

Connecting Compton and Gravitational Compton Scattering

Barry R. Holstein
UMass Amherst

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Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.*^{*}

(LIGO Scientific Collaboration and Virgo Collaboration)

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On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21} . It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. The signal was observed with a matched-filter signal-to-noise ratio of 24 and a false alarm rate estimated to be less than 1 event per 203 000 years, equivalent to a significance greater than 5.1σ . The source lies at a luminosity distance of 410^{+160}_{-180} Mpc corresponding to a redshift $z = 0.09^{+0.03}_{-0.04}$. In the source frame, the initial black hole masses are $36^{+5}_{-4} M_{\odot}$ and $29^{+4}_{-4} M_{\odot}$, and the final black hole mass is $62^{+4}_{-4} M_{\odot}$, with $3.0^{+0.5}_{-0.5} M_{\odot}c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.

Outline

- I. Compton Scattering
 - a) Model-independent results
 - b) Model-dependent results

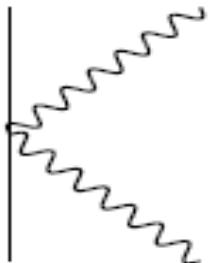
- II. Gravitational Compton Scattering
 - a) Model-independent results
 - b) Relation to Compton
 - c) Model-independent results

Low's Theorem: S=0

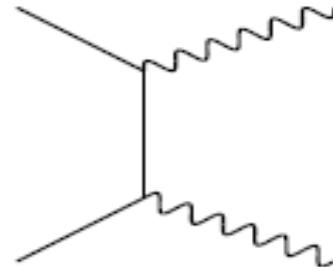
$$\text{Amp} = \epsilon_f^{*\mu} \epsilon_i^\nu [T_{\mu\nu}^{Born} + T_{\mu\nu}^{NB}]$$

$$T_{\mu\nu}^{Born} = 2e^2 \left[\frac{p_f^\mu p_i^\nu}{p_i \cdot k_i} - \frac{p_f^\mu p_i^\nu}{p_i \cdot k_f} - \eta_{\mu\nu} \right]$$

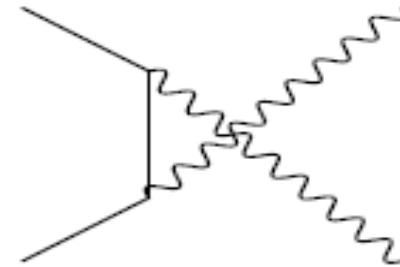
from



a)



b)



c)

and

$$T_{\mu\nu}^{NB} \sim \mathcal{O}(\omega^2).$$

From gauge invariance

$$\begin{aligned} T_{\mu\nu}^{NB} = & C_1 [k_{f\mu} k_{i\nu} - \eta_{\mu\nu} k_f \cdot k_i] \\ & + C_2 [P_\mu P_\nu k_f \cdot k_i - P \cdot K (k_{f\mu} P_\nu + k_{i\nu} P_\mu) + \eta_{\mu\nu} (P \cdot K)^2] \end{aligned}$$

where $K = \frac{1}{2}(k_i + k_f)$ and $P = \frac{1}{2}(p_i + p_f)$ so derives from

$$\mathcal{L} = -\frac{C_1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{C_2}{2}P^\mu F_{\mu\nu}F^{\nu\lambda}P_\lambda$$

$$= -\frac{C_1}{2}(\mathbf{B}^2 - \mathbf{E}^2) - \frac{C_2}{2}\mathbf{E}^2$$

Write as

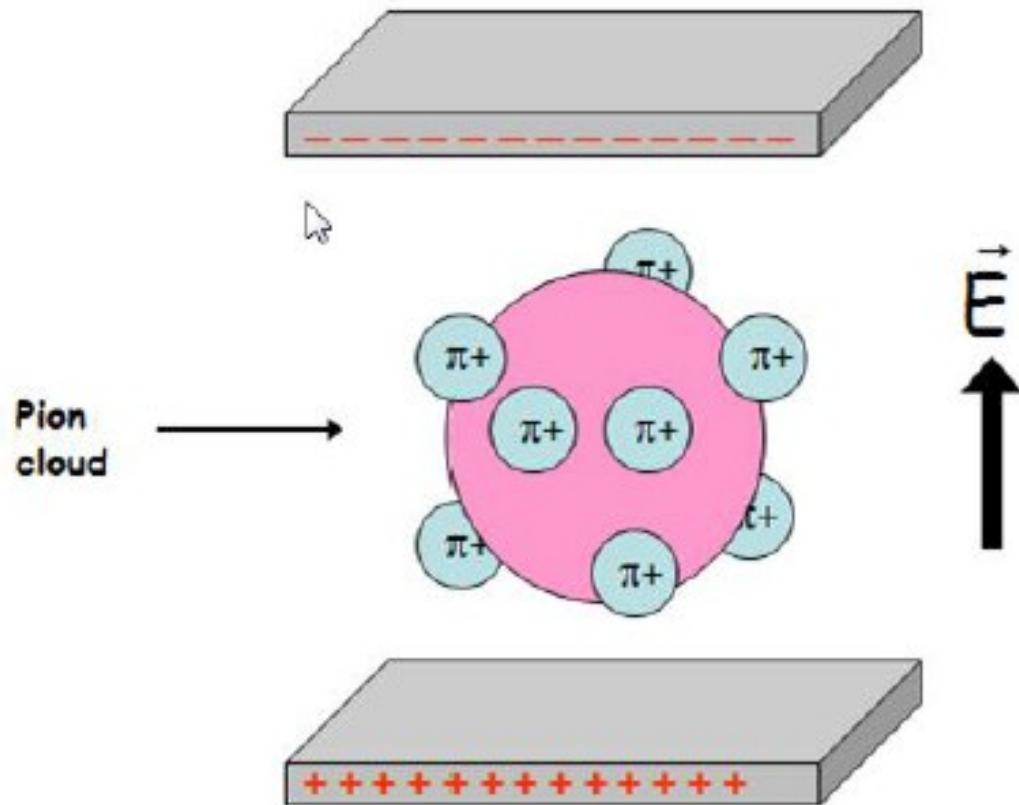
$$H_{eff} = -\frac{1}{2}4\pi\alpha_E \mathbf{E}^2 - \frac{1}{2}4\pi\beta_M \mathbf{B}^2$$

