The Status Quo of Self-Organised Criticality

History, Models, Universality Classes, Tools

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Outline

1. SOC: Past and Present
2. Universality Classes
3. Theory of SOC
4. Summary: Any Answers?
Prelude: The physics of fractals

Question: Where does scale invariant behaviour in nature come from?

Answer: Due to a phase transition, self-organised to the critical point.
Prelude: The physics of fractals

- Anderson, 1972: *More is different*
  Correlation, cooperation, emergence
- 1/f noise “everywhere” (van der Ziel, 1950; Dutta and Horn, 1981)
- Kadanoff, 1986: *Fractals: Where’s the Physics?*
The BTW Model

The sandpile model:

- Simple (randomly driven) cellular automaton → avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng et al.)

- The physics of fractals.
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Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.
Why is SOC important?

SOC today: Slowly driven, avalanching (intermittent) systems with non-linear interactions, that display non-trivial power-law correlations (cutoff by the system size) as known from ordinary critical phenomena, but with internal, self-organised, rather than external tuning of a control parameter (to a non-trivial value).

Emergence!

- Explanation of emergent,
- ... cooperative,
- ... long time and length scale
- ... phenomena,
- ... as signalled by power laws.
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Universality!

- Understanding and classifying natural phenomena
- ... using *Micky Mouse Models*
- ... on a small scale (in the lab or on the computer).
- (Triggering critical points?)
- But: Where is the evidence for scale invariance in nature (dirty power laws)?
Experiments:
Granular media, superconductors, rain . . .

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Ricepiles experiments by Frette et al. (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

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More models

- Initial intention for more models: Expand BTW universality class.
- Later: Provide more evidence for SOC as a whole.
- More models...
More models

- Zhang Model (1989) [scaling questioned]
- Dhar-Ramaswamy Model (1989) [solved, directed]
- Forest Fire Model (1990, 1992) [no proper scaling]
- Manna Model (1991) [solid!]
- Olami-Feder-Christensen Model (1992) [scaling questioned, \( \alpha \approx 0.05 \) (localisation), \( \alpha = 0.22 \) (jump)]
- Bak-Sneppen Model (1993) [scaling questioned]
- Zaitsev Model (1992)
- Sneppen Model (1992)
- Oslo Model (1996) [solid!]
The Bak-Chen-Tang Forest Fire Model

- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate $p$.
- Occasional re-lighting.
  Deterministic pattern, scale given by $1/p$. 
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The Drossel-Schwabl Forest Fire Model

- Fires **instantaneous**, explicit lightning mechanism with θ trees grown between two lightning attempts.
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- Fires *instantaneous*, explicit lightning mechanism with $\theta$ trees grown between two lightnings attempts.
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Lack of scaling

- Finite size not the only scale.
- Scale invariance possible only in the limit of $\theta \to \infty$.
- Lower cutoff moves as well.
Manna Model

Manna Model (1991)
- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.
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Collapse with Oslo

The Manna Model is in the same universality class as the Oslo model.
Manna on different lattices

One and two dimensions

(a) The simple chain. $L = 10, N = 10$.

(b) The rope ladder. $L = 10, N = 20$.

(c) The next nearest neighbour (nnn) chain. $L = 10, N = 20$.

(d) The Futatsubishi lattice. $L = 7, N = 22$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.
Manna on different lattices
One and two dimensions

(a) The square lattice. \( L_x = L_y = 6, N = 36 \).
(b) The jagged lattice. \( L_x = 4, L_y = 9, N = 36 \).
(c) The triangular lattice. \( L_x = 5, L_y = 7, N = 36 \).
(d) The Kagomé lattice. \( L_x = 10, L_y = 4, N = 40 \).
(e) The Archimedes lattice. \( L_x = 8, L_y = 4, N = 32 \).
(f) The non-crossing (nc) diagonal square lattice. \( L_x = L_y = 5, N = 25 \).
(g) The honeycomb lattice. \( L_x = 9, L_y = 4, N = 36 \).
(h) The Mitsubishi lattice. \( L_x = 5, L_y = 7, N = 35 \).

