Upadhyaya, X. Campman, V. F. Nesterenko, and V. Vitelli, Physical Review Letters, vol. 111, 048001 (2013)

Continuum →discrete→continuum



Momentum ratios $P_{2,n}/P_{2,1}$ between the n-th solitary wave and the leading one in the train for $A = m_2/m_1 = 0.125$. Red circles are the theoretical predictions while the black squares are the numerical values from the simulations.

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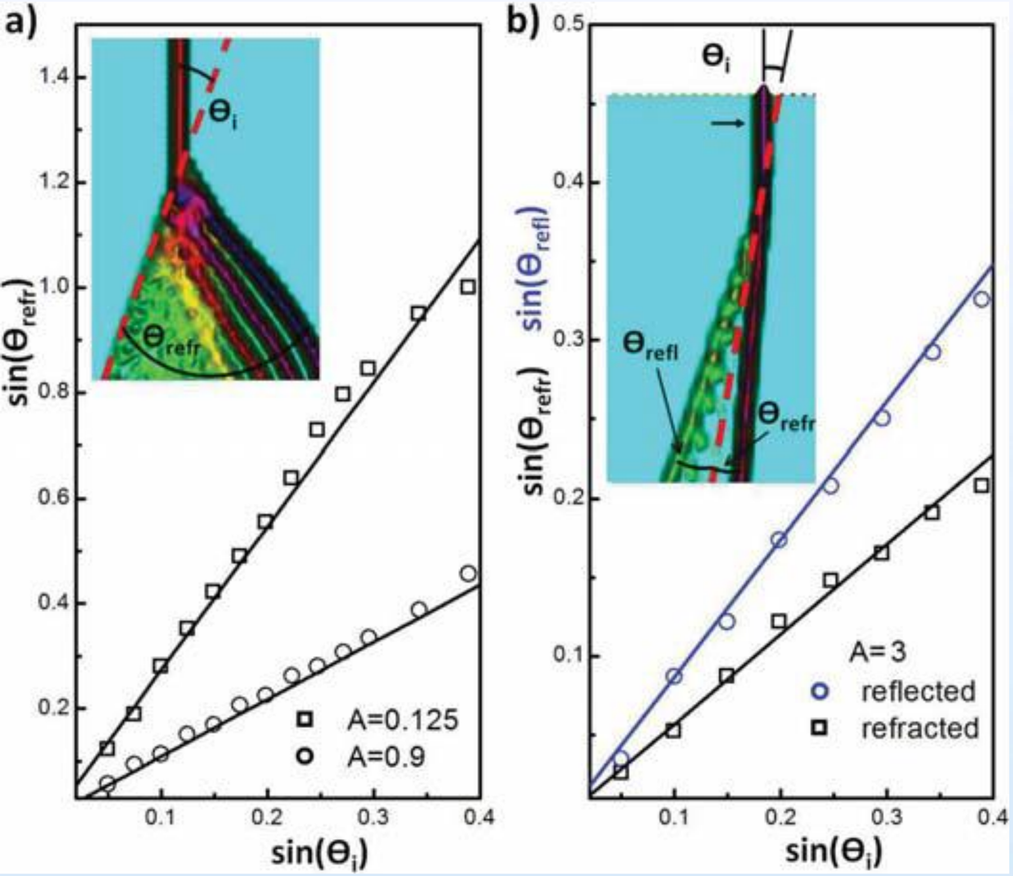
Angle of refraction θ_{refr} vs angle of incidence θ_i for the hexagonal lattice when A<1 (a) and A>1 (b)

From: A. M. Tichler, L. R. Gomez, N. Upadhyaya, X. Campman, V. F. Nesterenko, and V. Vitelli, Physical Review Letters, vol. 111, 048001 (2013)



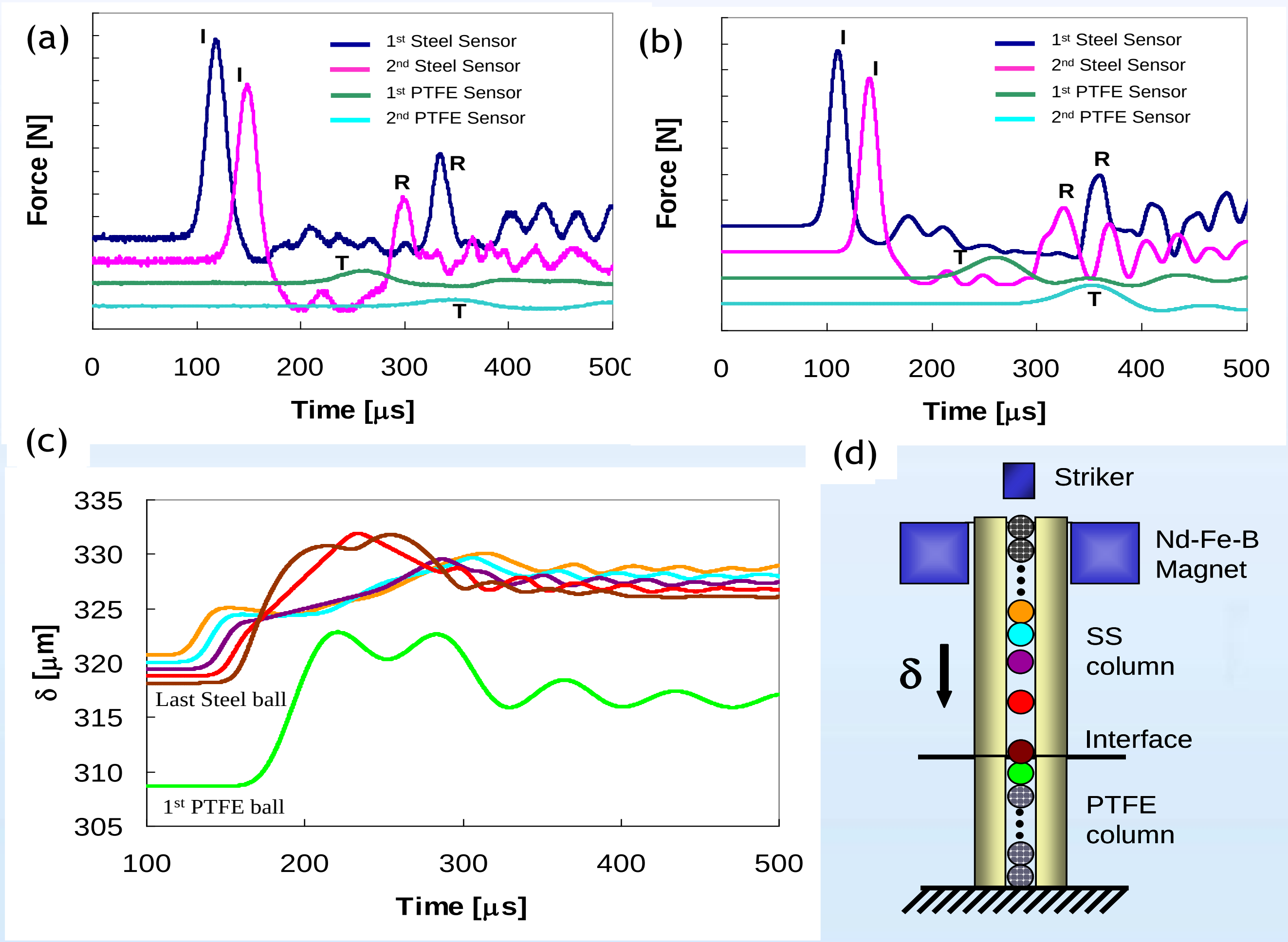
Numerical data (symbols) and the analytical estimate (solid curves) for the angle of refraction (black data) and angle of reflection (blue data) vs the angle of incidence for the hexagonal lattice. The interface is shown as the dashed (red) line and arrows represent the direction of propagation of the solitary wave front (thick dark region).

