Statistical size effects on compressive strength

from an interpretation of failure as a critical transition

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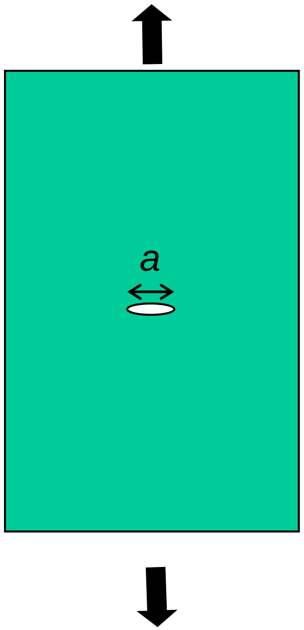
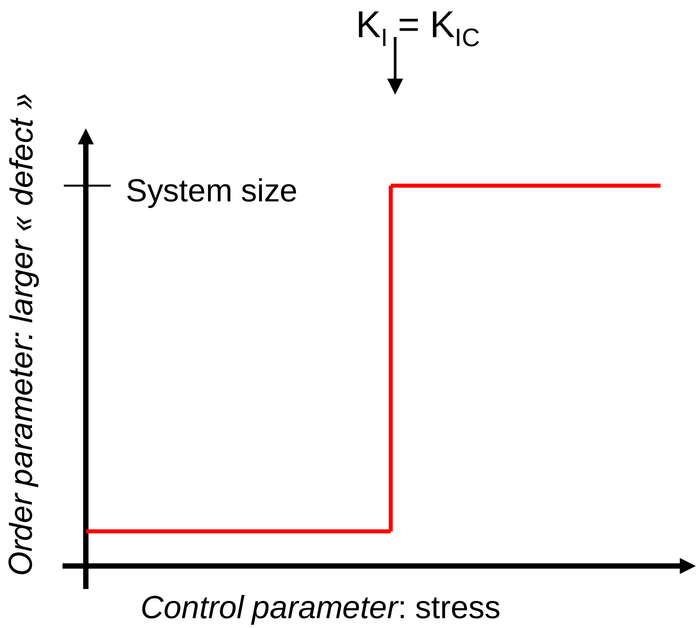
Failure under tension





Classical fracture mechanics « à la Griffith »

➢ **First-order transition, without precursory phenomenon**

$$\sigma_f \sim a^{-1/2}$$

A simple approach to failure statistics and size effects: The weakest link hypothesis

1D chain of independent links :
Failure controlled by weakest link

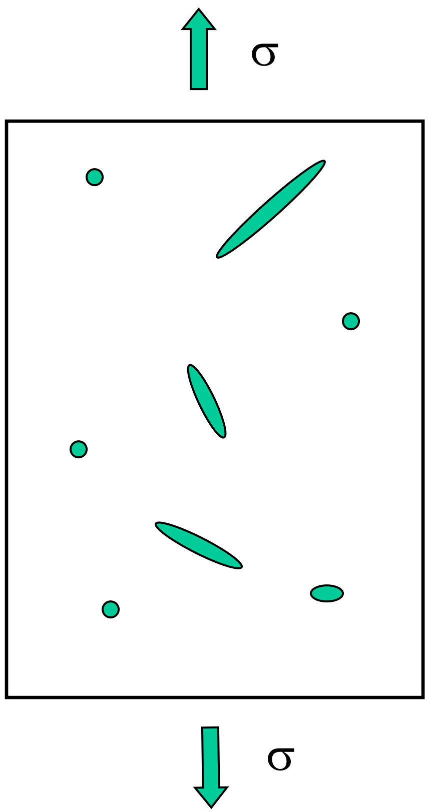
If survival probability under σ of one link : $q_o(\sigma)$
Then survival probability of a chain of *N* links :

$$[q_o(\sigma)]^N = [q_o(\sigma)]^{L/lo}$$

→ Extremal value statistics: Gumbel, Weibull, etc

Weibull statistics $$Q(\sigma, L) = \exp\left[-\left(\frac{L}{l_0}\right)\left(\frac{\sigma}{\sigma_0}\right)^m\right]$$

Weakest link approach: Extension to 3D



Weakest-link approach – Assumptions :

1. Defects do not interact with one another
2. Global failure is dictated by the activation of the largest flaw
3. The strength can be related to the critical defect size

Extremal statistics for failure strength (Weibull)

$$\langle \sigma_f \rangle \sim \delta(\sigma_f) \sim L^{-d/m}$$

Remarks :

1. Vanishing strength towards large sizes !

2. Failure statistics reduced to defect statistics; Has mechanics completely disappeared ?

Weakest link approach: How does it work ?

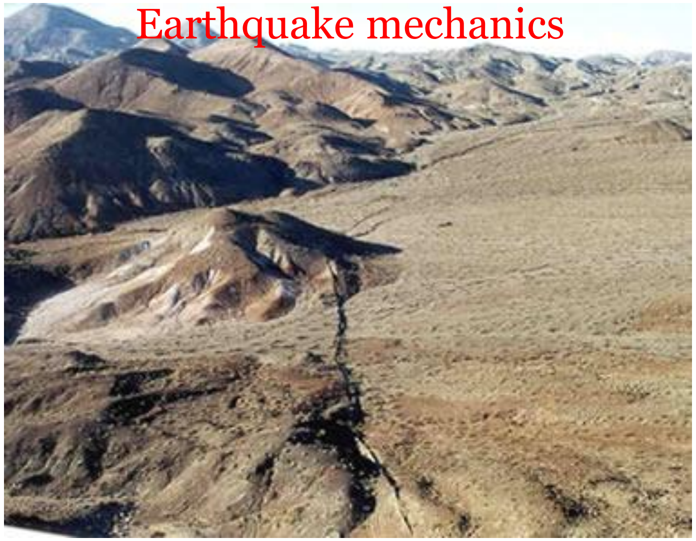
- Not too bad for brittle materials under tension (Glass, ceramics, fibers)

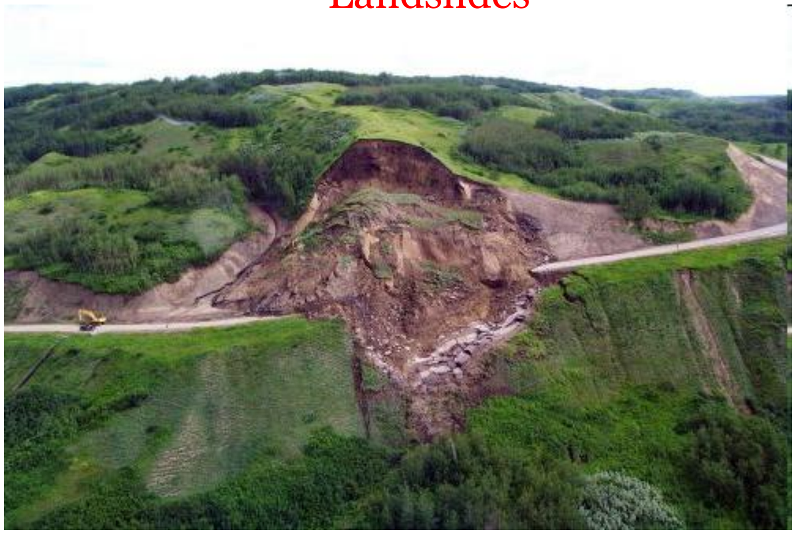
- Not too well for quasi brittle materials (concrete, etc): diffuse damage before nucleation and growth of a crack

- Here, focus on compressive strength of brittle materials

Compressive failure

Why is it so important (in Earth Science)?

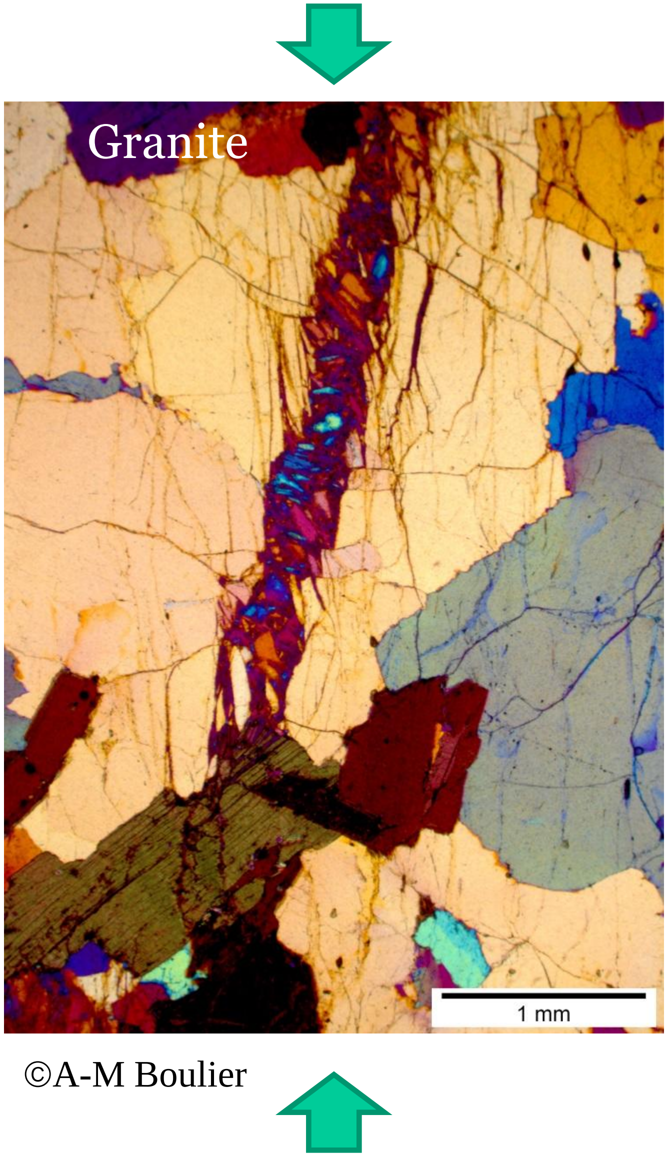




Compressive failure

Why is it so important (structural materials)?