Then α_E, β_M are the usual electric and magnetic polarizabilities, defined by

$$\mathbf{d}_E = 4\pi\alpha_E \mathbf{E} \quad \text{and} \quad \mathbf{d}_B = 4\pi\beta_M \mathbf{B}$$

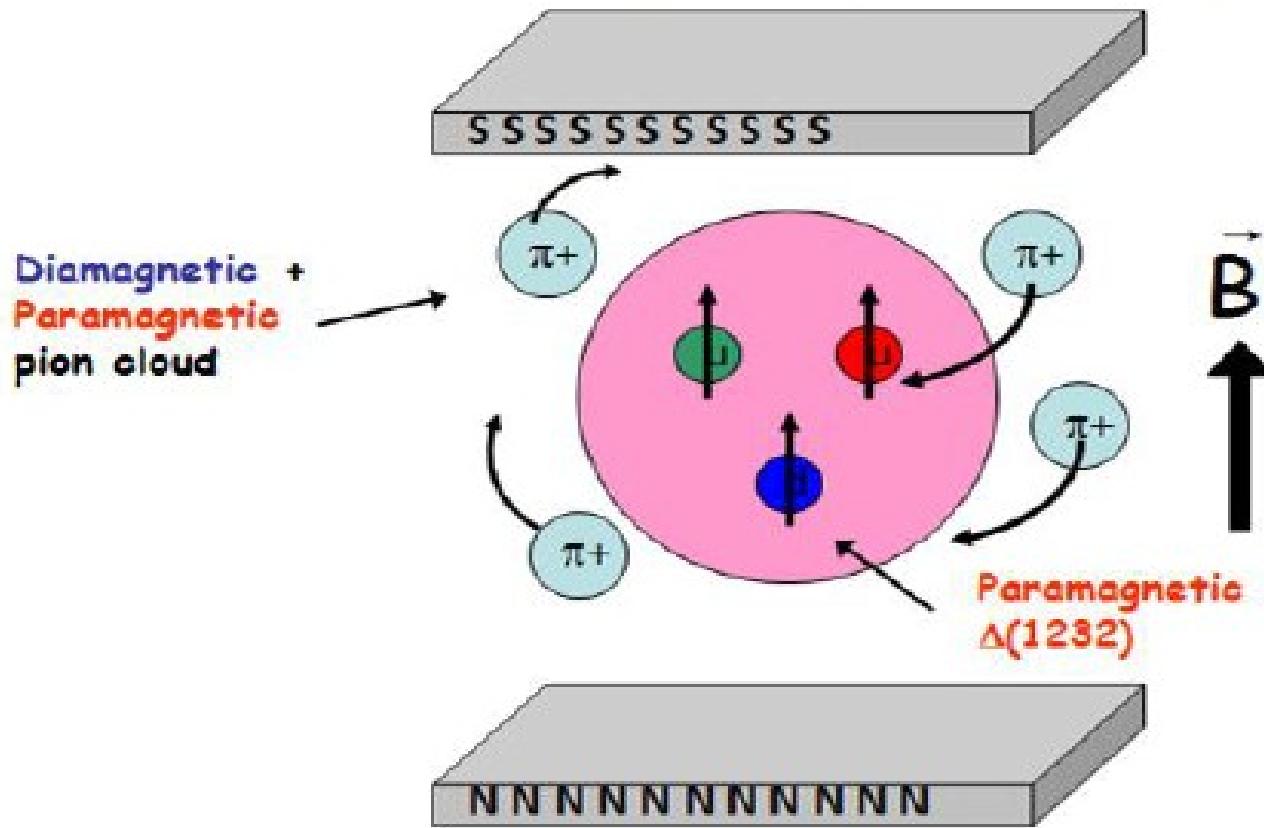
Nice physical picture

Proton electric polarizability



Electric polarizability: proton between charged parallel plates

Proton magnetic polarizability



Magnetic polarizability: proton between poles of a magnetic

Write

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - \frac{1}{2}4\pi\alpha_E E^2 - \frac{1}{2}4\pi\beta_M H^2 + \dots$$

so

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{2m^2} \frac{\omega_f^2}{\omega_i^2} \left[(1 + \cos^2 \theta) - \frac{m\omega_f\omega_i}{\alpha} \right. \\ &\times ((\alpha_E + \beta_M)(1 + \cos \theta)^2 + (\alpha_E - \beta_M)(1 - \cos \theta)^2) \\ &\quad \left. + \dots \right]\end{aligned}$$

Forward scattering amplitude is

$$\text{Amp}(\theta = 0) = \epsilon_f^* \cdot \epsilon_i f(\omega)$$

with

$$f(\omega) = f(-\omega) = -\frac{e^2}{4\pi m} + (\alpha_E + \beta_M)\omega^2 + \dots$$

Obeys subtracted dispersion relation

$$f(\omega) = -\frac{e^2}{4\pi m} + \frac{\omega^2}{2\pi^2} \int_0^\infty \frac{d\omega' \sigma_{tot}(\omega')}{\omega'^2 - \omega^2}$$

where $\sigma_{tot}(\omega')$ is the photoabsorption cross section.