Completely different behavior in cases when A<1 (a) and A>1 (b).

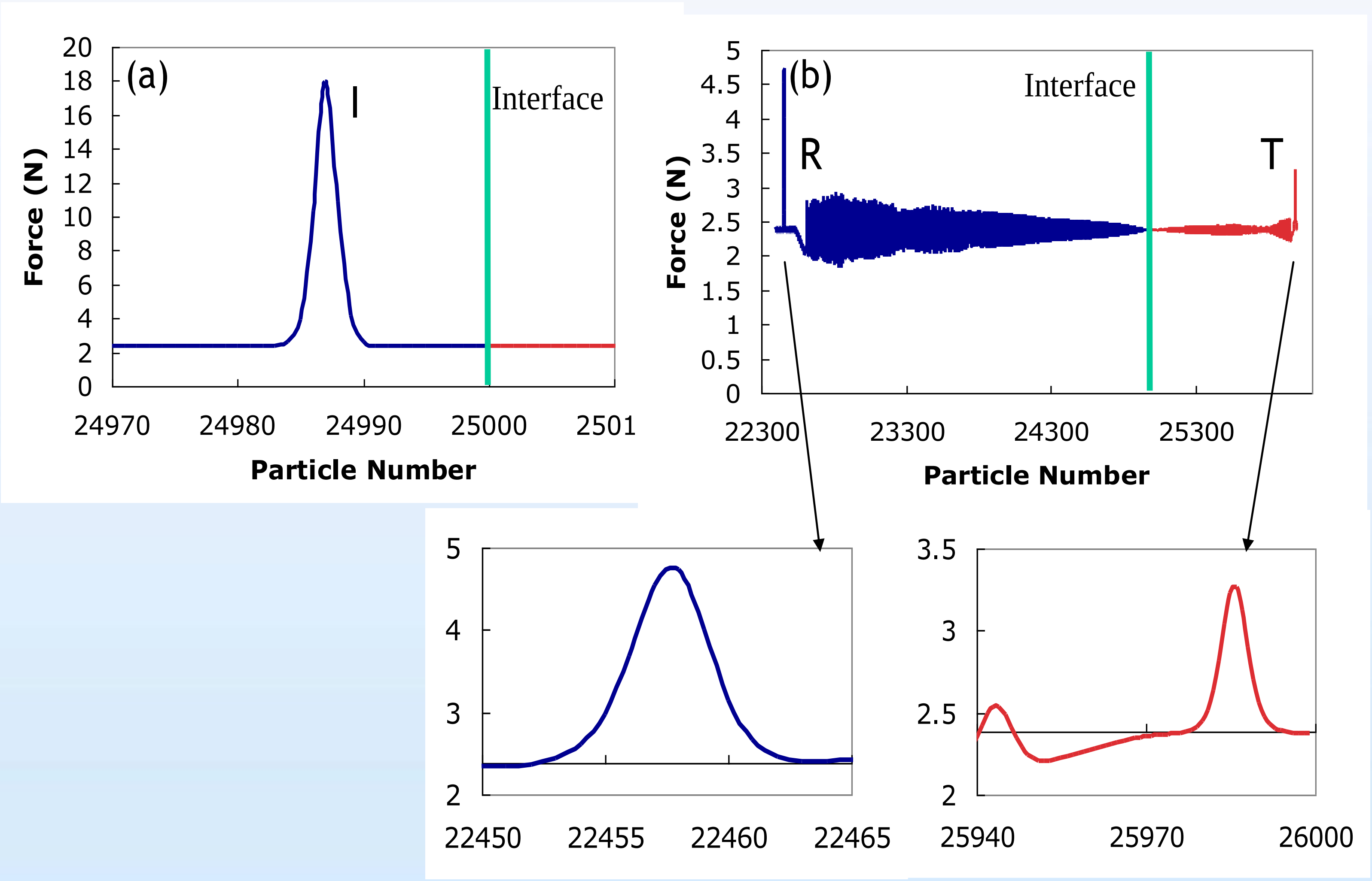
Anomalous reflection of strongly nonlinear pulse from magnetically preloaded interface of two sonic vacua,

acoustic "diode"-transmission through interface is regulated by static force

(*From: V.F. Nesterenko, C. Daraio, E. Herbold, S. Jin, PRL, 95, 158702 (2005)*)