Compressive failure



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A complex *process* :

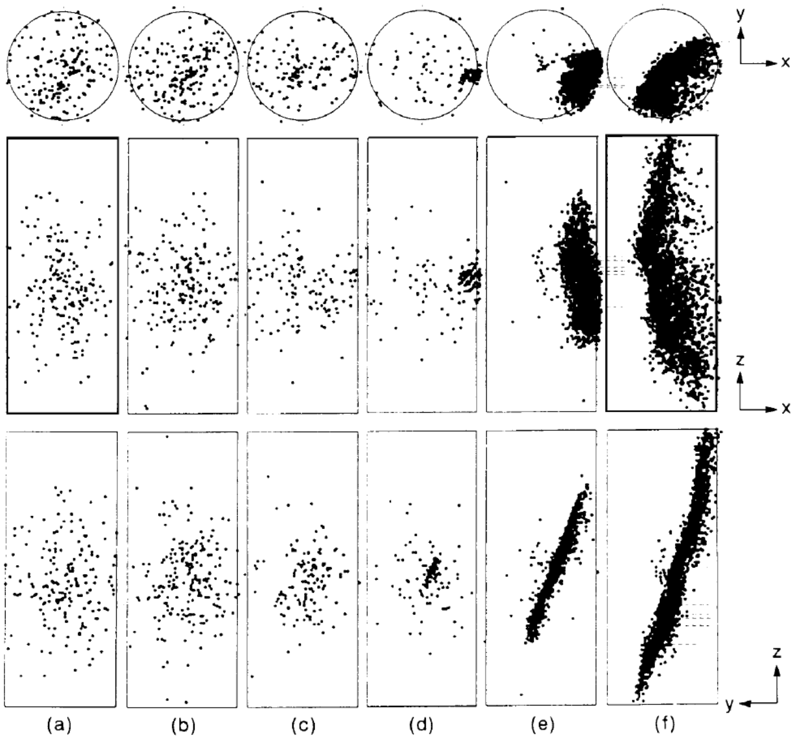
- Initiation of secondary (mode I) microcracks from frictional sliding along defects (GB, joints, cracks)

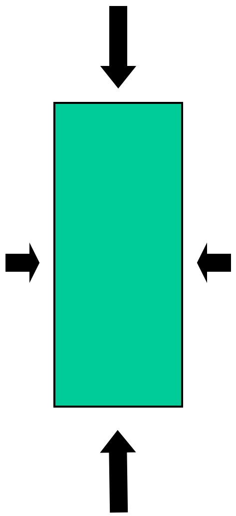
- Local softening → stress redistribution → new initiation

- Linking up along a macroscopic shear fault, with gouge formation, ..

Compressive failure

Lockner and Byerlee, 1992



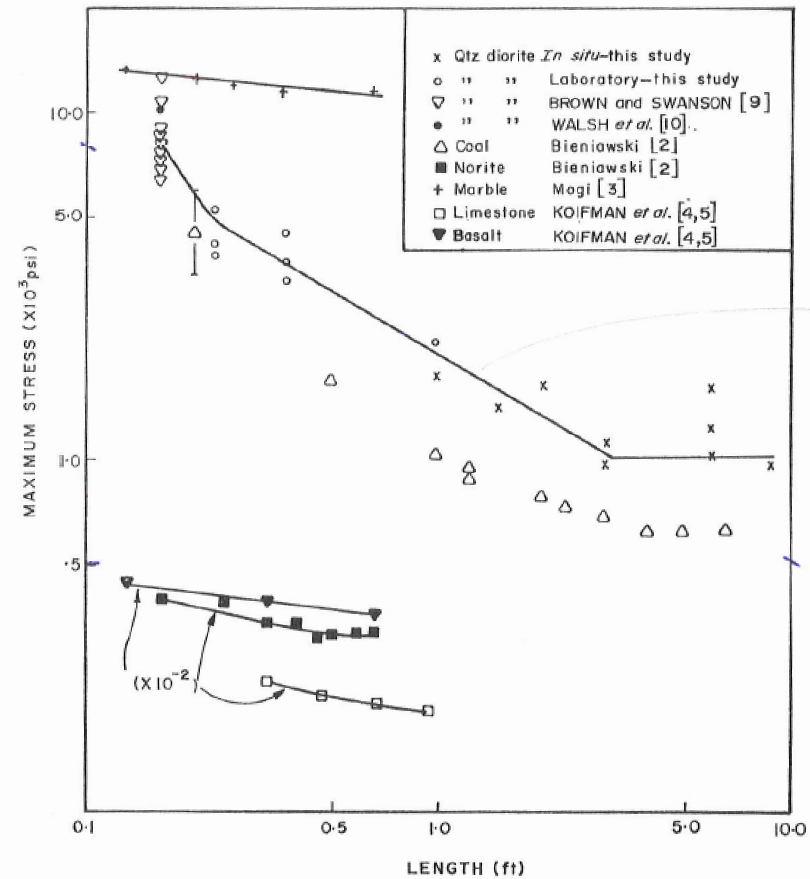


Precursory phenomena before the failure of rocks:

- **Increasing spatial clustering**
- **Increase of the size of the larger « quake »**
  (Changes in the tail of the PDF of AE energies)

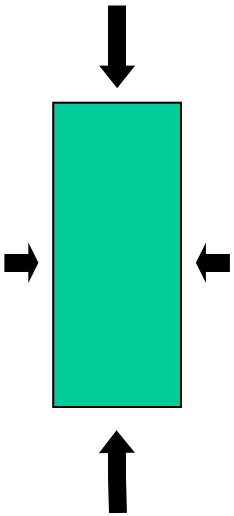
Compressive failure cohesive materials: Size effects

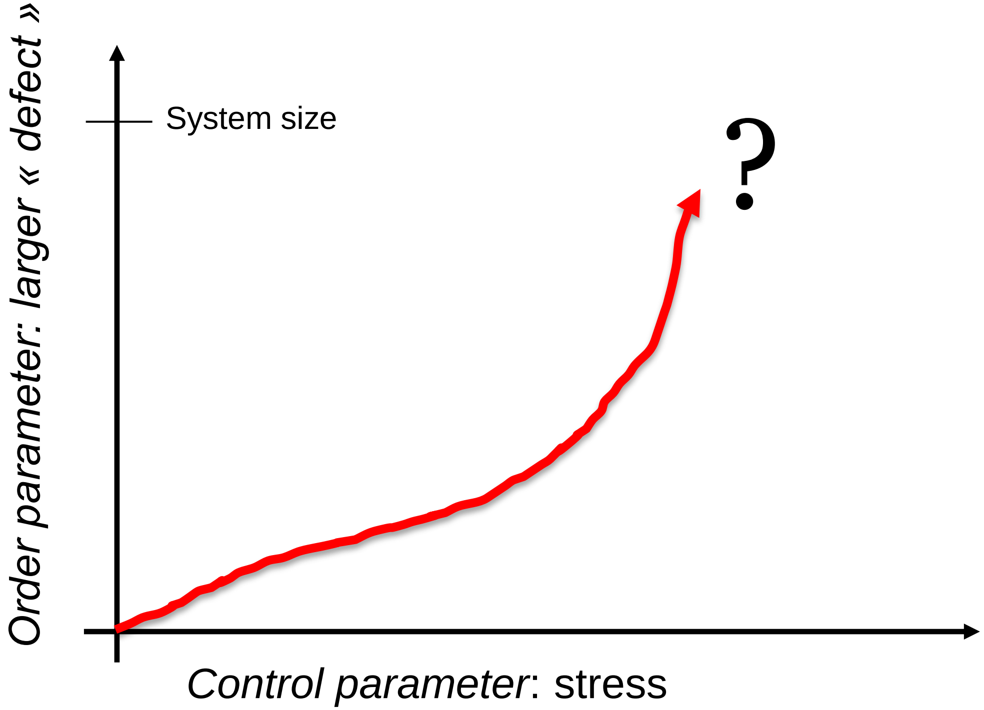
Granodiorite (Pratt et al., 1972)



- **Power law – like decay at small scales**
- **A non-vanishing asymptotic strength for *L*→+∞**
- **Decreasing variability towards large scales**

Compressive failure





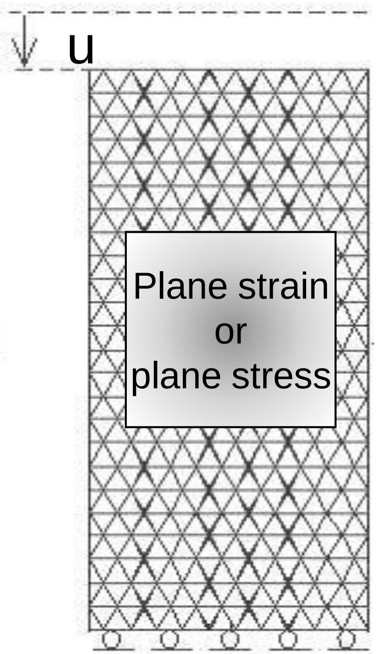
Questions:

- The route towards the failure ?
- Consequences in terms of size effects on strength ?