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### Manna on different lattices

#### One and two dimensions

<table>
<thead>
<tr>
<th>lattice</th>
<th>$d$</th>
<th>$D$</th>
<th>$\tau$</th>
<th>$z$</th>
<th>$\alpha$</th>
<th>$D_a$</th>
<th>$\tau_a$</th>
<th>$\mu^{(s)}_1$</th>
<th>$-\Sigma_\alpha$</th>
<th>$-\Sigma_\lambda$</th>
<th>$-\Sigma_\Sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>simple chain</td>
<td>1</td>
<td>2.27(2)</td>
<td>1.117(8)</td>
<td>1.450(12)</td>
<td>1.19(2)</td>
<td>0.998(4)</td>
<td>1.260(13)</td>
<td>2.000(4)</td>
<td>0.27(2)</td>
<td>0.27(3)</td>
<td>0.259(14)</td>
</tr>
<tr>
<td>rope ladder</td>
<td>1</td>
<td>2.24(2)</td>
<td>1.108(9)</td>
<td>1.44(2)</td>
<td>1.18(3)</td>
<td>0.998(7)</td>
<td>1.26(2)</td>
<td>1.989(5)</td>
<td>0.24(2)</td>
<td>0.26(5)</td>
<td>0.26(2)</td>
</tr>
<tr>
<td>nnn chain</td>
<td>1</td>
<td>2.33(11)</td>
<td>1.14(4)</td>
<td>1.48(11)</td>
<td>1.22(14)</td>
<td>0.997(15)</td>
<td>1.27(5)</td>
<td>1.991(11)</td>
<td>0.33(11)</td>
<td>0.3(2)</td>
<td>0.27(5)</td>
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<tr>
<td>Futatsubishi</td>
<td>1</td>
<td>2.24(3)</td>
<td>1.105(14)</td>
<td>1.43(3)</td>
<td>1.16(6)</td>
<td>0.999(15)</td>
<td>1.24(5)</td>
<td>2.008(11)</td>
<td>0.24(3)</td>
<td>0.23(9)</td>
<td>0.24(5)</td>
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<tr>
<td>square</td>
<td>2</td>
<td>2.748(13)</td>
<td>1.272(3)</td>
<td>1.52(2)</td>
<td>1.48(2)</td>
<td>1.992(8)</td>
<td>1.380(8)</td>
<td>1.9975(11)</td>
<td>0.748(13)</td>
<td>0.73(4)</td>
<td>0.76(2)</td>
</tr>
<tr>
<td>jagged</td>
<td>2</td>
<td>2.764(15)</td>
<td>1.276(4)</td>
<td>1.54(2)</td>
<td>1.49(3)</td>
<td>1.995(7)</td>
<td>1.384(8)</td>
<td>2.0007(12)</td>
<td>0.764(15)</td>
<td>0.76(5)</td>
<td>0.77(2)</td>
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<td>Archimedes</td>
<td>2</td>
<td>2.76(2)</td>
<td>1.275(6)</td>
<td>1.54(3)</td>
<td>1.50(3)</td>
<td>1.997(10)</td>
<td>1.382(11)</td>
<td>2.001(2)</td>
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<td>0.78(6)</td>
<td>0.76(3)</td>
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<td>nc diagonal square</td>
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<td>2.750(14)</td>
<td>1.273(4)</td>
<td>1.53(2)</td>
<td>1.49(2)</td>
<td>1.992(7)</td>
<td>1.381(8)</td>
<td>2.0005(12)</td>
<td>0.750(14)</td>
<td>0.75(4)</td>
<td>0.76(2)</td>
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<tr>
<td>triangular</td>
<td>2</td>
<td>2.76(2)</td>
<td>1.275(5)</td>
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<td>1.47(3)</td>
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<td>1.388(12)</td>
<td>1.997(2)</td>
<td>0.76(2)</td>
<td>0.71(6)</td>
<td>0.78(3)</td>
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<tr>
<td>Kagomé</td>
<td>2</td>
<td>2.741(13)</td>
<td>1.270(4)</td>
<td>1.53(2)</td>
<td>1.49(2)</td>
<td>1.993(8)</td>
<td>1.381(9)</td>
<td>1.9994(12)</td>
<td>0.741(13)</td>
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<tr>
<td>honeycomb</td>
<td>2</td>
<td>2.73(2)</td>
<td>1.268(6)</td>
<td>1.55(4)</td>
<td>1.51(4)</td>
<td>1.990(13)</td>
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<td>0.75(3)</td>
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The Manna Model has been investigated numerically in great detail.
Manna on different lattices