Solitary pulse reflection from interface of two sonic vacua
(stainless steel beads(high acoustic impedance)/PTFE beads (low acoustic impedance) beads)



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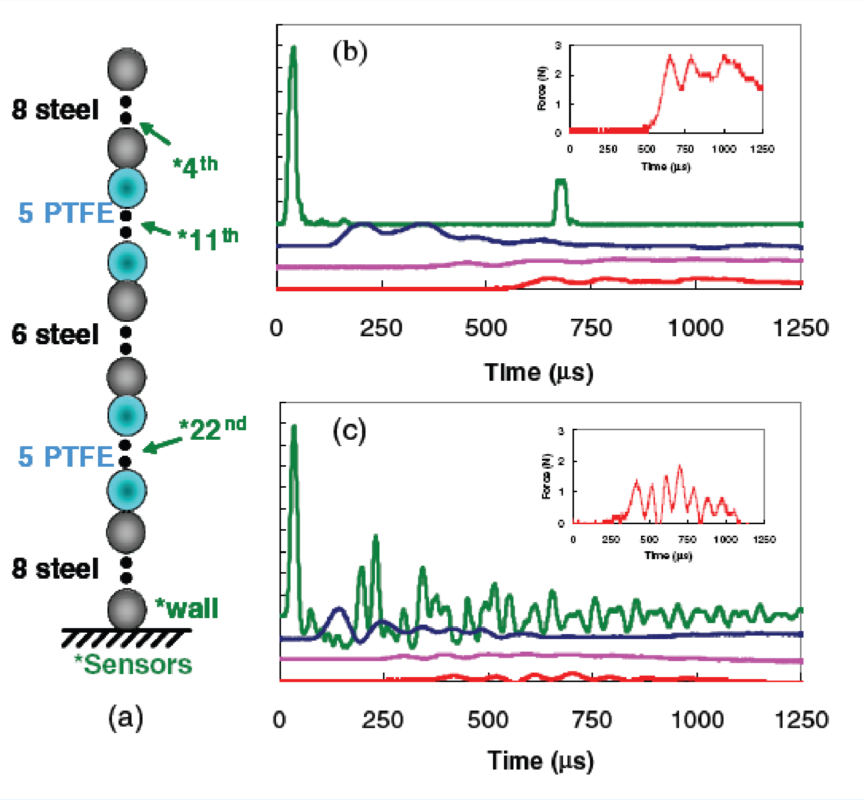
Energy Trapping and Pulse Mitigation by a Composite Discrete Medium

(C. Dario, V.F. Nesterenko, E.B. Herbold, S. Jin, Phys. PRL, vol. 96, 058002, 2006)

The introduction of the preload significantly reduced the force impulse acting on the wall, facilitating the splitting of the signal into a train of low-amplitude waves.



Strongly nonlinear two mass chains, tunable band gaps

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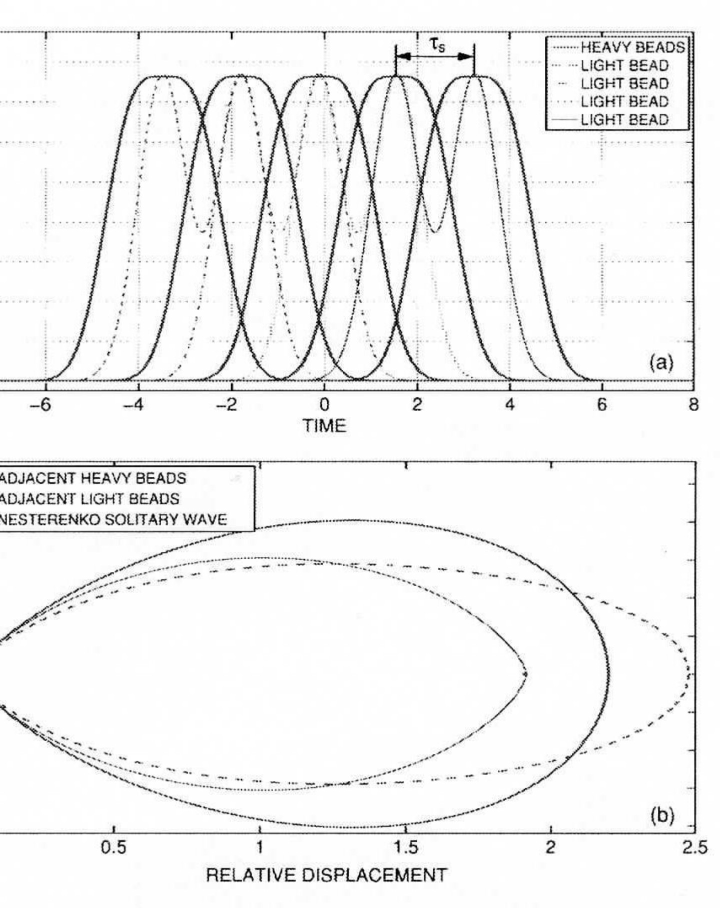
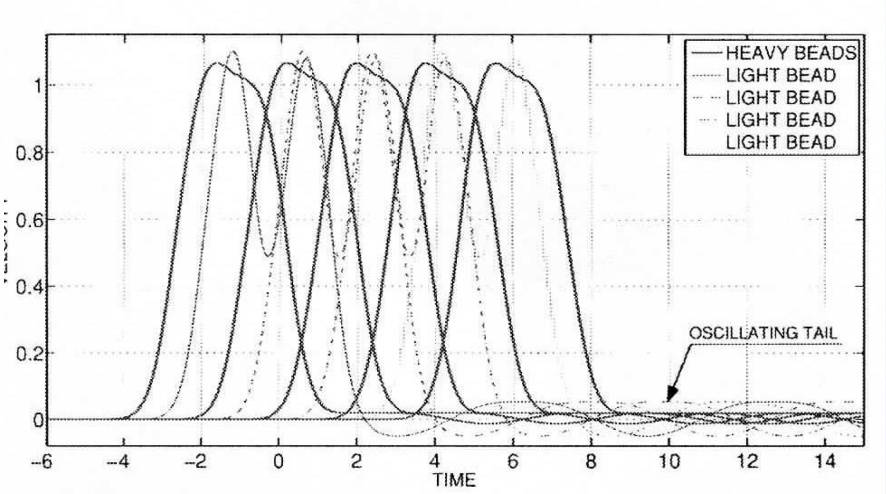
October 23, 2014