Model of progressive damage (Amitrano et al., 1999)

2D finite elements

- controlled strain or stress (compression)



Behavior of each element:

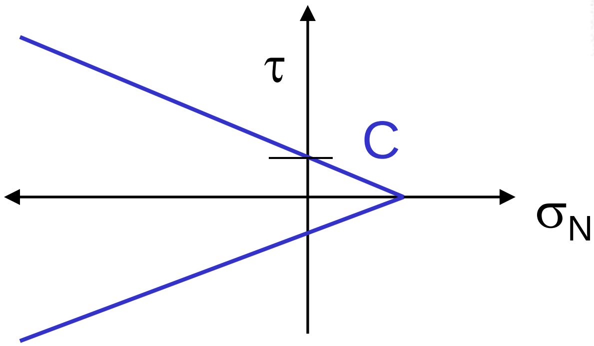
- The effective modulus decreases when the stress state reaches a local criterion : $\tilde{E} = E(1-d)$

- *d* models damage at the micro scale

- Coulomb damage criterion

$$\tau = C - \mu\sigma_N$$

μ: internal friction coefficient



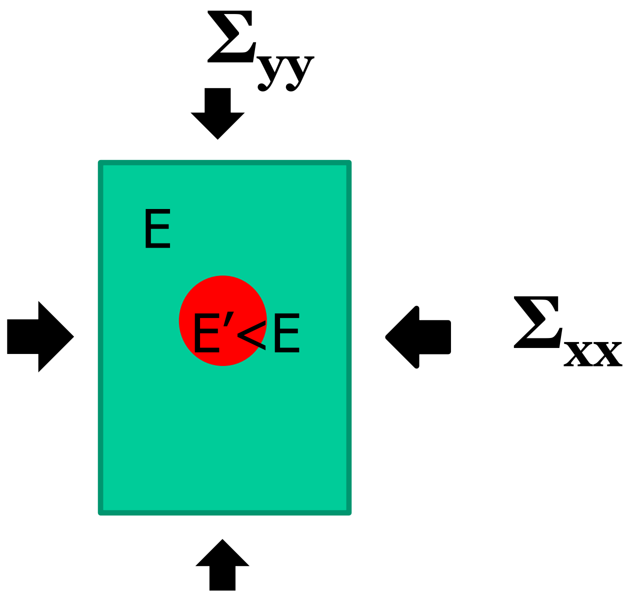
- some disorder on *C* and/or E_0

Girard et al., JSTAT, 2010

Girard et al., PRL, 2012

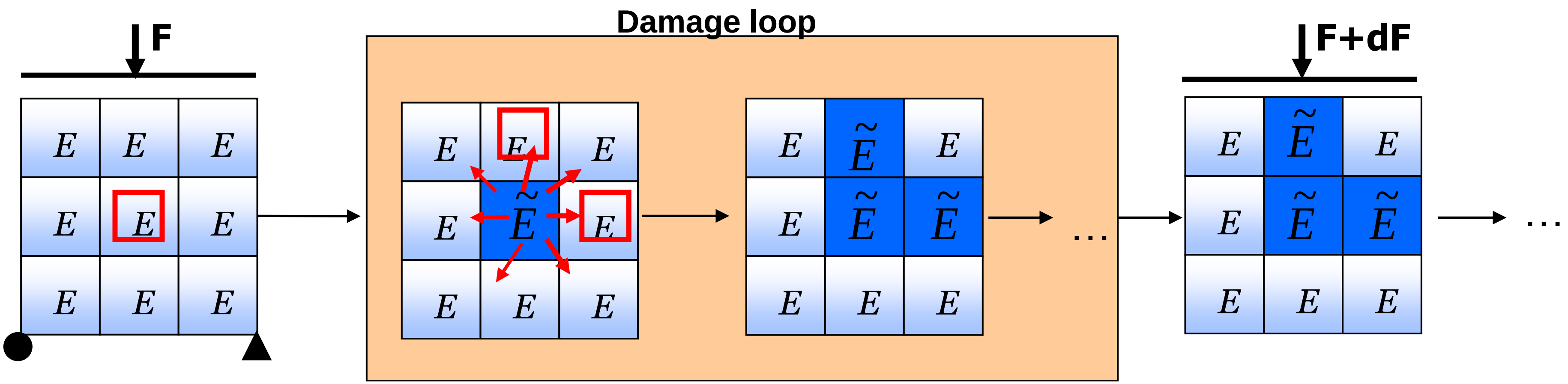
Eshelby inclusions
and quadrupolar stress interaction

Elastic inclusion



$$\sigma_{ij}(r,\theta) \approx \frac{\delta E}{E}(\Sigma_{xx} - \Sigma_{yy}) V f_{ij}(r,\theta)$$
$$+ \frac{\delta E}{E}(\Sigma_{xx} + \Sigma_{yy}) V g_{ij}(r,\theta)$$

Schematic view of damage propagation



Loading step (i) :

- Static equilibrium $\mathbf{\Sigma F=0}$
- Element stress state

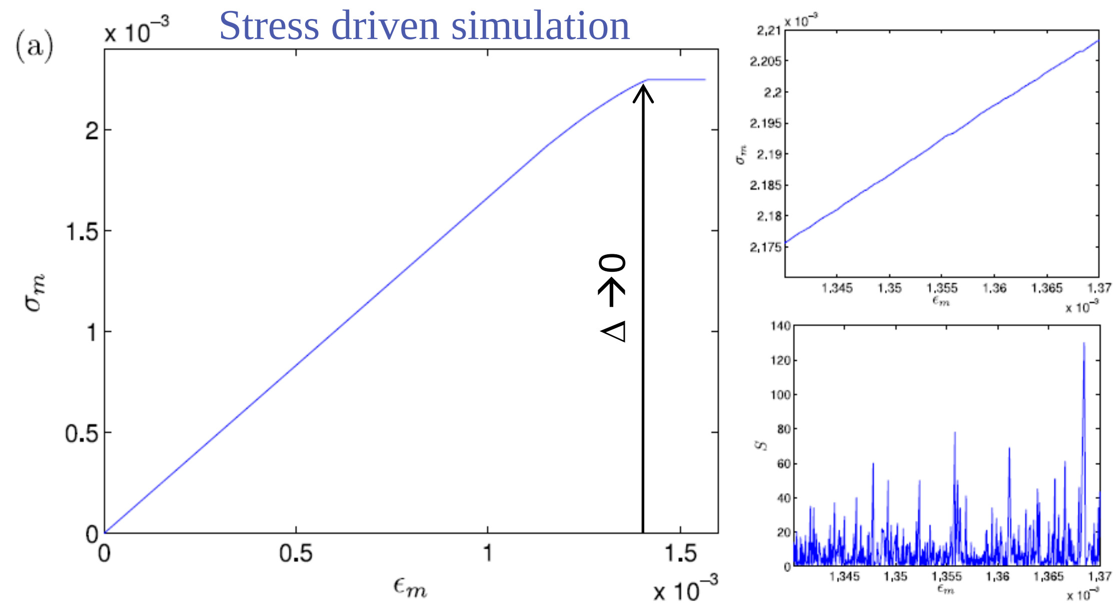
- Static-equilibrium:
  Stress relaxation locally
  $\rightarrow$ **Stress redistribution**

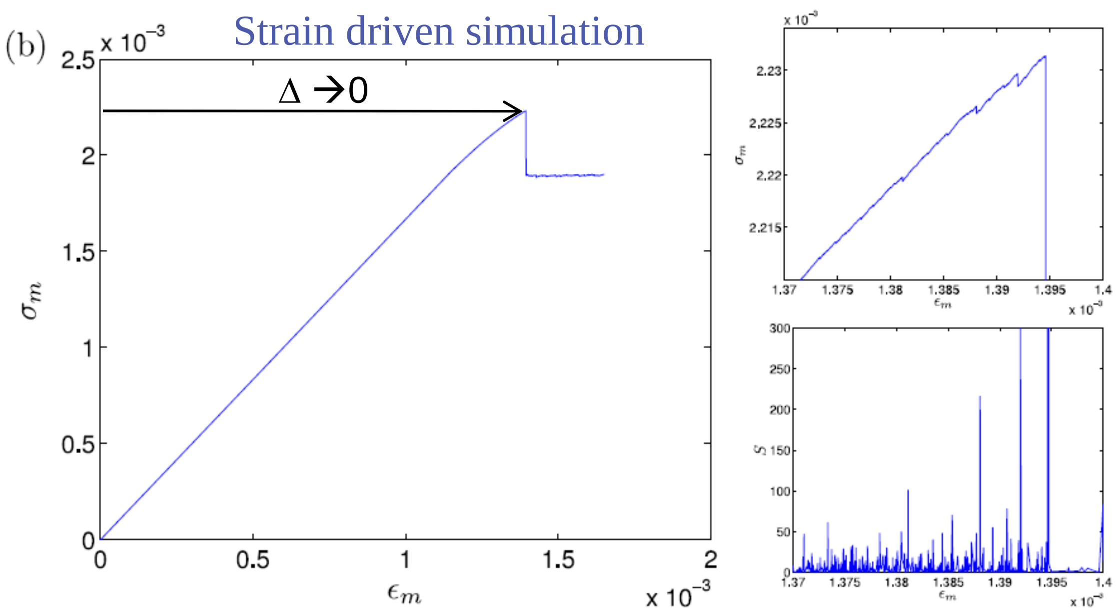
- The avalanche stops when the criterion is not fulfilled

Loading step (i+1)