Find then the Baldin sum rule

$$\alpha_E + \beta_M = \frac{1}{2\pi^2} \int_0^\infty \frac{d\omega' \sigma_{tot}(\omega')}{\omega'^2}$$

Similarly Low's Theorem: $S = \frac{1}{2}$

$$\text{Amp} = \epsilon_f^{*\mu} \epsilon_i^\nu [T_{\mu\nu}^{Born} + T_{\mu\nu}^{NB}]$$

with

$$\begin{aligned} T_{\mu\nu}^{Born} &= e^2 \bar{u}(p') [\\ &\times (\gamma_\nu + i \frac{\kappa}{2m} \sigma_{\nu\beta} k_f^\beta) \frac{1}{p_i + k_i - m} (\gamma_\mu - i \frac{\kappa}{2m} \sigma_{\mu\alpha} k_i^\alpha) \\ &+ (\gamma_\mu - i \frac{\kappa}{2m} \sigma_{\mu\beta} k_i^\beta) \frac{1}{p_i - k_f - m} (\gamma_\nu + i \frac{\kappa}{2m} \sigma_{\nu\alpha} k_f^\alpha)] u(p) \end{aligned}$$

where κ is the anomalous dipole moment in units of the Bohr magneton and

$$T_{\mu\nu}^{NB} \sim \mathcal{O}(\omega^2).$$

Now in forward direction

$$\text{Amp}(\theta = 0) = \epsilon_f^* \cdot \epsilon_i f(\omega) + i g(\omega) \boldsymbol{\sigma} \cdot \epsilon_f^* \times \epsilon_i$$

with

$$f(\omega) = -\frac{e^2}{4\pi m} + \mathcal{O}(\omega)^2 \quad \text{and} \quad g(\omega) = \frac{e^2 \kappa^2 \omega}{4\pi m} + \mathcal{O}(\omega^3)$$

Still have Baldin sum rule

$$\alpha_E + \beta_M = \frac{1}{2\pi^2} \int_0^\infty \frac{d\omega' (\sigma_{\frac{3}{2}}(\omega') + \sigma_{\frac{1}{2}}(\omega'))}{\omega'^2}$$

but spin-flip amplitude $g(\omega)$ obeys an *unsubtracted* dispersion relation

$$\frac{e^2 \kappa^2}{4\pi m^2} = \frac{1}{\pi^2} \int_0^\infty \frac{d\omega' (\sigma_{\frac{3}{2}}(\omega') - \sigma_{\frac{1}{2}}(\omega'))}{\omega'}$$

which is the Gerasimov-Drell-Hearn (DGH) sum rule
and is well satisfied experimentally for the proton—

$$LHS = 205 \mu\text{b} \quad \text{vs.} \quad RHS = 210 \pm 15 \mu\text{b}$$

Non-Born terms carry information about particle structure. Can write more general polarizability Hamiltonian

$$\begin{aligned} H_{pol} = & -\frac{1}{2}4\pi (\alpha_E \mathbf{E}^2 + \beta_M \mathbf{H}^2) \\ & -\frac{1}{2}4\pi (\gamma_{E1E1} \boldsymbol{\sigma} \cdot (\mathbf{E} \times \dot{\mathbf{E}}) + \gamma_{M1M1} \boldsymbol{\sigma} \cdot (\mathbf{H} \times \dot{\mathbf{H}}) \\ & -2\gamma_{M1E2}\sigma_i H_j E_{ij} + 2\gamma_{E1M2}\sigma_i E_j H_{ij}) \\ & -\frac{1}{2}4\pi (\alpha_{E\nu} \dot{\mathbf{E}}^2 + \beta_{M\nu} \dot{\mathbf{H}}^2) \\ & -\frac{1}{12}4\pi (\alpha_{E2} E_{ij}^2 + \beta_{M2} H_{ij}^2) \end{aligned}$$

with

$$E_{ij} = \frac{1}{2}(\nabla_i E_j + \nabla_j E_i) \quad \text{and} \quad H_{ij} = \frac{1}{2}(\nabla_i H_j + \nabla_j H_i)$$

Then new sum rules

$$\alpha_{E1\nu} + \beta_{M1\nu} + \frac{1}{12}(\alpha_{E2} + \beta_{M2}) \\ = \frac{1}{2\pi^2} \int_0^\infty \frac{d\omega' (\sigma_{\frac{3}{2}}(\omega') + \sigma_{\frac{1}{2}}(\omega'))}{\omega'^4}$$

$$\gamma_0 = \frac{1}{2\pi^2} \int_0^\infty \frac{d\omega' (\sigma_{\frac{3}{2}}(\omega') - \sigma_{\frac{1}{2}}(\omega'))}{\omega'^3}$$

$$\bar{\gamma}_0 = \frac{1}{2\pi^2} \int_0^\infty \frac{d\omega' (\sigma_{\frac{3}{2}}(\omega') - \sigma_{\frac{1}{2}}(\omega'))}{\omega'^5}$$