New family of solitary waves in strongly nonlinear granular dimer chains

(From K. R. Jayaprakash, Y. Starosvetsky, and A. F. Vakakis, PRE, E 83, 036606, 2011)



The system can be optimized to provide fastest pulse attenuation at the same density



olitary waves in the dimer pairs for mass ratio M=0.3428, no oscillatory tails appear in the trail of the opagating pulse

ase plot of relative velocity versus relative dispalcemnt tween successive heavy and light beads compared to e solitary wave of the homogeneous chain of heavy ads (m/M=1)

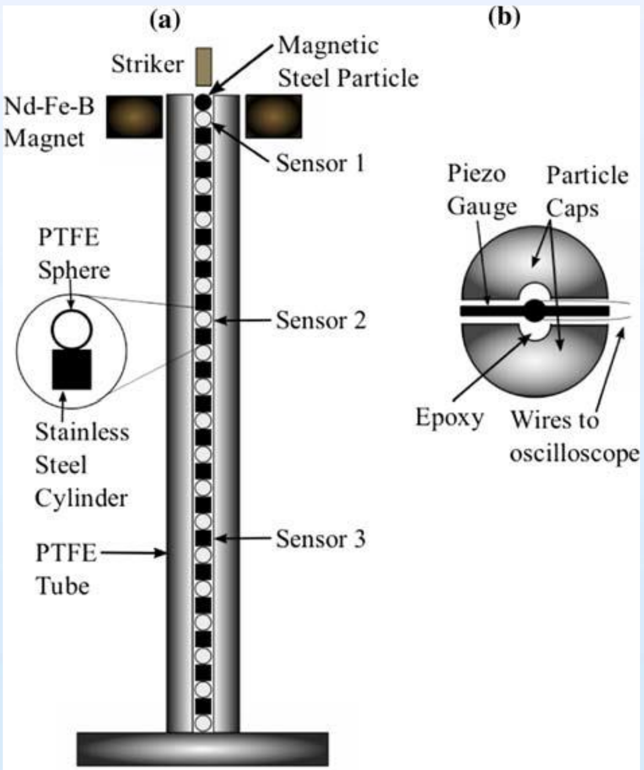
Velocity profiles of dimer pairs for the arbitrary mass ratio m/M=0.37, oscillatory tails appear in the trail of the propagating pulse causing their weak decay

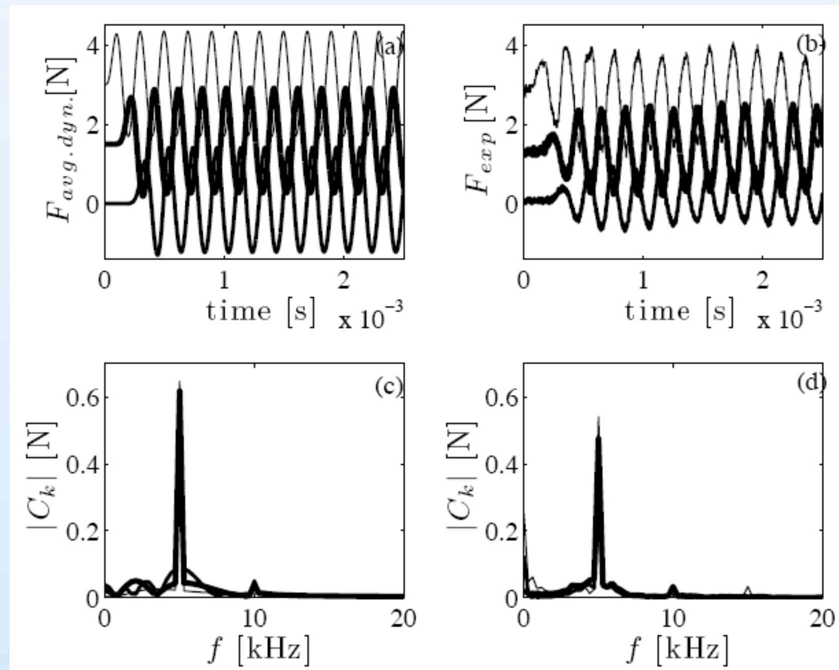
Two mass chains, band gap effects for nonlinear signals, frequency outside of band gap

(From: E. B. Herbold, J. Kim, V. F. Nesterenko, S. Wang, and C. Daraio, *Acta Mechanica*, 2009)

$$f_1 = \frac{1}{2\pi}\left(\frac{2\beta}{M}\right)^{1/2};\ f_2 = \frac{1}{2\pi}\left(\frac{2\beta}{m}\right)^{1/2},\ f \sim F_o^{1/6}$$

$$\beta \approx \frac{3}{2}A\delta_o^{1/2} = \frac{3}{2}A^{2/3}F_o^{1/3},\qquad A = \frac{4E_P E_S\left(1/R_S + 1/R_P\right)^{-1/2}}{3\left[E_S\left(1-\upsilon_P^2\right) + E_P\left(1-\upsilon_S^2\right)\right]}$$





(a), (c) Numerical calculations: (b), (d) Experimental results:.