Three dimensions

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<thead>
<tr>
<th>Lattice</th>
<th>$q$</th>
<th>$q^{(v)}$</th>
<th>$\langle z \rangle$</th>
<th>$D$</th>
<th>$\tau$</th>
<th>$z$</th>
<th>$\alpha$</th>
<th>$D_{a}$</th>
<th>$\tau_{a}$</th>
<th>$\mu_{1}^{(*)}$</th>
<th>$-\Sigma$</th>
<th>$-\Sigma_{t}$</th>
<th>$-\Sigma_{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC</td>
<td>6</td>
<td>1</td>
<td>$[0.622325(1)]$</td>
<td>3.38(2)</td>
<td>1.408(3)</td>
<td>1.779(7)</td>
<td>1.784(9)</td>
<td>3.04(5)</td>
<td>1.45(4)</td>
<td>2.0057(5)</td>
<td>1.38(2)</td>
<td>1.395(16)</td>
<td>1.36(13)</td>
</tr>
<tr>
<td>BCC</td>
<td>8</td>
<td>4</td>
<td>$[0.600620(2)]$</td>
<td>3.36(2)</td>
<td>1.404(4)</td>
<td>1.777(8)</td>
<td>1.78(1)</td>
<td>2.99(2)</td>
<td>1.444(18)</td>
<td>2.0030(5)</td>
<td>1.36(2)</td>
<td>1.390(19)</td>
<td>1.33(6)</td>
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<tr>
<td>BCCN</td>
<td>14</td>
<td>5</td>
<td>$[0.581502(1)]$</td>
<td>3.38(3)</td>
<td>1.408(4)</td>
<td>1.776(9)</td>
<td>1.783(11)</td>
<td>3.01(3)</td>
<td>1.44(3)</td>
<td>2.0041(6)</td>
<td>1.38(3)</td>
<td>1.39(2)</td>
<td>1.32(7)</td>
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<tr>
<td>FCC</td>
<td>12</td>
<td>4</td>
<td>$[0.589187(3)]$</td>
<td>3.35(4)</td>
<td>1.402(8)</td>
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<td>1.78(2)</td>
<td>3.1(2)</td>
<td>1.48(14)</td>
<td>2.0035(11)</td>
<td>1.35(4)</td>
<td>1.37(4)</td>
<td>1.5(5)</td>
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<tr>
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<td>5</td>
<td>$[0.566307(3)]$</td>
<td>3.38(4)</td>
<td>1.408(7)</td>
<td>1.781(14)</td>
<td>1.787(18)</td>
<td>3.00(4)</td>
<td>1.44(3)</td>
<td>2.0051(8)</td>
<td>1.38(4)</td>
<td>1.40(3)</td>
<td>1.32(9)</td>
</tr>
<tr>
<td>Overall</td>
<td></td>
<td></td>
<td></td>
<td>3.370(11)</td>
<td>1.407(2)</td>
<td>1.777(4)</td>
<td>1.783(5)</td>
<td>3.003(14)</td>
<td>1.442(12)</td>
<td>2.0042(3)</td>
<td>1.380(13)</td>
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From: Huynh, G P, 2012