Can use models to predict size of polarizabilities:

Chiral perturbation theory used to calculate polarizabilities—general agreement shows importance of pion cloud

	HB χ pt	B χ pt	PDG
α_E	10.7 ± 0.5	10.6 ± 0.5	11.2 ± 0.4
β_M	3.2 ± 0.5	3.2 ± 0.5	2.5 ± 0.4

	DR	$\text{HB}\chi\text{pt}$	$\text{B}\chi\text{pt}$	Expt.
γ_{E1E1}	-5.6	-1.1 ± 1.8	-3.3 ± 0.8	-3.5 ± 1.2
γ_{M1M1}	3.8	2.2 ± 0.6	2.9 ± 1.5	3.2 ± 0.9
γ_{E1M2}	-0.7	-0.4 ± 0.4	0.2 ± 0.2	-0.7 ± 1.2
γ_{M1E2}	2.9	1.9 ± 0.4	1.1 ± 0.4	2.0 ± 0.3

	$\text{HB}\chi\text{pt(NNLO)}$
α_{E2}	17.3 ± 3.9
β_{M2}	-15.5 ± 3.5
$\alpha_{E1\nu}$	-1.3 ± 0.4
$\beta_{M1\nu}$	7.1 ± 2.5

Gravitational Compton Scattering

Many parallels between gravitational and electromagnetic interactions

$$\mathcal{L}_{em} = -e J_\mu A^\mu \rightarrow \mathcal{L}_{grav} = -\frac{\kappa}{2} T_{\mu\nu} h^{\mu\nu}$$

with metric tensor given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$,

$$e^2 = 4\pi\alpha \longrightarrow \kappa^2 = 32\pi G$$

and, for S=0

$$\langle p_f | J_\mu(0) | p_i \rangle = G_1(q^2)(p_f + p_i)_\mu \rightarrow$$

$$\begin{aligned} \langle p_f | T_{\mu\nu} | p_i \rangle &= F_1(q^2)(p_f + p_i)_\mu(p_f + p_i)_\nu \\ &\quad + F_2(q^2)(q_\mu q_\nu - \eta_{\mu\nu} q^2) \end{aligned}$$

For electromagnetism $G_1(0) = 1$ because of charge conservation becomes in gravitational case $F_1(0) = 1$ because of energy-momentum conservation.

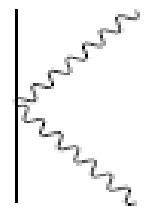
Also photon propagator

$$D_{\mu\nu}^{em}(q) = -i\eta_{\mu\nu}/q^2$$

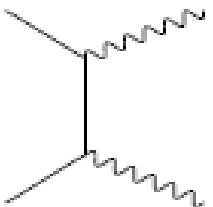
becomes graviton propagator

$$D_{\mu\nu;\alpha\beta}^{grav} = i(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})/2q^2$$

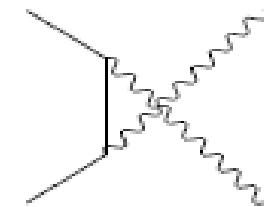
However, due to nonlinearity, there are now *four* diagrams



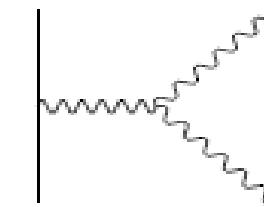
a)



b)



c)



d)

Contributions are

$$\text{Born - a} = -2e^2 \epsilon_f^* \cdot \epsilon_i \longrightarrow \kappa^2 \left[\epsilon_f^* \cdot \epsilon_i (\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f + \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i) - \frac{1}{2} k_i \cdot k_f (\epsilon_f^* \cdot \epsilon_i)^2 \right]$$

$$\text{Born - b} = 2e^2 \frac{\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f}{p_i \cdot k_i} \longrightarrow 2\kappa^2 \frac{(\epsilon_i \cdot p_i)^2 (\epsilon_f^* \cdot p_f)^2}{p_i \cdot k_i}$$

$$\text{Born - c} = -2e^2 \frac{\epsilon_f^* \cdot p_i \epsilon_i \cdot p_f}{p_i \cdot k_f} \longrightarrow -2\kappa^2 \frac{(\epsilon_f^* \cdot p_i)^2 (\epsilon_i \cdot p_f)^2}{p_i \cdot k_f}$$

$$\text{Born - d = 0} \longrightarrow \frac{4\kappa^2}{k_i \cdot k_f} [\epsilon_f^* \cdot p_f \epsilon_f^* \cdot p_i (\epsilon_i \cdot (p_i - p_f))^2$$