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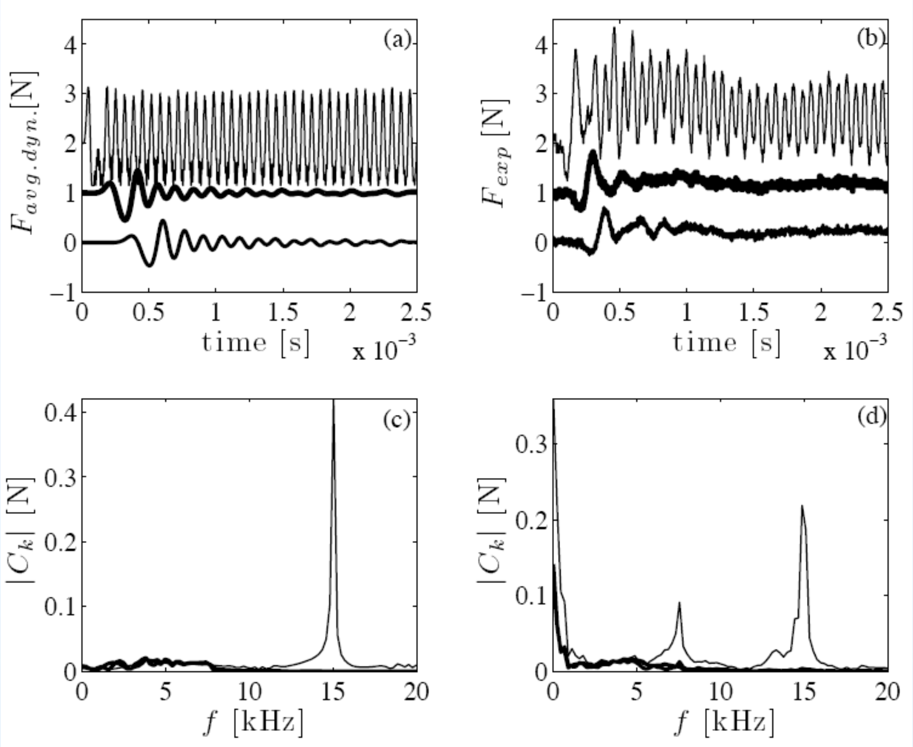
Two mass chains, band gap effects for strongly nonlinear signals, frequency inside band gap

(From: E. B. Herbold, J. Kim, V. F. Nesterenko, S. Wang, and C. Daraio, *Acta Mechanica*, 2009)

The lower range of the forbidden frequency is 7.7 kHz being lower than the frequency of the major harmonic in the input signal.



(a), (c) Numerical calculations; (b), (d) Experimental results.

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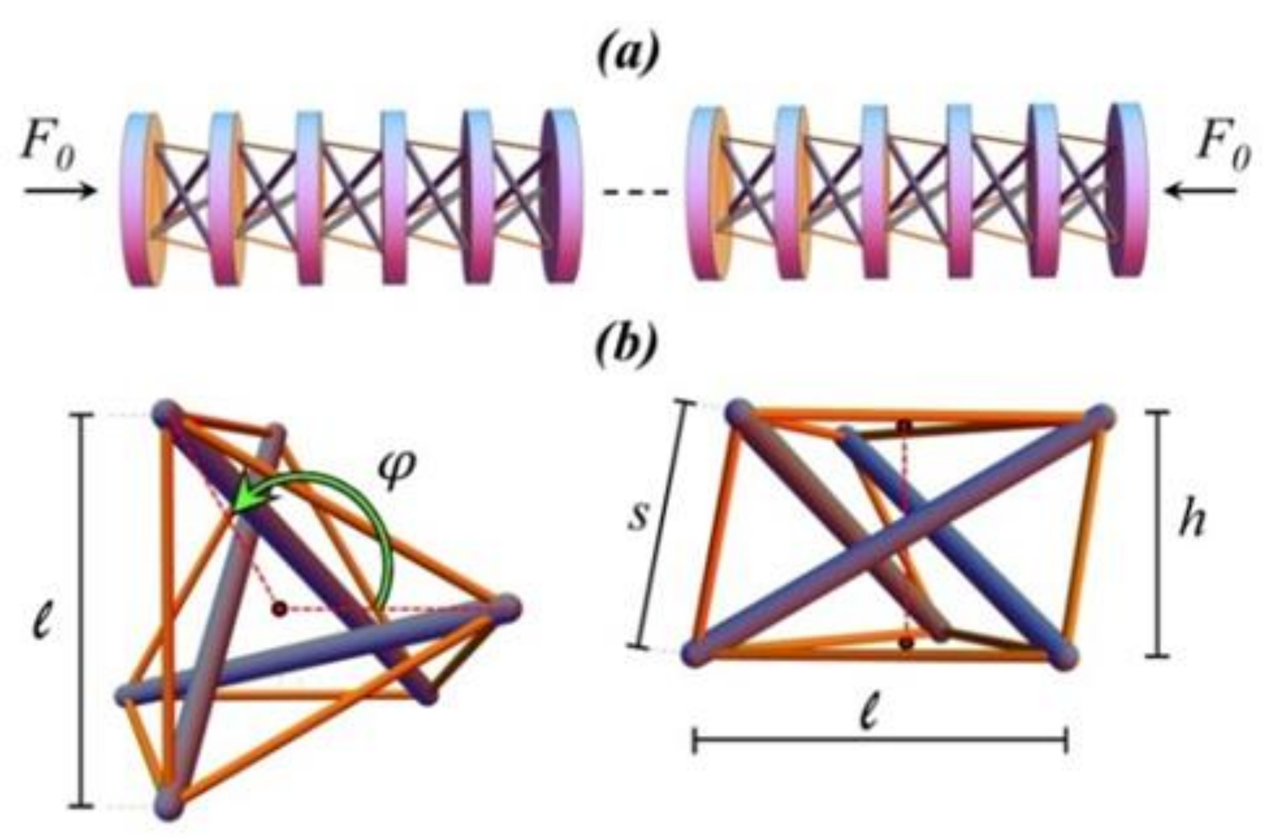
Strongly nonlinear discrete systems with unprecedented tenability: chain of tensegrity prisms and lumped masses