Avalanche size: number of damage events during a loading step

Loading curves





Framework of critical phase transitions theory

Δ: control parameter

$$\Delta = \frac{\sigma_c - \sigma}{\sigma_c}$$

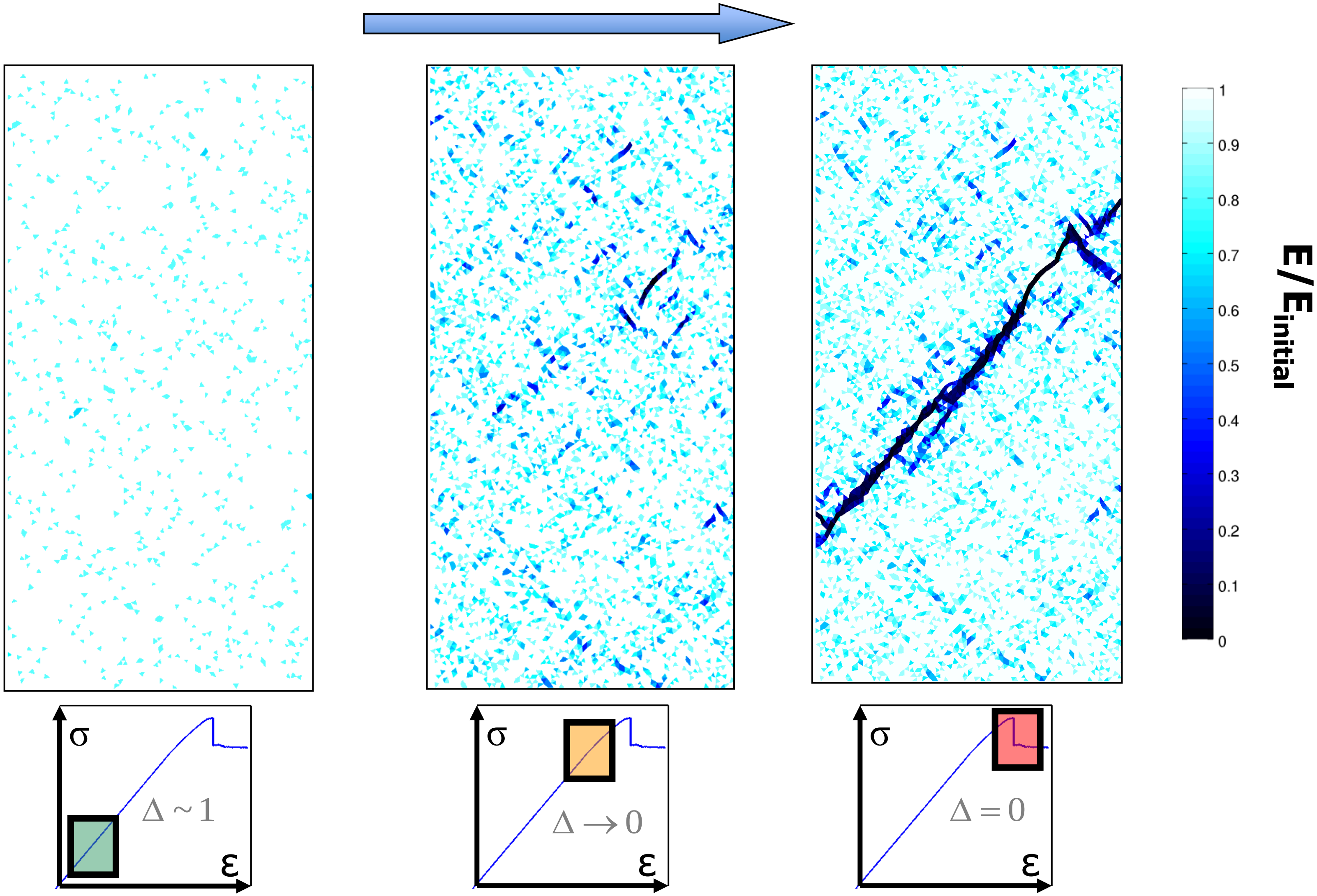
$$\Delta = \frac{\varepsilon_c - \varepsilon}{\varepsilon_c}$$

Critical point: $\Delta \rightarrow 0$

Girard et al, JSTAT 2010

Damage localisation

➢ **Evolution of the elastic stiffness: keeps memory of damage events**

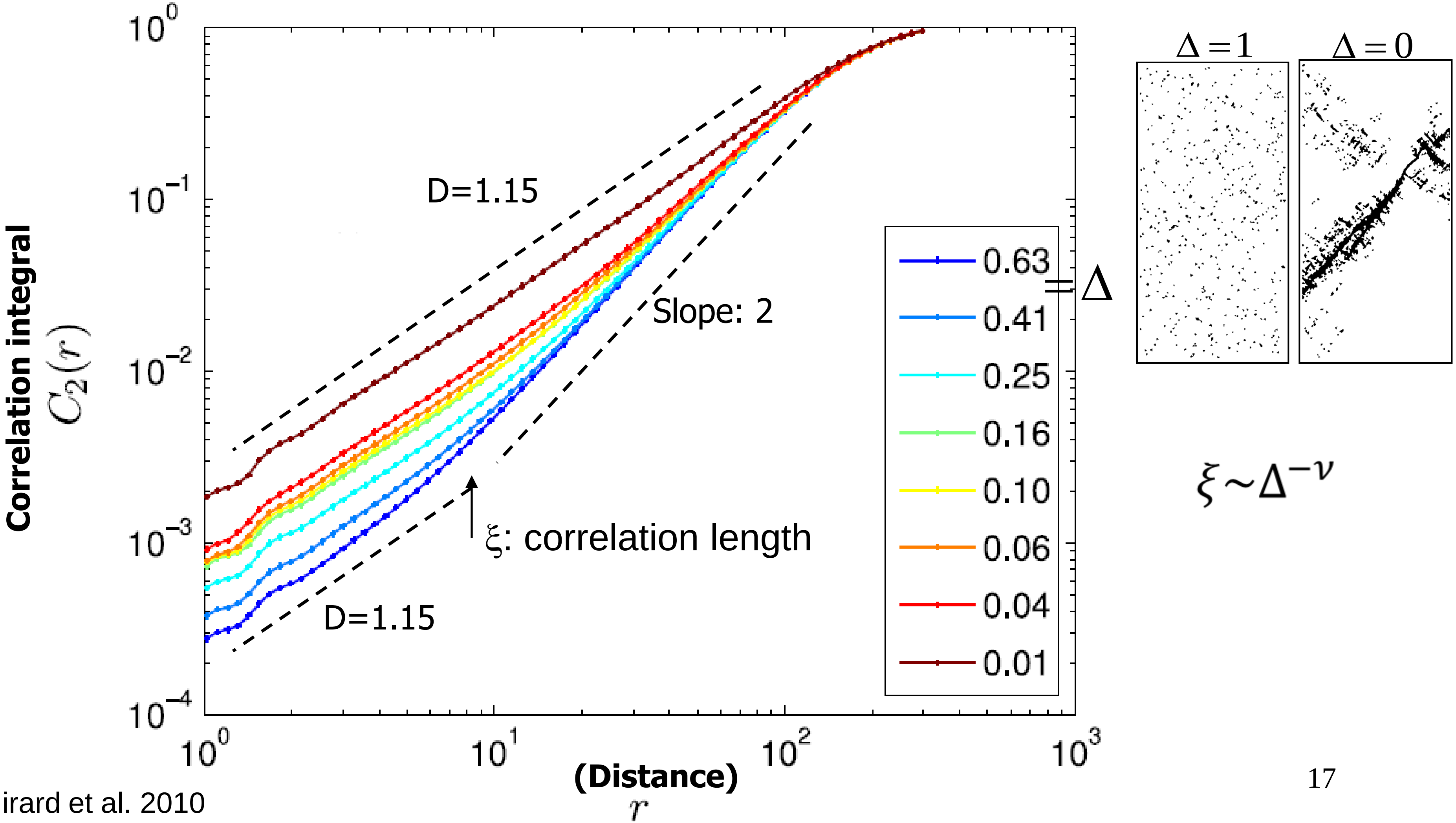


Spatial clustering

➢ **Correlation between damage events** as a function of their spatial distance

$C_2(r) \sim r^D$ **for $r < \xi$**

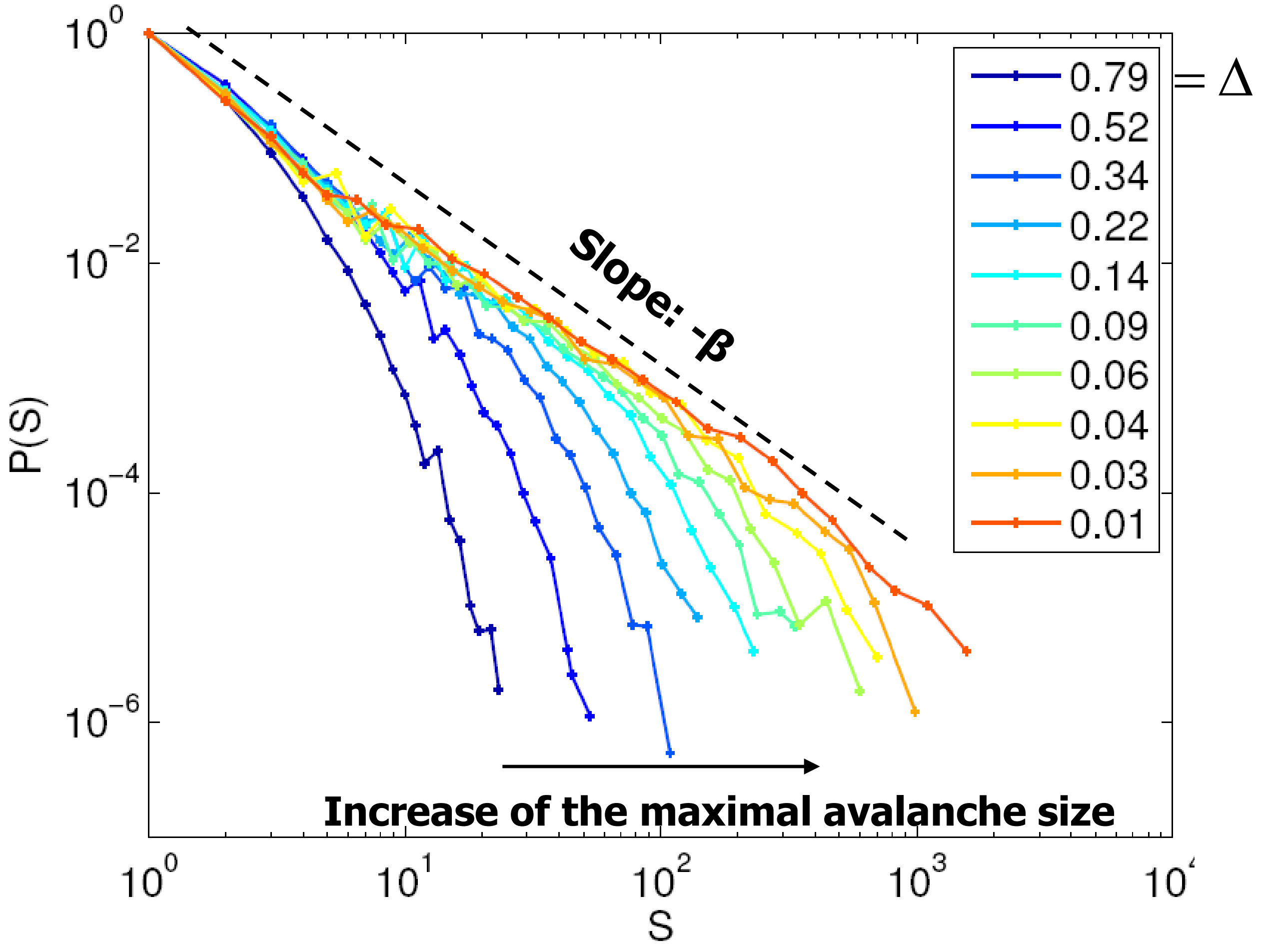
Correlation dimension D : dimensionality of clusters
Correlation length ξ : characteristic size of clusters