$$\begin{aligned}
& + \epsilon_i \cdot p_i \epsilon_i \cdot p_f (\epsilon_f^* \cdot (p_i + p_f))^2 \\
& + \epsilon_i \cdot (p_i - p_f) \epsilon_f^* \cdot (p_f - p_i) (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i + \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) \\
& + \epsilon_f^* \cdot \epsilon_i (\epsilon_i \cdot (p_i - p_f) \epsilon_f^* \cdot (p_f - p_i) (p_i \cdot p_f - m^2) \\
& \quad + k_i \cdot k_f (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i + \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f)) \\
& + \epsilon_i \cdot (p_i - p_f) (\epsilon_f^* \cdot p_f p_i \cdot k_f + \epsilon_f^* \cdot p_i p_f \cdot k_f) \\
& + \epsilon_f^* \cdot (p_f - p_i) (\epsilon_i \cdot p_i p_f \cdot k_i + \epsilon_i \cdot p_f p_i \cdot k_i)) \\
& + (\epsilon_f^* \cdot \epsilon_i)^2 (p_i \cdot k_i p_f \cdot k_i + p_i \cdot k_f p_f \cdot k_f \\
& \quad - \frac{1}{2} (p_i \cdot k_i p_f \cdot k_f + p_i \cdot k_f p_f \cdot k_i) \\
& \quad + \frac{3}{2} k_i \cdot k_f (p_i \cdot p_f - m^2)) \Big]
\end{aligned}$$

where complex form of Born-d comes from triple-graviton vertex

$$\begin{aligned}
\tau_{\alpha\beta,\gamma\delta}^{\mu\nu}(k, q) = & \frac{i\kappa}{2} \left\{ (I_{\alpha\beta,\gamma\delta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}) \left[k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2 \right] \right. \\
& + 2q_\lambda q_\sigma \left[I^{\lambda\sigma,}_{\alpha\beta} I^{\mu\nu,}_{\gamma\delta} + I^{\lambda\sigma,}_{\gamma\delta} I^{\mu\nu,}_{\alpha\beta} - I^{\lambda\mu,}_{\alpha\beta} I^{\sigma\nu,}_{\gamma\delta} - I^{\sigma\nu,}_{\alpha\beta} I^{\lambda\mu,}_{\gamma\delta} \right] \\
& + [q_\lambda q^\mu (\eta_{\alpha\beta} I^{\lambda\nu,}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\nu,}_{\alpha\beta}) + q_\lambda q^\nu (\eta_{\alpha\beta} I^{\lambda\mu,}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\mu,}_{\alpha\beta}) \\
& - q^2 (\eta_{\alpha\beta} I^{\mu\nu,}_{\gamma\delta} + \eta_{\gamma\delta} I^{\mu\nu,}_{\alpha\beta}) - \eta^{\mu\nu} q^\lambda q^\sigma (\eta_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta} I_{\alpha\beta,\lambda\sigma})] \\
& + [2q^\lambda (I^{\sigma\nu,}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\mu + I^{\sigma\mu,}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\nu \\
& - I^{\sigma\nu,}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\mu - I^{\sigma\mu,}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\nu)] \\
& + q^2 (I^{\sigma\mu,}_{\alpha\beta} I_{\gamma\delta,\sigma}^\nu + I_{\alpha\beta,\sigma}^\nu I^{\sigma\mu,}_{\gamma\delta}) + \eta^{\mu\nu} q^\lambda q_\sigma (I_{\alpha\beta,\lambda\rho} I^{\rho\sigma,}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho} I^{\rho\sigma,}_{\alpha\beta})] \\
& + [(k^2 + (k-q)^2) \left(I^{\sigma\mu,}_{\alpha\beta} I_{\gamma\delta,\sigma}^\nu + I^{\sigma\nu,}_{\alpha\beta} I_{\gamma\delta,\sigma}^\mu - \frac{1}{2}\eta^{\mu\nu} P_{\alpha\beta,\gamma\delta} \right) \\
& \left. - (k^2 \eta_{\gamma\delta} I^{\mu\nu,}_{\alpha\beta} + (k-q)^2 \eta_{\alpha\beta} I^{\mu\nu,}_{\gamma\delta}) \right] \}
\end{aligned}$$

where

$$I_{\alpha\beta;\gamma\delta} = \frac{1}{2} [\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma}]$$

Summing, we find

$$\begin{aligned}
 \text{Amp}_{GC}(S = 0) &= \kappa^2 \\
 &\times \left\{ -\frac{(\epsilon_i \cdot p_i)^2 (\epsilon_f^* \cdot p_f)^2}{p_i \cdot k_i} + \frac{(\epsilon_f^* \cdot p_i)^2 (\epsilon_i \cdot p_f)^2}{p_i \cdot k_f} \right. \\
 &+ [\epsilon_f^* \cdot \epsilon_i (\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f + \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i) - \frac{1}{2} k_i \cdot k_f (\epsilon_f^* \cdot \epsilon_i)^2] \\
 &+ \frac{1}{2k_i \cdot k_f} \left[\epsilon_f^* \cdot p_f \epsilon_f^* \cdot p_i (\epsilon_i \cdot (p_i - p_f))^2 \right. \\
 &\quad \left. + \epsilon_i \cdot p_i \epsilon_i \cdot p_f (\epsilon_f^* \cdot (p_i - p_f))^2 \right] \\
 &+ \epsilon_i \cdot (p_i - p_f) \epsilon_f^* \cdot (p_f - p_i) (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i + \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) \\
 &- \epsilon_f^* \cdot \epsilon_i \left(\epsilon_i \cdot (p_i - p_f) \epsilon_f^* \cdot (p_f - p_i) (p_i \cdot p_f - m^2) \right. \\
 &\quad \left. + k_i \cdot k_f (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i + \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) \right] \\
 &+ \epsilon_i \cdot (p_i - p_f) (\epsilon_f^* \cdot p_f p_i \cdot k_f + \epsilon_f^* \cdot p_i p_f \cdot k_f) \\
 &\quad \left. \searrow \right]
 \end{aligned}$$