F. Fraternali, G. Carpentieri, A. Amendola, R. E. Skelton, and V. F. Nesterenko, Multiscale tunability of solitary wave dynamics in tensegrity metamaterials, arXiv 1409.7097



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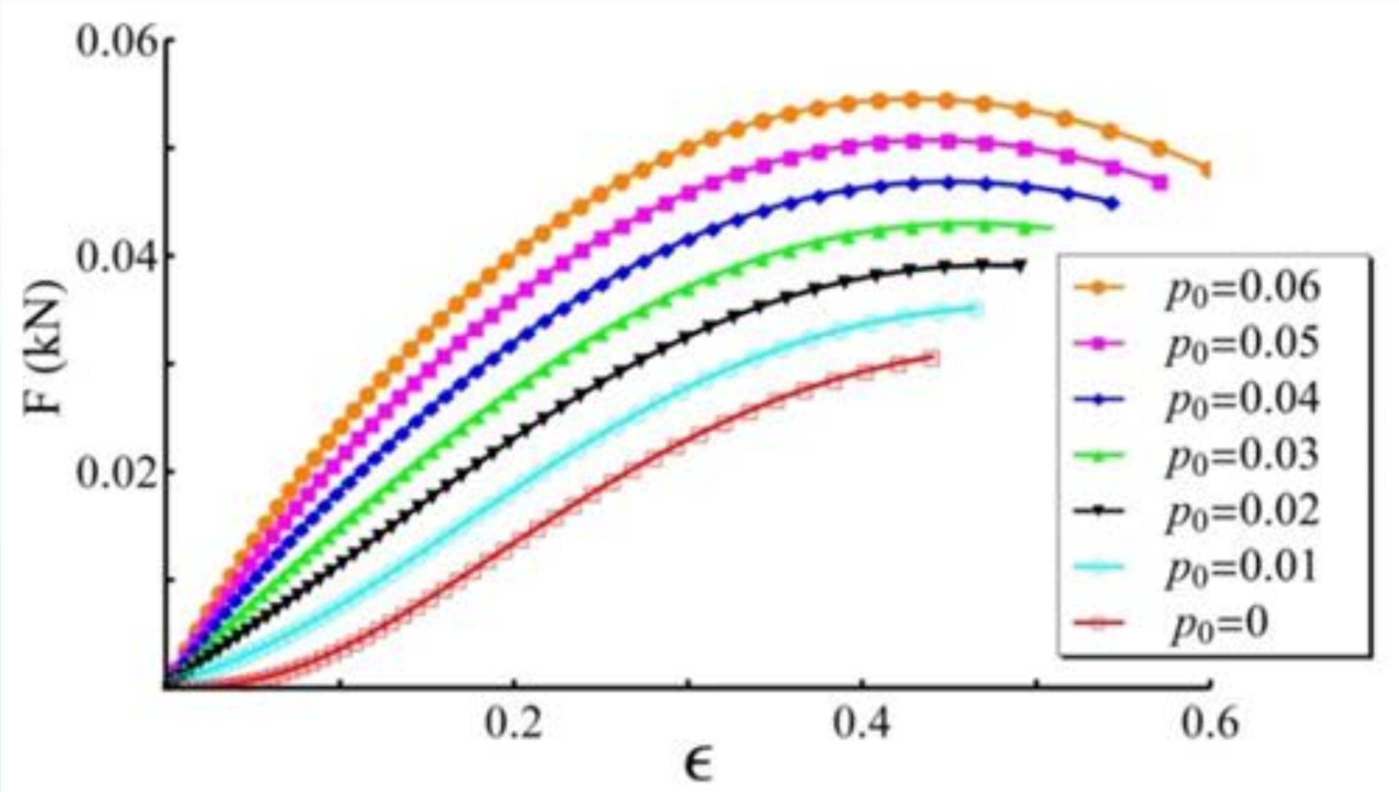
October 23, 2014

Multiscale tunability of the tensegrity unit

From: F. Fraternali, G. Carpentieri, A. Amendola, R. E. Skelton, and V. F. Nesterenko,' Multiscale tunability of solitary wave dynamics in tensegrity metamaterials, arXiv 1409.7097