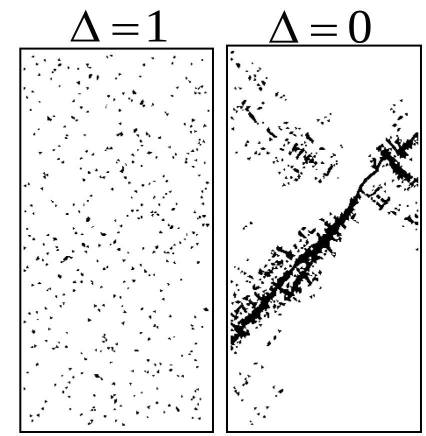


Girard et al. 2010

Damage avalanches

- **Avalanche size: number of damage events during a loading step**

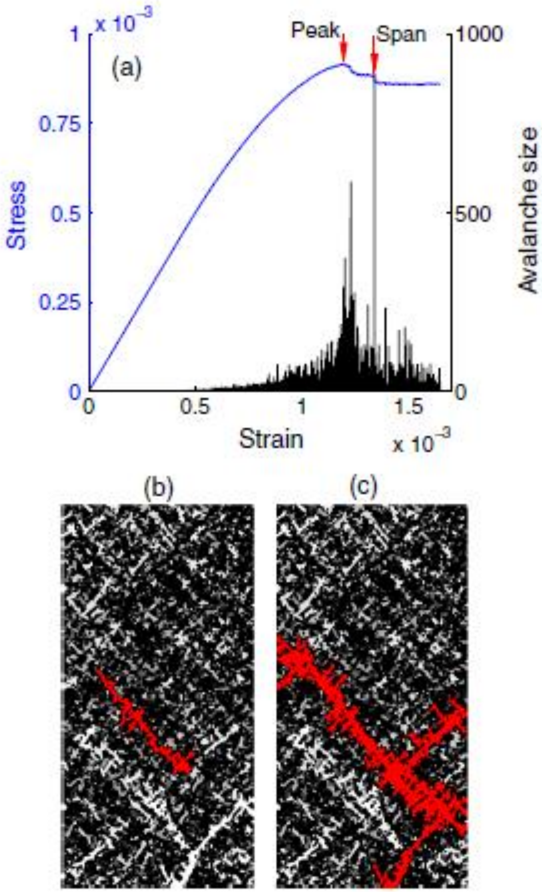




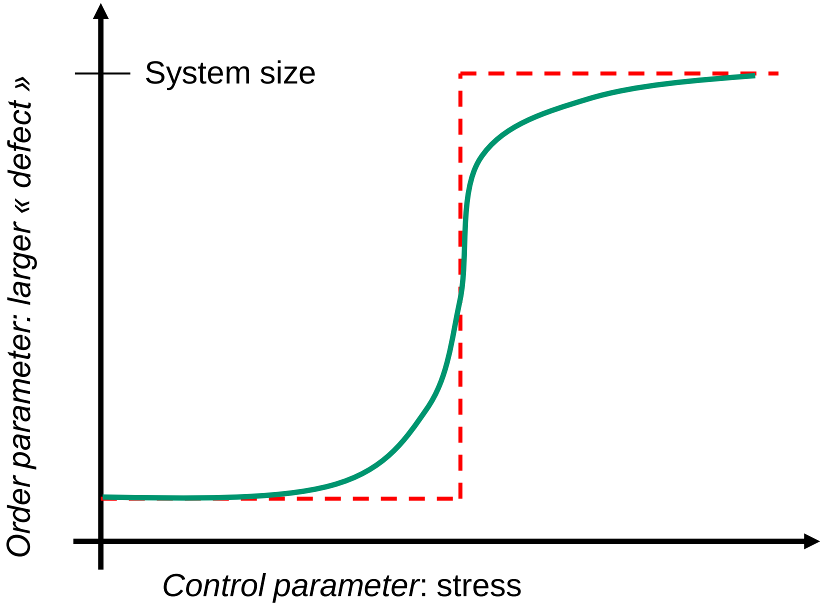
$$P(S) \sim S^{-\beta} \exp\left(-\frac{S}{S^*}\right)$$

For an infinitely large system: $S^* \sim \Delta^{-\gamma}$

Evolution of the order parameter



Π: size of the largest damage "cluster"

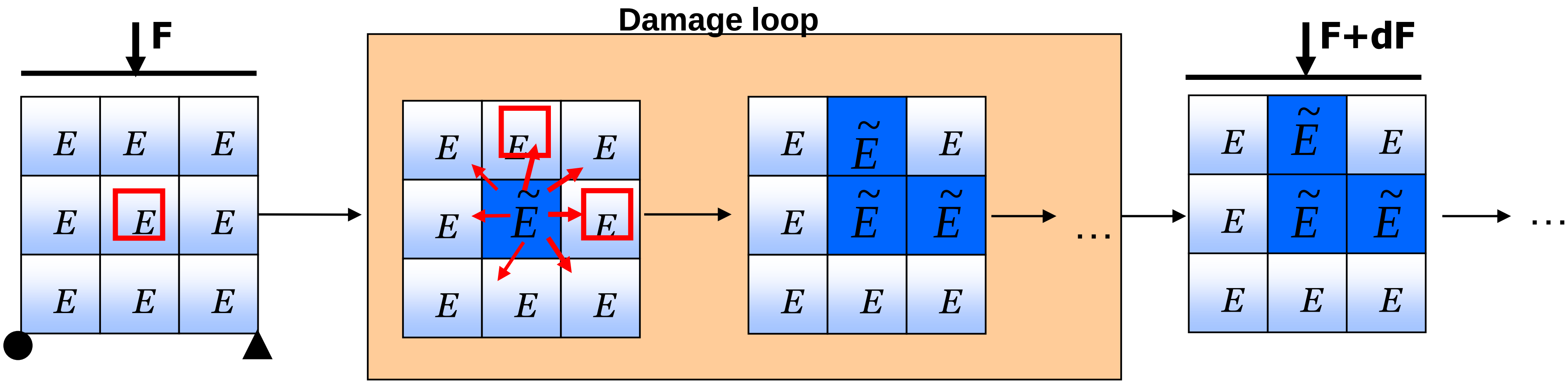


Girard et al., PRL, 2012

Compressive failure can be interpreted as a critical transition

→ **Consequences in terms of size effects on strength ?**

Mapping onto the depinning transition



Loading step (i) :
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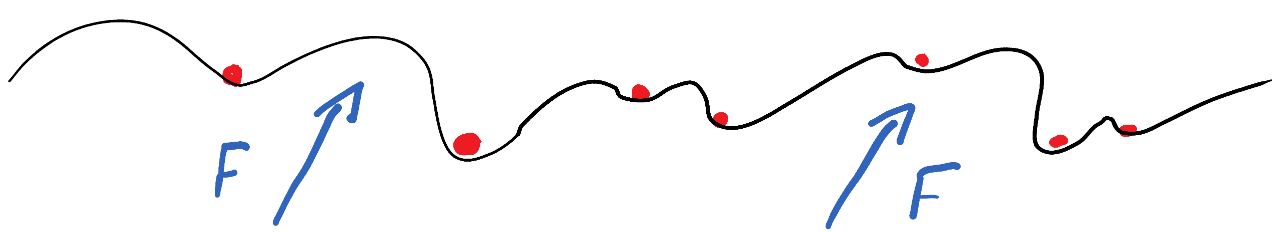
- Static-equilibrium:
  Stress relaxation locally
  $\rightarrow$ **Stress redistribution**

- The avalanche stops when the criterion is not fulfilled

Loading step (i+1)

Avalanche size: number of damage events during a loading step

➢ **Analogy with the depinning of an elastic manifold**



Mapping onto the depinning transition

Weiss et al, PNAS, 111, 6231 (2014)

Evolution of the damage field: $$\frac{\partial D}{\partial t} = H\left(\sigma_{ext+el}^{Coulomb}\left(\sigma_{ext}(t), \{D(t)\}\right) - \tau_c(r,t)\right)$$

Disorder on cohesion

where the coulomb stress $$\sigma^{Coulomb} = |\tau| - \mu\sigma_N$$

is calculated from $$\boldsymbol{\sigma}(r,t) = \boldsymbol{\sigma}_{\boldsymbol{ext}}(t) + \boldsymbol{\sigma}_{\boldsymbol{el}}(\{D(t)\})$$

External stress

Contribution of elastic interactions

Mapping to the depinning transition

Finite size scaling of the threshold force

$$\begin{cases} \delta(f_c) = A L^{-1/\nu_{FS}} \\ \langle f_c \rangle = f_c^{(\infty)} + B L^{-1/\nu_{FS}} \end{cases}$$

↓

Finite size scaling of compressive strength

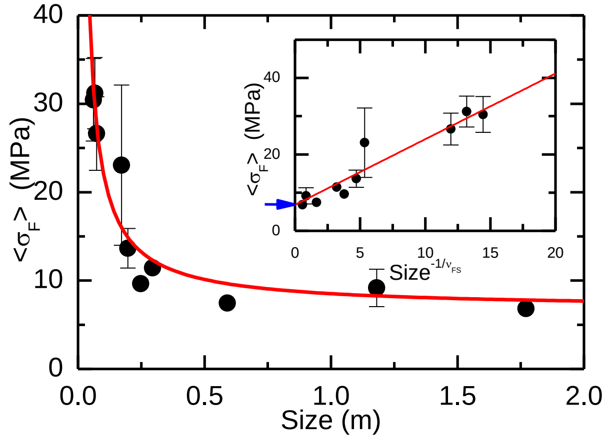
$$\begin{cases} \delta(\sigma_f) = A L^{-1/\nu_{FS}} \\ \langle \sigma_f \rangle = \sigma_f^{(\infty)} + B L^{-1/\nu_{FS}} \end{cases}$$

Might explain:

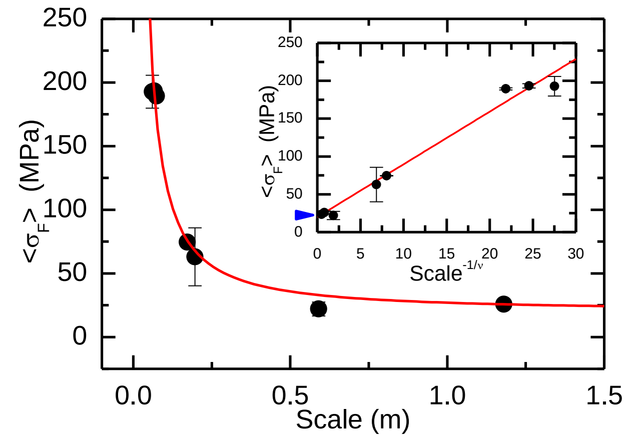
- **Power law – like decay at small scales**
- **A non-vanishing asymptotic strength for $L \to +\infty$**
- **Decreasing variability towards large scales**

Comeback to experiments

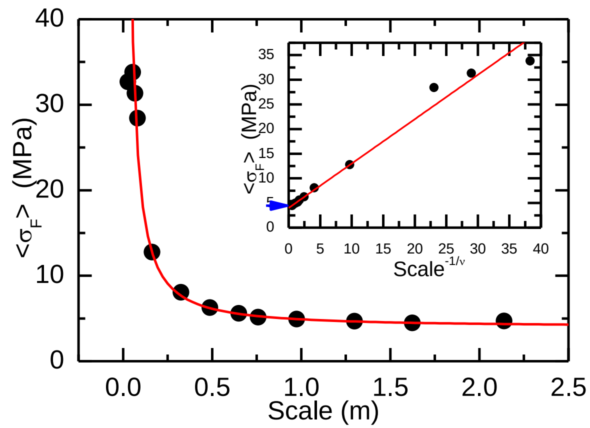
$$\langle \sigma_f \rangle = \sigma_f^{(\infty)} + B L^{-1/\nu_{FS}}$$



Quartz (Pratt et al., 1972): ν_{FS} = 1.05, $\sigma^{(\infty)}$=6.8 MPa



Granodiorite (Pratt et al., 1972): ν_{FS} = 0.85, $\sigma^{(\infty)}$=20 MPa



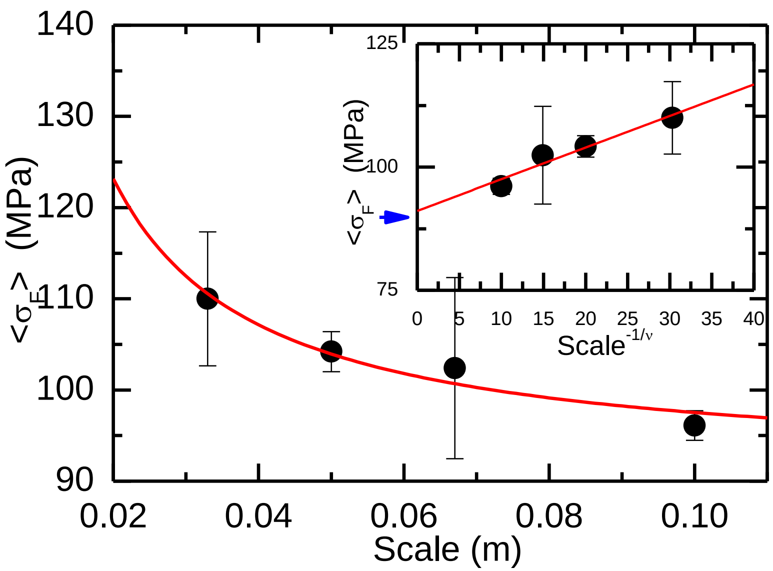
Coal (Bieniawski, 1968): ν_{FS} = 0.8, $\sigma^{(\infty)}$=4 MPa

$\nu_{FS} \approx 1.0$

Comeback to experiments

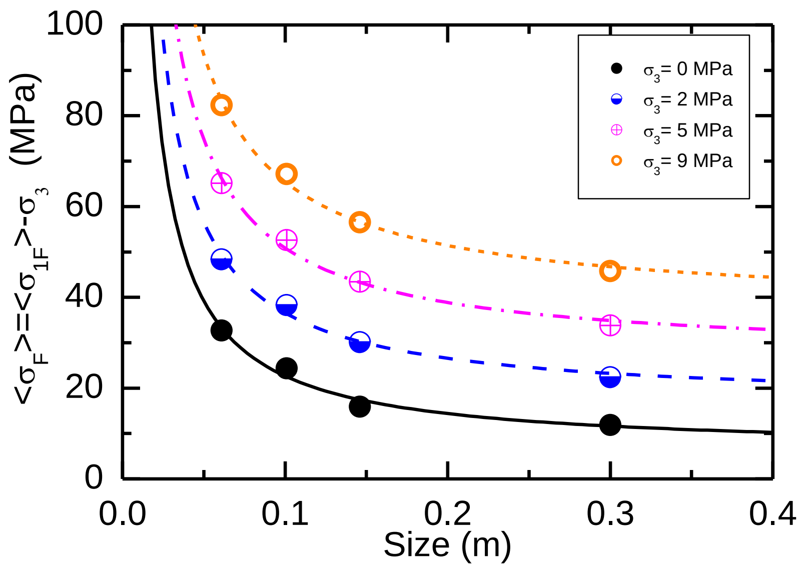
$$\langle \sigma_f \rangle = \sigma_f^{(\infty)} + B L^{-1/\nu_{FS}}$$

Structural materials (concrete)



HP Concrete (del Viso, 2008): ν_{FS} fixed at 1.0, $\sigma^{(\infty)}$=91 MPa

Effect of confining pressure



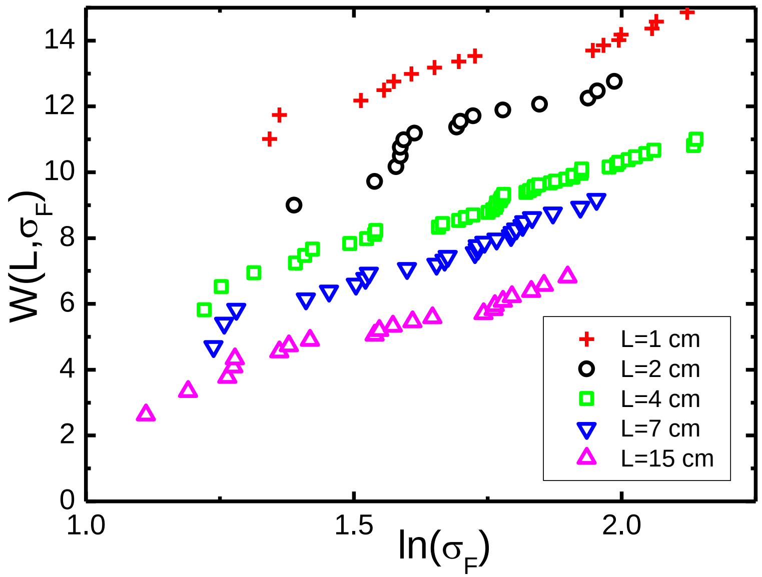
Coal (Hoek and Brown, 1997): ν_{FS} fixed at 1.0

$\sigma^{(\infty)}$ increases with increasing confinement

Probability density function

Ice samples (Kuehn et al., 1992)

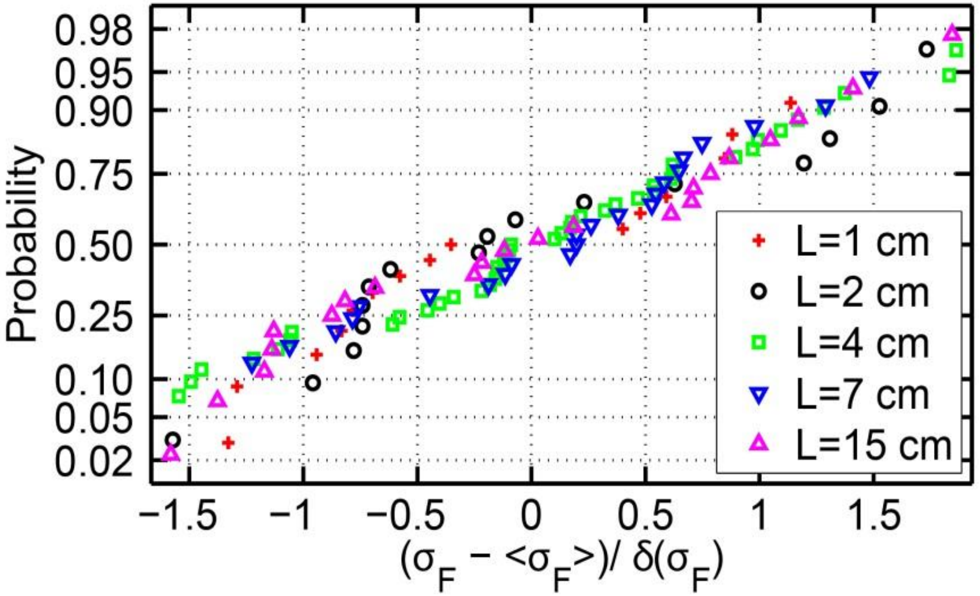
$$W(L,\sigma_f) = ln\left(\frac{-ln(1 - P_F(\sigma))}{L^3}\right)$$



Extreme value statistics (Weibull or Gumbel) are irrelevant

Probability density function

Ice samples (Kuehn et al., 1992)