$$\begin{aligned}
& + \epsilon_f^* \cdot (p_f - p_i)(\epsilon_i \cdot p_i p_f \cdot k_i + \epsilon_i \cdot p_f p_i \cdot k_i) \Big) \\
& + (\epsilon_f^* \cdot \epsilon_i)^2 (p_i \cdot k_i p_f \cdot k_i + p_i \cdot k_f p_f \cdot k_f \\
& - \frac{1}{2}(p_i \cdot k_i p_f \cdot k_f + p_i \cdot k_f p_f \cdot k_i) \\
& + \frac{3}{2}k_i \cdot k_f(p_i \cdot p_f - m^2)) \Big] \Big] \Big\}
\end{aligned}$$

MIRACLE occurs (double copy theorem)

$$\text{Amp}_{GC}^{Born}(S=0) = F \times \left(\text{Amp}_C^{Born}(S=0) \right)^2$$

where

$$\text{Amp}_C^{Born}(S=0) = 2e^2 \left[\frac{\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i}{p_i \cdot k_i} - \frac{\epsilon_i \cdot p_f \epsilon_f^* \cdot p_i}{p_i \cdot k_f} - \epsilon_f^* \cdot \epsilon_i \right]$$

and

$$F = \frac{\kappa^2}{8e^4} \frac{p_i \cdot k_i p_i \cdot k_f}{k_i \cdot k_f}$$

For spin- $\frac{1}{2}$ we have

$$\langle p_f | T_{\mu\nu} | p_i \rangle = \bar{u}(p_f) [F_1(q^2) (\gamma_\mu(p_f + p_i)_\nu + \gamma_\nu(p_f + p_i)_\mu)$$

$$- F_2(q^2) \frac{i}{2m} ((p_f + p_i)_\mu \sigma_{\nu\alpha} q^\alpha + (p_f + p_i)_\nu \sigma_{\mu\alpha} q^\alpha) \\ + F_3(q^2) (q_\mu q_\nu - \eta_{\mu\nu} q^2)] u(p_i)$$

where now $F_1(0) = 1$ because of energy-momentum conservation and $F_2(0) = 1$ because of angular momentum conservation. That is, there is no anomalous gravitomagnetic moment— $\kappa_g = 0$.

Graviton Scattering: Spin 1/2

$$\begin{aligned}
 \text{Born - a} &= 0 \longrightarrow \kappa^2 \bar{u}(p_f) \\
 &\times \left[\frac{3}{16} \epsilon_f^* \cdot \epsilon_i (\not{v}_i \epsilon_f^* \cdot (p_i + p_f) + \not{v}_f^* \epsilon_i \cdot (p_i + p_f)) \right. \\
 &\quad \left. + \frac{i}{8} \epsilon_f^* \cdot \epsilon_i \epsilon^{\rho\sigma\eta\lambda} \gamma_\lambda \gamma_5 (\epsilon_{i\eta} \epsilon_{f\sigma}^* k_{f\rho} - \epsilon_{f\eta}^* \epsilon_{i\sigma} k_{i\rho}) \right] u(p_i) \\
 \text{Born - b} &= \frac{e^2}{2p_i \cdot k_i} \bar{u}(p_f) \not{v}_f^*(\not{p}_i + \not{k}_i + m) \not{v}_i u(p_i) \\
 &\longrightarrow \kappa^2 \frac{\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i}{8p_i \cdot k_i} \bar{u}(p_f) [\not{v}_f^*(\not{p}_i + \not{k}_i + m) \not{v}_i] u(p_i) \\
 \text{Born - c} &= -\frac{e^2}{2p_i \cdot k_f} \bar{u}(p_f) \not{v}_i(\not{p}_i - \not{k}_f + m) \not{v}_f u(p_i) \\
 &\longrightarrow -\kappa^2 \frac{\epsilon_f^* \cdot p_i \epsilon_i \cdot p_f}{8p_i \cdot k_f} \bar{u}(p_f) [\not{v}_i(\not{p}_i - \not{k}_f + m) \not{v}_f] u(p_i)
 \end{aligned}$$

$$\begin{aligned}
& \text{Born - d} = 0 \rightarrow \frac{\kappa^2}{k_i \cdot k_f} \bar{u}(p_f) \\
& \times \left[(\not{k}_i \epsilon_f^* \cdot k_i + \not{k}_f \epsilon_i \cdot k_f) (\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f - \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i) \right. \\
& \quad - (\epsilon_f^* \cdot \epsilon_i) \left(k_i \cdot k_f (\not{k}_f \epsilon_i \cdot k_f + \not{k}_i \epsilon_f^* \cdot p_i) \right. \\
& \quad + \not{k}_i (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i - \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) + p_i \cdot k_i (\not{k}_i \epsilon_f^* \cdot k_i + \not{k}_f \epsilon_i \cdot k_f) \Big) \\
& \quad \left. \left. + (\epsilon_f^* \cdot \epsilon_i)^2 \not{k}_i (p_i \cdot k_i - \frac{1}{2} k_i \cdot k_f) \right] u(p_i) \right]
\end{aligned}$$