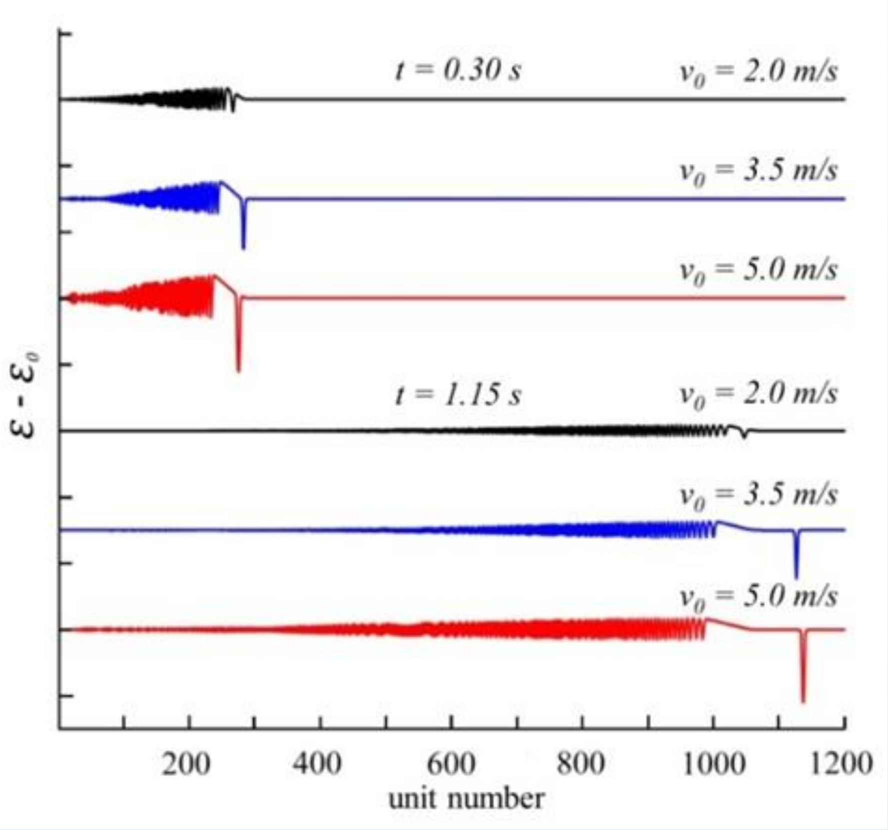


Curves correspond to the quasi-static force response of the tensegrity unit to global strain ε at different values of the local prestrain p_0.

Vitali F. Nesterenko

Impact on elastically-softening chain

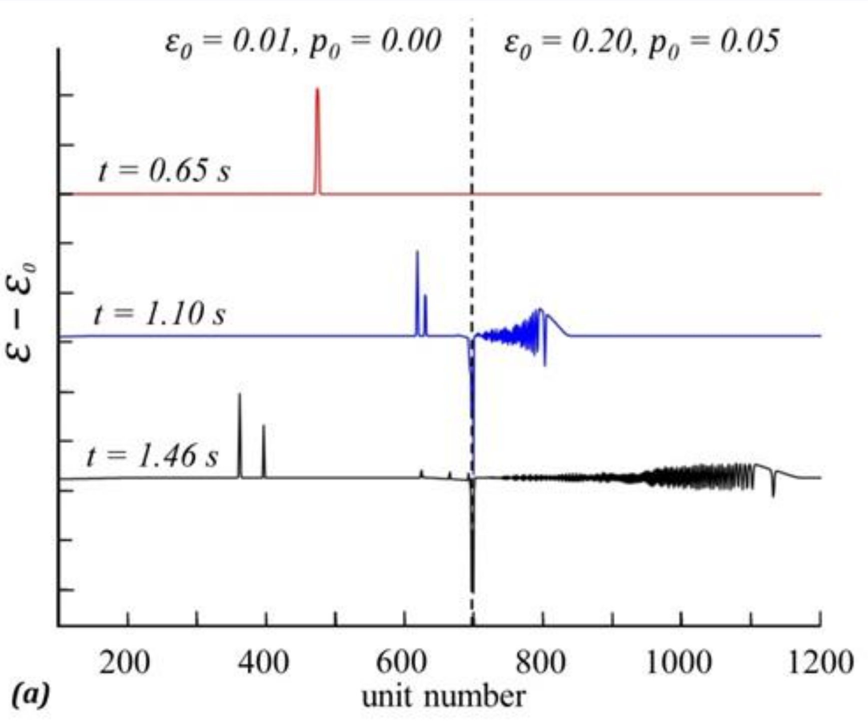
From: F. Fraternali, G. Carpentieri, A. Amendola, R. E. Skelton, and V. F. Nesterenko, Multiscale tunability of solitary wave dynamics in tensegrity metamaterials, arXiv 1409.7097

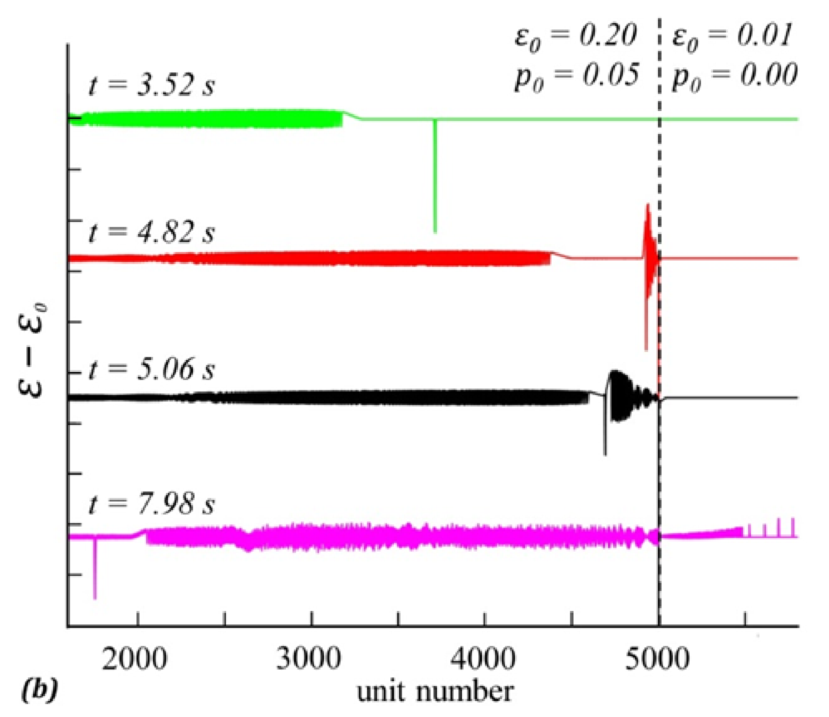


Evolution of initial compression pulse into rarefaction wave and periodic trail at different impact velocities (global and local prestrain are correspondingly equal to ε_0=0.20, p_0=0.04).

