Oscillatory particle banding in a rotating fluid

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Particle levitation

A particle in a fluid can be “levitated” at an off-axis point if the fluid rotates

\[ F_v = 6\pi \eta a r \Omega \]

For a spherical particle (Stokes)
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Experimental Apparatus for growing crystals by levitation

- Experimentally, this position is stable
- Crystals were grown in 1991-2 from seeds in super-saturated solution using dynamic levitation.
When there are many particles, a new phenomenon occurs:

The particles accumulate in bands, with separation proportional to the tube diameter ($\Lambda/R \sim 2$)

Band at the tube end

No band at the tube end

$\Omega = 9.4\text{rad/s}$, 220 plexiglass cylinders; $\Lambda/R \sim 4$ in this case
Dependence on parameters:

- Not dependent on type of particle, size, shape
- Not dependent on $\Omega$ if it is in range between settling and centrifuging
- Not dependent on viscosity up to 10X water
- Dependent on buoyancy: bubbles interleave particles.
Hydrodynamic numbers for fluid=water

- Reynolds # for particles $\sim 100$-300
- Eckman # (viscous /Coriolis) $\sim 10^{-3}$
- Rossby # (non-linear/linear) $\sim 0.5$

*These show that the problem is not so simple, and will involve non-linear and viscous effects in the long run*

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Previous work on suspensions in horizontally rotating fluids:

- Observation of banding of 100$\mu$m SiO$_2$ particles in glycerol solutions
  

- Theory to explain banding period observed by Matson et al by means of averaging hydrodynamic interactions between particles
  

- Observation of banding of sand particles in water: work concentrated on orbits of single particles
  

- Observation of banding in crystal growth from supersaturated solutions: attributed possibly to growth dynamics
  

- Observation of banding of different types of particles and bubbles, suggestion of a wave interaction mechanism
  

- Observation of banding of neutrally-buoyant particles in partially-filled rotating tubes- the free surface makes it a different problem
  
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Dependence on the tube length

A model for a resonator with a finite amplification band and wavelength determined by boundary conditions at the ends (e.g. a laser resonator)
Oscillations

- There are two equivalent patterns for each tube length:
  - If the tube has length $n\Lambda$, there can be particle at both ends, or neither
  - If the tube has length $(n + \frac{1}{2})\Lambda$, there is a band at one end and none at the other
- Oscillations between the two possibilities slowly take place.

Short film of the oscillations

oscillations.mpg
Bands at intervals of 3.5 sec. in a tube of length $2\Lambda$ (20 sec oscillation period)

Inertial waves

- In the atmosphere they are related to Rossby waves, which are created when an easterly airstream crosses an obstacle.
- Coriolis force is important (restoring force)
- Solution of Navier-Stokes in a rotating frame of reference.

Navier-Stokes with Coriolis force

\[ \frac{\partial (\rho v)}{\partial t} + \nabla (\rho v) + \Omega \times (\rho v) + 2\Omega \times (\Omega \times \rho r) = -\eta \nabla p + \rho g + F \]

Includes non-linear term, centrifugal force, gravity, viscosity and an external force F

*Note: g is oscillatory at \( \Omega \) in this frame!

Now simplify by ignoring non-linear term and viscosity, and replacing \( p \) by an effective pressure:

\[ p^* = p - \frac{1}{2} \rho (\Omega \times r)^2 - \rho g \cdot r \]

Whence:

\[ \frac{\partial (\rho v)}{\partial t} + 2\Omega \times \rho v = -\nabla p^* + F \]
Now with an oscillatory dependence of $p^*$ on time:

$$p^* = \exp(i\omega t)$$

we get a wave equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p^*}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p^*}{\partial \theta^2} + \left( 1 - \frac{4\Omega^2}{\omega^2} \right) \frac{\partial^2 p^*}{\partial z^2} = f(F)$$

which we solve like the waveguide equation in electro-magnetic theory.

If we assume the particles to be driving the fluid at $\omega=\Omega$ (which is reasonable, but has to be inspected), the homogeneous solution is $p_m^*(r, \theta, z, t)$:

$$p^* = p_0 J_m(\gamma r) \cos(m\theta - \Omega t) \cos(kz)$$

Where $k = \frac{\gamma}{\sqrt{3}}$
The velocity components for the m=−1 mode are

\[ v_r = V_0 \left[ J'_1(\gamma r) + \frac{2J_1(\gamma r)}{\gamma r} \right] \sin(\theta + \Omega t) \cos(kz); \]

\[ v_\theta = V_0 \left[ 2J'_1(\gamma r) + \frac{J_1(\gamma r)}{\gamma r} \right] \cos(\theta + \Omega t) \cos(kz); \]

\[ v_z = \frac{\gamma}{k} V_0 J_1(\gamma r) \sin(\theta + \Omega t) \sin(kz). \]

Why m=-1? Because it is stationary in the lab frame!

Boundary conditions: \( v_r = 0 \) on the curved walls (\( r=R \)) and \( v_z = 0 \) at the ends lead to an eigenvalue equation:

\[ J'_1(\gamma r) + \frac{2J_1(\gamma r)}{\gamma r} = 0 \]

With a first solution \( \gamma R = 2.74 \) and wavelength \( \lambda = 3.97 R \).

Compare with experimental results....
But note that this means \( \lambda = \Lambda \), i.e. the bands are spaced by one wavelength, and not a half wavelength as you might expect. Why?

Fluid velocity in the plane \( kz = 2n\pi \)

(a) shows \( v \) in the lab frame, and (b) in the rotating frame: (a) matches the particle motion in the rotating fluid, (c).
Fluid velocity in the plane $kz=(2n+1)\pi$

A bubble plane

Doesn’t match- so particles don’t accumulate here; but bubbles do match.

In the center plane $x=0$

Ends of tube could be here: $v_z=0$
The oscillations

- In alternate nodal planes the particle accumulate. But any nodal plane satisfies the end boundary conditions.
- If $\omega$ is not exactly equal to $\Omega$, there can be beats between the two patterns: hence the oscillations?
- We have not yet dealt with this satisfactorily.

Viscosity

- Effect of viscosity: what is the viscous damping length of inertial waves?
- As the viscosity is increased, the 3-D particle motion becomes very clear, but the banding gets more complicated. No clear results yet.
- The interaction between the particles and the fluid will have to be introduced as a perturbation.
- At some stage particles are trapped in both types of node (remember the crystal bands, $\Lambda/R \sim 2$, not 4)
• How does viscosity affect the resonance condition?
• If we put the interaction force $F=-\alpha \rho \nu$, we find

$$\frac{\gamma^2}{k^2} = \frac{4\Omega^2}{\Omega^2 + \alpha^2} - 1$$

instead of 3, which increases $\omega$ and decreases $\Lambda$, as seen in the experiments

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Conclusions

• Banding was observed when several particles are suspended in a rotating inviscid fluid.
• The banding period $\Lambda$ is determined approximately by the radius of the tube and exactly by its length.
• Interpreted as excitation of inertial waves in the rotating fluid.
• Particles accumulate in alternate nodes of the standing wave, and bubbles in the interleaving planes.
• Oscillations occur between two degenerate possibilities, the origin of which is not yet clear.