Summing, we find again a MIRACLE:

$$\text{Amp}_{GC}^{Born}(S = \frac{1}{2}) = F \times (\text{Amp}_C^{Born}(S = 0))$$

$$\times \left(\text{Amp}_C^{Born}(S = \frac{1}{2}) \right)$$

where

$$\text{Amp}_C^{Born}(S = \frac{1}{2}) = e^2 \bar{u}(p_f) \left[\frac{\not{e}_f^*(\not{p}_i + \not{k}_i + m) \not{e}_i}{2p_i \cdot k_i} \right.$$

$$\left. - \frac{\not{e}_i(\not{p}_i - \not{k}_f + m) \not{e}_f^*}{2p_i \cdot k_f} \right] u(p_i)$$

Now generalized Low's theorem gives

$$\text{Amp} = \epsilon_f^{*\alpha} \epsilon_f^{*\beta} \epsilon_i^\gamma \epsilon_i^\delta (T_{\alpha\beta;\gamma\delta}^{\text{Born}} + T_{\alpha\beta;\gamma\delta}^{NB})$$

with $T_{\alpha\beta;\gamma\delta}^{NB} = \mathcal{O}(\omega^4)$

Physical significance of NB terms

Electromagnetic case—external electric field leads to electric dipole moment

$$\mathbf{d}_E = 4\pi\alpha_E \mathbf{E}$$

while in gravitational case—gravitational field gradient leads to quadrupole moment

$$Q_{ij} = \frac{Q}{3r^2} (3r_i r_j - \delta_{ij} r^2) = \alpha_G R(0i; 0j)$$

where α_G is the quadrupole polarizability, characterized by Hamiltonian

$$H = -\frac{1}{2}\alpha_G^A \sum_{ij} R_{0i;0j}^2 = -\frac{\alpha_G^A}{2m_A^4} p_1^\alpha p_1^\gamma R_{\alpha\beta;\gamma\delta} R^{\rho\beta;\sigma\delta} p_{1\rho} p_{1\sigma}$$

Then just as electromagnetic polarizability lead to long range electromagnetic interaction between systems (Casimir-Polder)

$$V_{em} = -\frac{23(\alpha_E^{(1)}\alpha_E^{(2)} + \beta_M^{(1)}\beta_M^{(2)}) - 7(\alpha_E^{(1)}\beta_M^{(2)} + \alpha_E^{(2)}\beta_M^{(1)})}{4\pi r^7}$$

quadrupole polarizability leads to long range gravitational interaction between systems

$$V_{grav} = -\frac{3987\alpha_G^{(1)}\alpha_G^{(2)}}{1024\pi^3 r^{11}}$$

Are there dispersion relations? Yes.

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Low-Energy Theorem for Graviton Scattering

DAVID J. GROSS*

Lyman Laboratory, Harvard University, Cambridge, Massachusetts

AND

ROMAN JACKIW*

Lyman Laboratory, Harvard University, Cambridge, Massachusetts

and

CERN, Geneva, Switzerland

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A low-energy theorem for the scattering of gravitons from spin-0 particles is derived. We use the dispersion-theoretic method, recently utilized by Abarbanel and Goldberger to derive low-energy theorems for the Compton scattering of photons, to write unsubtracted dispersion relations for physical helicity amplitudes. The scattering amplitude at fixed angle is shown to be given by the Born approximation up to fourth-order terms in the graviton energy.

Fixed-angle dispersion relations for helicity amplitudes
of the form

$$\text{Re Amp} = \frac{1}{\pi} \int_0^\infty \frac{d\omega' \text{Im Amp}(\omega')}{\omega' - \omega}$$

BUT electromagnetic sum rules given above are found
by going to forward direction and using optical theorem

$$\text{Im Amp}(\theta = 0) = \frac{\omega}{4\pi} \sigma(\omega)$$

to write in terms of cross sections.

In gravitational case, however, amplitudes are proportional to $1/t$ and diverge in forward scattering limit, so cannot be written in terms of cross sections. In particular, there is NO gravitational GDH sum rule, which might expect to read

$$0 = \frac{1}{\pi^2} \int_0^\infty \frac{d\omega' (\sigma_{\frac{5}{2}}(\omega') - \sigma_{\frac{3}{2}}(\omega'))}{\omega' - \omega}$$

Another way to see this is from factorization. The Compton amplitudes behave in forward direction as

$$\text{Amp}_C^{S=0} = 2e^2(\epsilon_2^* \cdot \epsilon_1)(1 + \dots)$$

$$\begin{aligned}\text{Amp}_C^{S=\frac{1}{2}} &= 2e^2 [(\epsilon_2^* \cdot \epsilon_1)(1 + \dots) \\ &\quad + i\boldsymbol{\sigma} \cdot \epsilon_2^* \times \epsilon_1 \left(\frac{t}{2(s - m^2)} + \dots \right)]\end{aligned}$$

Then

$$\text{Amp}_{GC}^{S=\frac{1}{2}} = 8\pi G \left[(\epsilon_2^* \cdot \epsilon_1)^2 \left(\frac{(s - m^2)^2}{t} + \dots \right) \right.$$

$$\left. + i(\epsilon_2^* \cdot \epsilon_1) \boldsymbol{\sigma} \cdot \epsilon_2^* \times \epsilon_1 \left(\frac{s - m^2}{2} + \dots \right) \right]$$

so spin-flip amplitude is non-vanishing.