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Outline

1. SOC: Past and Present
2. Universality Classes
   - Early themes
   - Relevant fields
   - Universality classes
3. Theory of SOC
4. Summary: Any Answers?
Early themes

- Initially the BTW Model was conceived as the paradigm of SOC and maybe the **SOC universality class**.
- Zhang and Manna Models were initially suggested to be in that BTW/SOC universality class.
- Starting from the mid-ninties, new universality classes proposed.
- Universality requires (some) robustness.
Dividing lines between models

The following features are generally considered as relevant fields:\(^1\)

- stochastic vs deterministic
- directed vs undirected (isotropy generally)
- Abelian vs non-Abelian (note initial confusion of stochastic=non-Abelian)
- conservative vs non-conservative

Most observations made in variations of BTW and Manna Models.

---

\(^1\) e.g. Ben-Hur and Biham, 1996; Milshtein, Biham, Solomon, 1998; Karmakar, Manna, Stella, 2005
Universality classes

Widely accepted universality classes are:

- **Directed sandpiles (stochastic and deterministic).**
- **Manna universality class in** \( d = 1, 2, 3, 4, \) **free above.**
- **BTW (multiscaling) in** \( d = 2, 3, 4, \) **free above?**, includes possibly the Zhang Model.
- **OFC Model** (somewhat robust if conservative, class of its own?).
- **Forest Fire Model** (not robust, class of its own?).
- **Bak-Sneppen Model** (not robust, class of its own?).
Directed Models

- Typically solved by mapping to random walker (time is equivalent to one spatial dimension, $d = d_\perp + 1$).
- Exact solutions and controlled approximations.
- $d_\perp = 0, 1, 2$, upper critical dimension is $d_\perp = 2$.

From Pruessner 2012, p.287
Directed Models

- Plethora of models.
- Two classes: Random distribution to downstream neighbours vs deterministic distribution to downstream neighbours.
- Directedness results in no (or short-ranged or trivial) spatial correlations.
- Fully characterised (Dhar and Ramaswamy, 1989; Paczuski and Bassler, 2000; Bunzarova 2010).

From Pruessner 2012, p.287
The Manna Universality Class

- The only large universality class in SOC.
- Includes large number of models, which seemingly are very different.
- Spatially isotropic.
- Numerically characterised in $d = 1, 2, 3, 4, 5$ (e.g. Luebeck and Heger, 2003).
- Little known analytically, no proper mean field theory.
Outline

1. SOC: Past and Present
2. Universality Classes
3. Theory of SOC
   - Tools in SOC
   - The Absorbing State Mechanism
   - Field theory for SOC
   - The SOC mechanism
4. Summary: Any Answers?
Tools in SOC

- (Extensive) numerics (BTW, FFM, BS, Manna, Oslo).
- Analytical tools:
  - Exact solutions (so far: directed models only).
  - Mappings to known (understood?) phenomena.
  - Growth processes and field theories.
Link to growth phenomena (generic scale invariance)

Stochastic evolution of sandpile surface.

\[ \partial_t \phi(r, t) = (\nu_\parallel \partial^2_\parallel + \nu_\perp \partial^2_\perp) \phi + \eta(r, t) \]

- **Generic** scale invariance (Hwa and Kardar, 1989, and Grinstein, Lee and Sachdev 1990)
- No mass term $-\epsilon \phi$ on the right \(\longrightarrow\) conservative dynamics (finiteness generates $\epsilon$).
- Anisotropy (boundaries?) required in the presence of conserved noise.
- Non-trivial exponents in the presence of non-linearities and non-conserved noise.
- Concept abandoned with the arrival of non-conservative models (FFM [1990], OFC [1992], BS [1993]).
Effect of a mass term

Mass term

\[ \partial_t \phi = \nu \nabla^2 \phi - \epsilon \phi + \ldots + \eta \]

represents dissipation

\[ \partial_t \int_V d^d x \phi = \text{surface terms} - \epsilon \int_V d^d x \phi \]

and correlation length

\[ \phi = \ldots e^{-|x|\sqrt{\epsilon/\nu}}. \]

But: How can a renormalised \( \epsilon = 0 \) be maintained without trivialising (no additive renormalisation, \( \epsilon = 0 \) is the critical point in mean field) the phenomenon?
Field theories for Manna and Oslo

Number of charges interpreted as an interface.