Interaction of solitary waves with interfaces LIEH and HIES

From: F. Fraternali, G. Carpentieri, A. Amendola, R. E. Skelton, and V. F. Nesterenko, Multiscale tunability of solitary wave dynamics in tensegrity metamaterials, arXiv 1409.7097





Interaction of a compression solitary wave with a LIEH-HIES interface: instead expected single reflected solitary wave we have two and transmitted wave is a rarefaction wave with periodic tail

Interaction of a rarefaction solitary wave with a HIES-LIEH interface: instead of reflected compression wave we observe reflected rarefaction wave and transmitted compression wave

Conclusions

- Strongly nonlinear, discrete “soft” condensed materials (e.g., granular materials) represent a new class of medium where new wave dynamics should be developed.
- Strongly nonlinear systems are highly tunable, including tunabilty of band gaps by small external force. Metal plates separated by polymer o-rings are more tunable than Herzian systems
- A strongly nonlinear wave equation is based on only one small parameter—ratio of the particle size to the wave length. It is more general than weakly nonlinear Korteweg–de Vries equation, the latter being a partial case of the former for small amplitude waves
- Strongly nonlinear periodic waves, compression solitary, and shock waves are qualitatively different from the weakly nonlinear KdV case.
- The spatial width of compression strongly nonlinear solitary waves in “sonic vacuum” and their shape do not depend on amplitude, initial sound speed does not determine the soliton parameters, speed has strong dependence on amplitude.
- Long wave approximation, considering discrete chain as a continuum, satisfactory predicts spatial scale of solitary waves, being comparable to particle size a (for n=3/2 it is $5a$, for larger n it is approaching a), type of shock wave (oscillatory versus monotonous depending on viscosity) and parameters of periodic waves

Conclusions

- In “sonic vacuum” initial impulse is split into a soliton train quickly on very short distances from the entrance
- Strongly nonlinear solitary waves demonstrate new behavior and anomalous reflection from interfaces, pulse trapping inside protecting layer is possible
- The solitary waves are observed in experiments including metal, polymers and polymer coated metal beads by different group of researchers
- Quantum effects in classical systems: the steady solitary waves propagate only at specific discrete values of mass ratio (0.3428, 0.1548, 0.0901, 0.0615, 0.04537…) in “diatomic” particulate chains
- Viscous dissipation may qualitatively change shock wave structure or result in two wave structure under δ-force excitation
- In the case of strongly nonlinear materials with abnormal behavior (e.g., tensegrity based), there are periodic waves, rarefaction shock waves, and rarefaction solitons. They are supersonic, relative to the initial state. No stationary compression waves are allowed in such materials and initial compression pulse quickly desintegrates into rarefaction wave and oscillatory tail. Strongly nonlinear materials with abnormal behavior might provide ultimate impact protection systems

Questions/suggestions for future

Wave dynamics in discrete, strongly nonlinear mechanical systems (first example was granular chain) is in the very early stage of development with many open questions:

- Continuum versus discrete description. Limits of strongly nonlinear continuum approach.
- Nonstationary analysis of transients, dependence of number of solitary waves on parameters of initial disturbance
- Optimization of properties of dissipative strongly nonlinear systems, specifically, periodic systems of toroidal rubber elements (highly nonlinear) and metal plates
- Assembling and testing of a new strongly nonlinear *metamaterials*, particularly for high energy absorption, e.g., tensegrity based
- Experiments with strongly nonlinear systems on smaller scales, from mesoscale to atomic scale
- Anomalous strongly nonlinear systems, experimental realization, additive manufacturing is a possible approach for their manufacturing
- Optimized strongly nonlinear systems for targeted energy transfer
- Electromagnetic and molecular analogs of strongly nonlinear systems
- Chaos and thermalization in a system without phonons
- Nanoscale systems without phonons. How heat propagates in a systems without phonons?
- Anderson localization in disordered strongly nonlinear systems
- Optimization of “soft” condensed materials for impact/blast mitigation, development of lenses for focusing/defocusing, delay lines

Detailed information can be found in the following book and papers

1. V.F. Nesterenko, *Dynamics of Heterogeneous Materials*, Springer-Verlag, 2001, 522 pages, New York, Chapter 1.
2. V.F. Nesterenko, New Wave Dynamics in Granular State, in *The Granular State*, MRS Symp. Proc., edited by S. Sen and M.L. Hunt, 2001, vol. 627, pp. BB3.1.1 – BB3.1.12.
3. V.F. Nesterenko, (Shock (Blast) Mitigation by “Soft” Condensed Matter, in "Granular Material-Based Technologies”, edited by S. Sen, M.L. Hunt, and A.J. Hurd, *MRS Symp. Proc*., vol. 759 (MRS, Pittsburgh, PA, 2003), pp. MM4.3.1- 4.3.12 (arXiv preprint cond-mat/0303332).
4. C. Daraio, V.F. Nesterenko and S. Jin “Highly Nonlinear Contact Interaction and Dynamic Energy Dissipation by Forest of Carbon Nanotubes”, *Appl. Phys. Lett.*, **85**, no.23, p. 5724, 2004.
5. C. Daraio, V.F. Nesterenko, E. Herbold, and S. Jin “Strongly Nonlinear Waves in a Chain of Teflon Beads”, *Phys. Rev E,* **72**, 016603, 2005.
6. V.F. Nesterenko, C. Daraio, E. Herbold, and S. Jin “Anomalous wave reflection at the interface of two strongly nonlinear granular media”, *Physical Review Letters*, **95**, 58702, 2005.
7. C. Daraio, V.F. Nesterenko, E.B. Herbold, S. Jin “Tunability of solitary wave properties in one-dimensional strongly nonlinear phononic crystals”, *Phys. Rev E*, **73**, 026610, 2006.
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Thank you for your attention!