Alternate factorization gives photon-graviton scattering

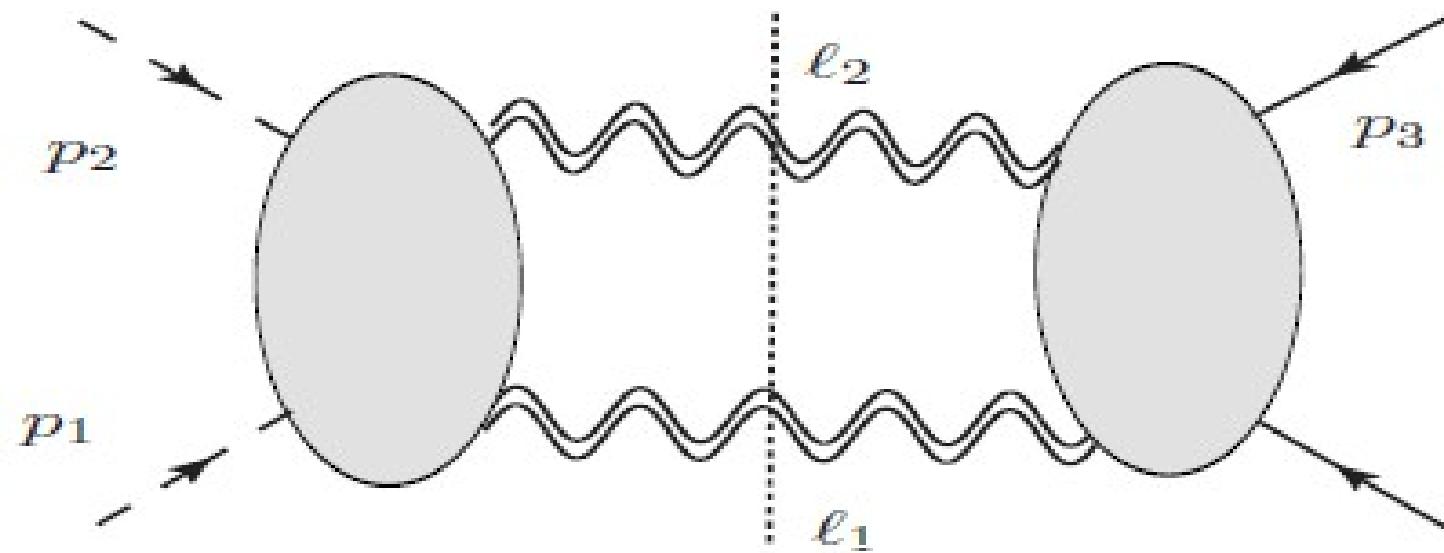
$$\text{Amp}_{GC}(\gamma-g) \sim \lim_{m \rightarrow 0} \frac{\kappa^2}{8e^4} F(\text{Amp}_C(S=1))^2 \cdot \text{Amp}_C(S=0)$$

and even graviton-graviton scattering

$$\text{Amp}(g-g) \sim \lim_{m \rightarrow 0} Y \cdot (\text{Amp}_C(S=1))^2 \cdot \text{Amp}_C(S=1)$$

but look at more magic...

MORE MAGIC—use gravitational Compton amplitude to calculate higher order (one-loop) gravitational interaction via discontinuity across two-graviton t-channel cut



From unitarity

$$\begin{aligned} \text{DiscAmp}_2^{grav}(q) = & \frac{-i}{2!} \frac{1}{4m_A m_B} \int \frac{d^3 \ell_1}{(2\pi)^3 2\ell_1^0} \int \frac{d^3 \ell_2}{(2\pi)^3 2\ell_2^0} \\ & \times (2\pi)^4 \delta^4(p_1 + p_2 - \ell_1 - \ell_2) T_i^{\alpha\beta;\gamma\delta}(p_1, p_2; \ell_1, \ell_2) \\ & \times P_{\alpha\beta;\rho\sigma} P_{\gamma\delta;\xi\zeta} T_f^{*\rho\sigma;\xi\zeta}(p_3, p_4, \ell_1 \ell_2) \end{aligned}$$

Using factorization can be related to electromagnetic Compton amplitudes and, since intermediate state is on-shell there are no ghosts! Then two year calculation becomes a two day calculation! Find

$$\text{DiscAmp}_2^{grav}(q) = -i \frac{\kappa^4 m_A m_B}{128\pi} \\ \times \left[\frac{41}{5} + 6 \frac{\pi(m_A + m_B)}{\sqrt{-q^2}} + 4\pi i \frac{m_A m_B m_r}{p_0 q^2} + \dots \right]$$

$$\text{Amp}_2^{grav}(q) = -\frac{\kappa^4 m_A m_B}{1024\pi^2} \left[\frac{41}{5}L - 6S(m_A + m_B) + 4\pi i \frac{m_A m_B m_r}{p_0 t} L + \dots \right].$$

where $L = \log -q^2$ and $S = \pi^2/\sqrt{-q^2}$. Agrees with Bjerrum-Bohr et al. and Kirilin and Khriplovich.

Simplification comes from factorization and reordering the integration and sum over diagrams. That is, in usual method four-dimensional integration yields many gauge-dependent diagrams which are then summed. Here summation occurs first so two-dimensional integration is over gauge-independent forms.

Lesson is that quantum gravity can now be done fairly simply and can even be put into the classroom. Much more to be done!