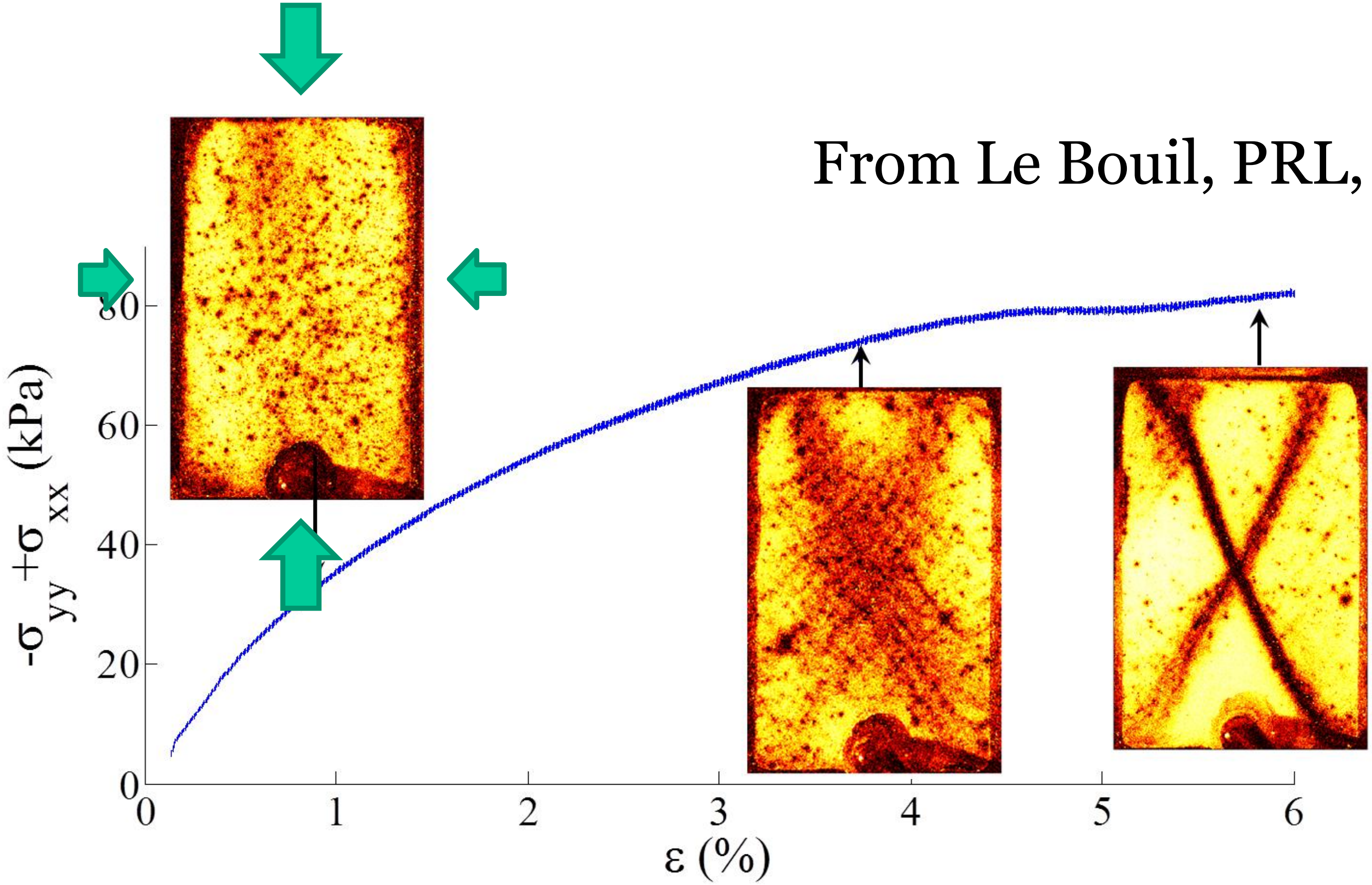


Gaussian distribution !

$$P_F(\sigma, L) = \frac{1}{2}\left[1 + erf\left(\frac{\sigma - \sigma_\infty - B\,L^{-1/\nu_{FS}}}{\sqrt{2}\,A\,L^{-1/\nu_{FS}}}\right)\right]$$

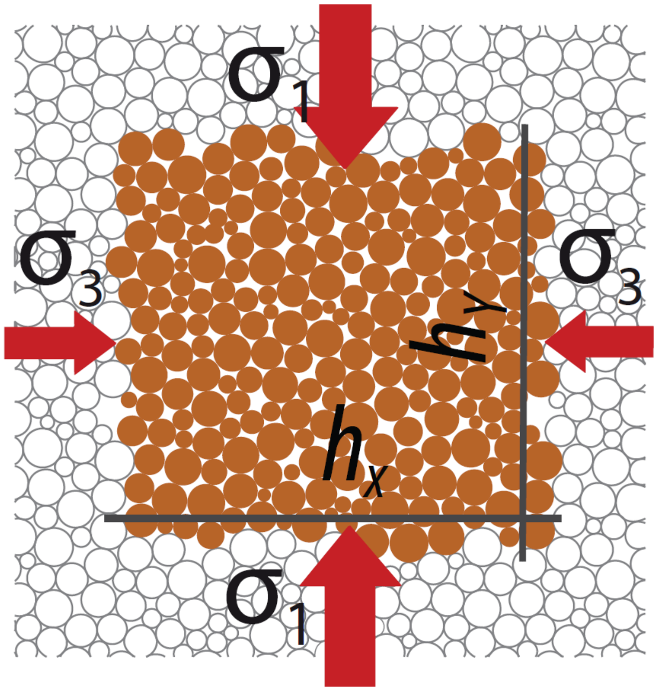
Progressive strain localization in compressed granular media

From Le Bouil, PRL, 2014



Numerical Simulations: Molecular Dynamics

DEM code developed by Gael Combe (3S-R laboratory)



Sample properties

Polydispersity : $D_{max} = 3D_{min}$

Packing properties
- Density $\phi_{ini} = 0.85$
- Coordination number $Z^*_{ini} = 4$

Microscopic Laws

Normal force: Elastic contact with viscous damping.

Tangential Force: Perfect Elasto-plastic incremental force

Coulombic friction with $\mu_{micro} = 1$.

Setup configuration

Strain and Stress Controlled Biaxial tests

2 D Periodic Boundary Conditions

Samples from **100** to **45000** grains.

Dimensionless parameters

Grain rigidity

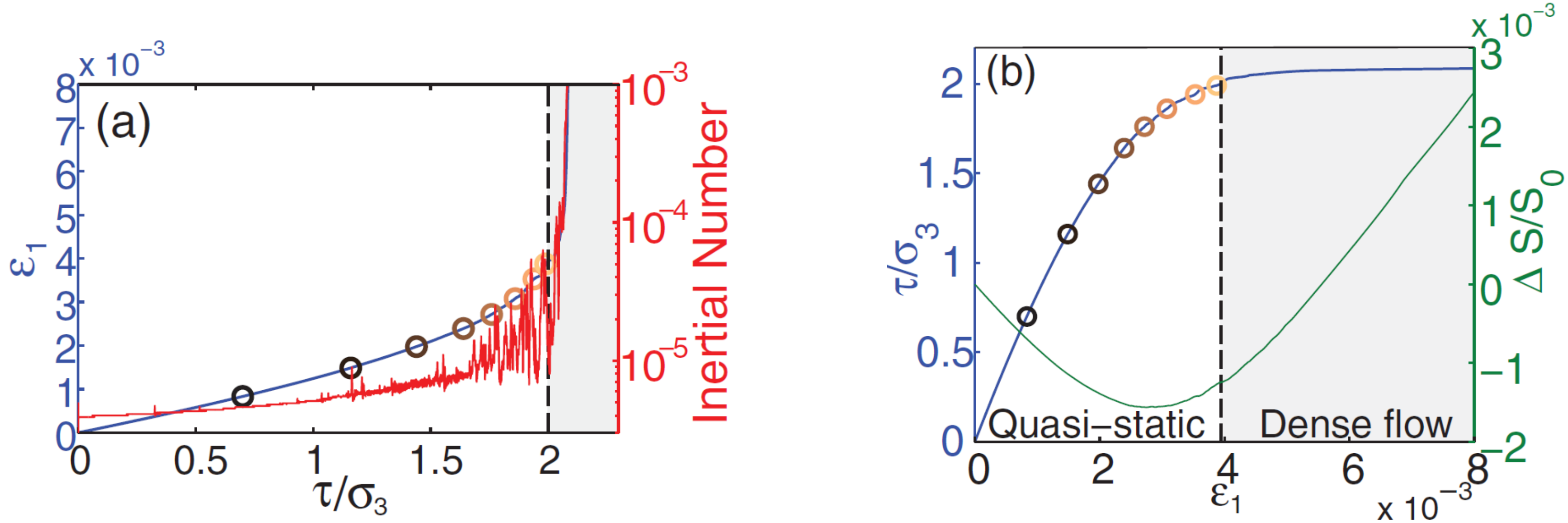
$\kappa = K_N/P = 1000$

Inertial Number

$I = \dot{\epsilon}\sqrt{m/Pd}$

Macroscopic behaviour

Stress control tests $\dot{\sigma}_1 = constant$



A transition from quasi-static deformation to dense flow

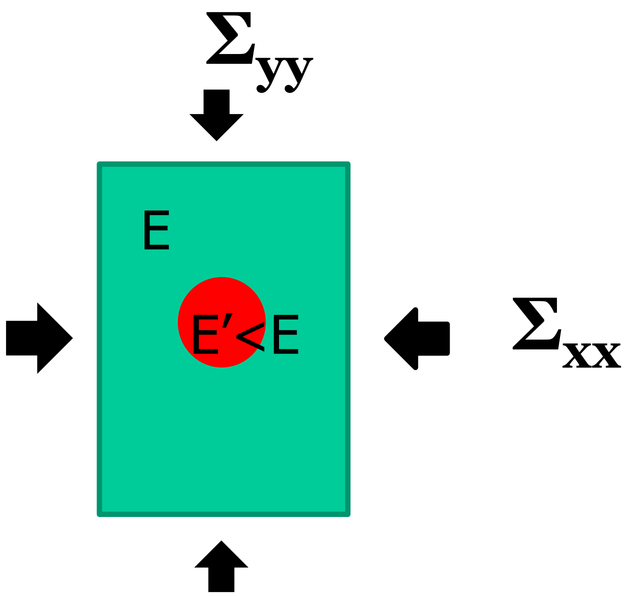
Divergence of the correlation length at the onset of dense flow

$$\xi \sim \Delta^{-\nu}$$

Gimbert et al, EPL, 104, 46001 (2013)

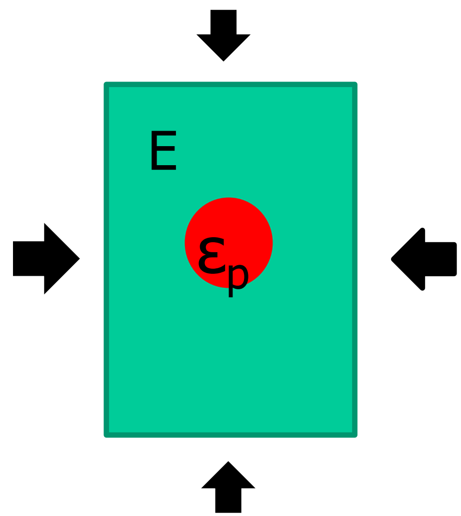
Eshelby inclusions and quadrupolar stress interaction