- **Manna model** has a (weird!) Langevin equation.
- **Oslo model** implements *quenched Edwards Wilkinson equation* → interfaces!

Field theories for both still investigated.

Mechanism of self-organisation still investigated.

Link to known universality classes.

Link to **directed percolation**?
The Absorbing State Mechanism
Dickman, Vespignani, Zapperi 1998

- SOC model: activity $\rho_a$ leads to dissipation
- dissipation reduces particle density $\zeta$
- density is reduced until system is inactive
  $\rightarrow$ absorbing phase
- external drive increases particle density
  $\rightarrow$ back to active phase

An SOC model can be seen as an AS model that drives itself into the inactive phase by dissipation $\epsilon$ and is pushed back into the active phase by external drive $h$.

\[ \dot{\zeta} = h - \epsilon \rho_a \quad \text{stationarity} \quad \rho_a = h/\epsilon \]
The Absorbing State Mechanism

Idea: SOC drives $h/\epsilon = \rho_a$ to 0 as $L \to \infty$

Leading orders: $h(L) = h_0 L^{-\omega}$ and $\epsilon(L) = \epsilon_0 L^{-\kappa}$
The Absorbing State Mechanism

Problem: SOC exponents would be affected by the way how driving and dissipation are implemented \( \rightarrow \) no universality.

Fey, Levine and Wilson suggest that critical point is not reached.
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Field theory for SOC

The Manna Model

Field theoretic formulation of the time evolution of the Manna Model. Note: Before taking any limits, this theory is *exact*.

- Continuum limit
- Simplify...
- Diagrams (meaning?, process?, tree level?)
- Renormalisation
Simplification of the field theory

Bare propagators from field theory by inspection. Simplification by considering periodic boundary conditions in $d - 1$ directions. Surface appears in only one dimension.
Bare propagators

\begin{align*}
\frac{1}{-\omega + D(\mathbf{k}^2 + q_n^2)}
\end{align*}

where \( q_n = \frac{\pi}{L} n \) with \( n = 1, 2, \ldots \)

- \( d - 1 \) dimensions can be treated the “usual” way.
- Usually, the gap in the propagator is the mass \( r_0 \) in

\begin{align*}
\frac{1}{-\omega + D(\mathbf{k}^2 + r_0)}
\end{align*}

found by evaluating the inverse propagator at minimal momentum and frequency magnitude, \( \mathbf{k} = 0 \) and \( \omega = 0 \).

- Here, the gap is set by the minimum magnitude of \( q_n \) allowed. The effective mass is \( q_1^2 = (\pi/L)^2 \).
Bare propagators

Consider the system size as the effective mass of the system. Expect convergence as circumference is increased; critical point controlled by height ($L$) only.
Bare propagators

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Bare propagators

Exact first moments

Circumference does not enter into first moment.

Avalanche size: Total activity (total number of charges).

In one dimension (continuum limit):

$$\langle s \rangle = \frac{1}{6} L^2$$

and $$\langle s \rangle = \frac{1}{6} (L + 1)(L + 2)$$ discretely. In higher dimensions:

$$\langle s \rangle = \frac{d}{6} L^2$$

and $$\langle s \rangle = \frac{d}{6} (L + 1)(L + 2)$$ discretely.
Vertices

The interaction vertices are

- Spontaneous branching and substrate deposition:

- Substrate interaction resulting in attenuation or deposition:

All relevant for $d \leq d_c = 4$. Loops occur.
Vertices

The interaction vertices are

- Spontaneous branching and substrate deposition:

- Substrate interaction resulting in attenuation or deposition:

Only the former are relevant for $d > d_c = 4$; as in $\phi^4$ the latter enter only for the lowest mode. No loops.
Tree level

Tree level becomes exact above $d_c = 4$. Two vertices are relevant there:

For example:

$$\langle s^2 \rangle = 2 \left( \frac{2}{L} \right)^3 \sum_{n,m,l \text{ odd}} \frac{4}{q_l q_m} \frac{2}{q_n} = \frac{d^3}{140} L^6$$

Higher order moments follow similarly.
Tree level

Comparison to numerics

Tree level moments can be compared to the numerics of the Manna Model at $d > 4$, here $d = 5$:

<table>
<thead>
<tr>
<th>Observable</th>
<th>analytical</th>
<th>numerical (leading order)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle s \rangle$</td>
<td>$(d/6)L^2 = 0.833 \ldots L^2$</td>
<td>$0.83334(6)L^2$</td>
</tr>
<tr>
<td>$\langle s \rangle \langle s^3 \rangle / \langle s^2 \rangle^2$</td>
<td>$3.08754 \ldots$</td>
<td>$3.111(11)$</td>
</tr>
<tr>
<td>$\langle s^2 \rangle \langle s^4 \rangle / \langle s^3 \rangle^2$</td>
<td>$1.6693 \ldots$</td>
<td>$1.70(3)$</td>
</tr>
</tbody>
</table>

Note: Numerical fitting pretty *ad hoc*. 
Tree level: Mean Field Theory

The process corresponding to tree level is the \textit{effective} mean field theory of the Manna Model (random walk, not space-less!). Parameters are self-organised (see below).

- For that process, avalanche moments can be calculated easily\textsuperscript{2} directly (not via the field theory).
- Results coincide with those from field theory and numerics in $d = 5$.

This mean field theory identifies precisely the correlations and fluctuations to be ignored. Not an \textit{ad-hoc} approximantion. Mean field theories in SOC are usually effective theories of certain observables and do not incorporate space at any level.

\textsuperscript{2}Mathematica takes care of the mess
The SOC mechanism

How does SOC work?

→ Organisation to the critical point? Why are the propagators massless?

Mass is attenuation (loss of activity). At tree level:

\[
\begin{align*}
\text{mass} & \\
+ & \rightarrow + \rightarrow + \rightarrow + \\
+ & \rightarrow + \rightarrow + \rightarrow + \rightarrow + \rightarrow + \rightarrow + \rightarrow + \rightarrow + \\
\end{align*}
\]
The SOC mechanism

How does SOC work?

Attenuation leads to deposition by the external drive — diagrams have that symmetry.
Density of particles in the substrate:
The SOC mechanism

How does SOC work?

Attenuation leads to deposition by the external drive — diagrams have that symmetry.

Density of particles in the substrate:

\[
\text{Additional deposition by external drive vanishes at stationarity.}
\]
The SOC mechanism

How does SOC work?

Mass:

Additional deposition:

Only difference between the two diagrams: Left most vertex (coupling identical at renormalised and bare level).
The SOC mechanism

So how does it work then?

- Activity attenuation is mass.
- Conservation links attenuation to (additional) substrate deposition... 
- or equivalently, symmetry of vertices equates mass terms of activity and substrate deposition terms.
- Additional substrate deposition vanishes *as we choose to consider stationarity.*

Terms and conditions apply... 

Issue: Deposition without attenuation, by seemingly conservative terms.
The SOC mechanism
So how does it work then?

- **Stationarity causes criticality.** (qualification of Hwa and Kardar: Masslessness by conservation).
- Conservation is secondary to stationarity (links attenuation and deposition, the latter being stationary) — non-conservative SOC is possible!
- (Ward-Takahashi) symmetry of diagrams produces for self-tuning.
- Shift of stationary particle density understood.
- Innocent looking processes (such as “catalytic” diffusion in substrate) destroy critical state.
- Relation to absorbing state mechanism unclear.
Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

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