Elastic inclusion



$$\sigma_{ij}(r,\theta) \approx \frac{\delta E}{E}(\Sigma_{xx}-\Sigma_{yy})Vf_{ij}(r,\theta) + \frac{\delta E}{E}(\Sigma_{xx}+\Sigma_{yy})Vg_{ij}(r,\theta)$$

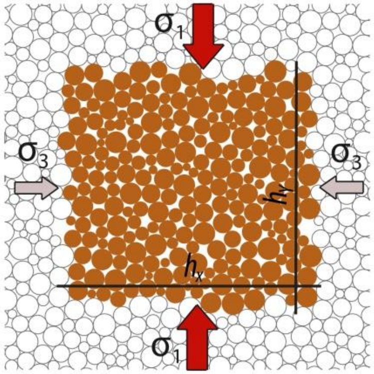
Plastic inclusion

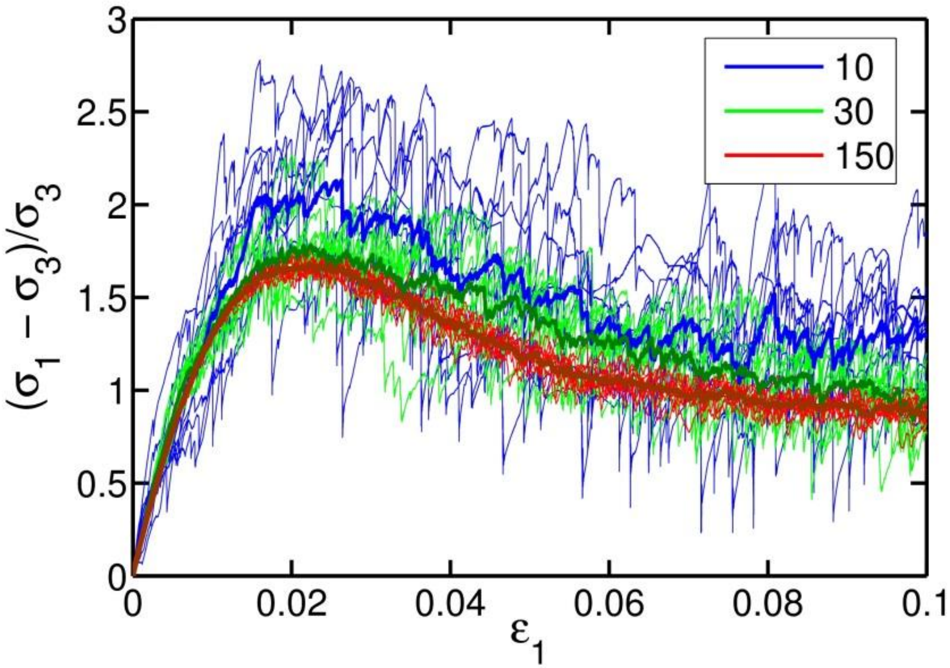


$$\sigma_{ij}(r,\theta) \sim EV\varepsilon_p f_{ij}(r,\theta) + EV\varepsilon_p g_{ij}(r,\theta)$$

Size effects on strength

Weiss et al, PNAS, 111, 6231 (2014)



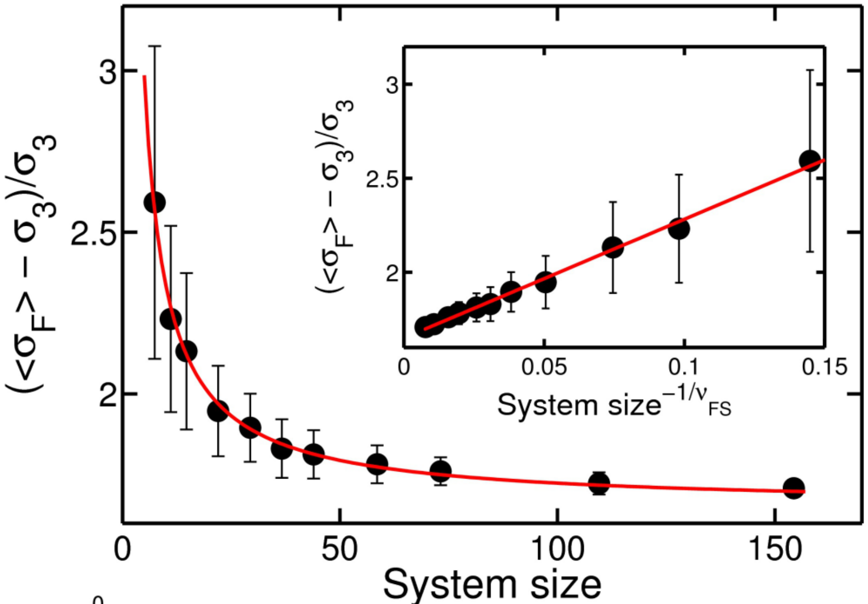


Size effects on strength

Weiss et al, PNAS, 111, 6231 (2014)

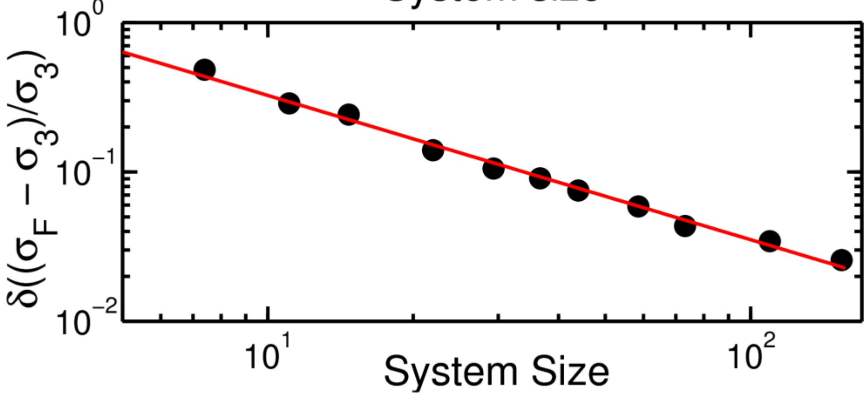
Mean strength

$$\langle \sigma_f \rangle = \sigma_f^{(\infty)} + B L^{-1/\nu_{FS}}$$

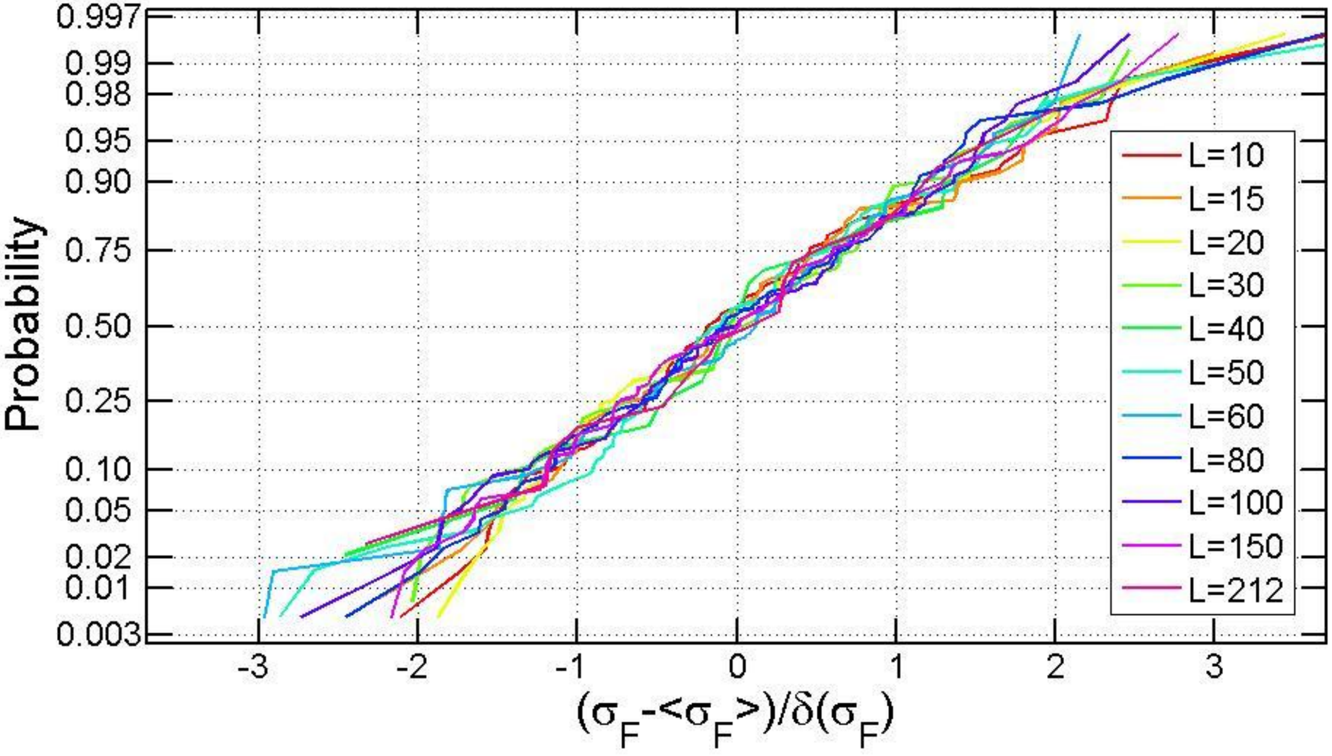


Variability

$$\delta(\sigma_f) = A L^{-1/\nu_{FS}}$$



Probability density function



Gaussian distribution !

Consequences / Conclusion

- Compressive failure *(of cohesive as well as granular media)* as a critical phase transition

- Practical consequences in termes of size effects

  → a correct estimate of compressive strength can generally be obtained from laboratory tests

  → variability is expected to decrease significantly towards large scales