CMB From CFT

Sandip Trivedi

Tata Institute of Fundamental Research, Mumbai.

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Ishan Mata, Suvrat Raju and SPT

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Outline

- Introduction
- The general approach (with motivation from AdS/CFT)
- The specific correlator and its calculation
- Conclusions

Introduction

During inflation universe was approximately deSitter space.

The symmetry group of 4-dim. deSitter space is SO(1,4)

Introduction

The question we will ask :

What constraints does this symmetry impose on correlation functions of the quantum perturbations produced inflation?

Introduction

The answer :

sometimes the constraints can be very powerful.

In one case it completely determines the correlation function.

Fixing the full functional form.

And the normalisation (in terms of the two-point functions).

Introduction

Motivation:

1) Arguments based on symmetries robust and model independent. Particularly important since there is no compelling model of inflation.

2) These arguments allow us to deal with situations where Einstein gravity is not a good approximation.

Introduction

•E.g., inflation could have occurred so that $H \sim M_{st}$

•In that case higher derivative corrections would have to be incorporated (while loop corrections would be small since $M_s < M_{Pl}$).

Introduction

- Given our poor understanding of string theory in time dependent situations we cannot directly calculate the resulting quantum perturbations in such a situation today.

- However symmetry considerations should still hold and conclusions obtained from them should apply here too.

Introduction

- Surprisingly the symmetry constraints have not been investigated thoroughly as yet.
- Maldacena and Piementel (1104.2846) building on Maldacena's earlier paper (astro-ph/0210603) laid down the basic framework and applied it to the three point tensor correlator $< \gamma_{ij}\gamma_{kl}\gamma_{mn} >$

Introduction

- We will extend this to include some scalar perturbations.

- Specifically we consider the correlator $< \zeta\zeta\gamma_{ij} >$

(Some comments on other correlators will be made at the end)

Antoniadis, Mazor, Mottola,
Kehagias, Riotto,
Bzowski, McFaden Skenderis
Larsen, McNees
Larsen Van der Shaar, Leigh
Weinberg
Cheung, Cremenelli, Fitzpatrik,
Kaplan, Senatore
Goldberger, Hui, Nicolis
K. Hinterbichler, L.Hui, J.~Khoury,

Introduction

The symmetries of deSitter space:

$$ds^2 = -dt^2 + e^{2Ht}(dx^i)^2$$

H: Hubble Constant

SO(1,4): $\frac{5.4}{2} = 10$ generators

Translations: 3 $x^i \to x^i + c^i$

Rotations: 3 $x^i \to R^i_j x^j$

Scaling: 1 $t \to t + \frac{1}{H}\ln(\lambda), x^i \to \frac{x^i}{\lambda}$

Symmetries of deSitter Space

- This leaves 3 more isometries. These are the special conformal transformations.

- These will play an especially important role in our subsequent discussion.

Symmetries of deSitter Space

$$x^i \to x^i - 2(b_j x^j)x^i + b^i(\sum_j (x^j)^2 - e^{-2Ht}),$$

$$t \to t + 2b_j x^j.$$

$b_i, i = 1, 2, 3$: parameters specifying the special conformal transformations

Symmetries of deSitter Space

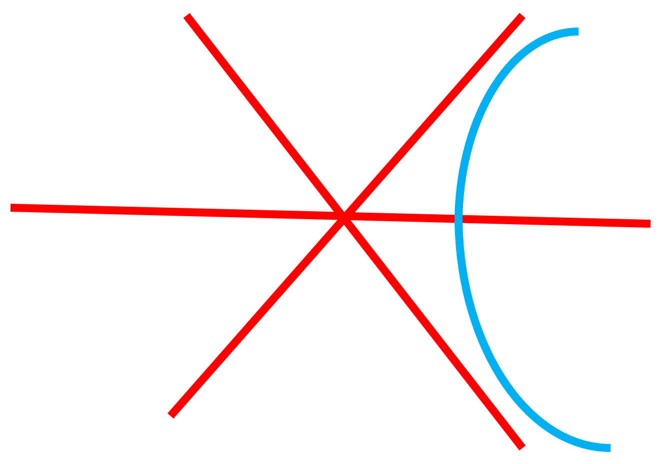
- The group SO(1,4) is also the group of symmetries of a 3 dimensional Euclidean Conformal Field Theory.

- We will call it the conformal group.

Symmetries of deSitter Space

deSitter space can be thought of as a hyperboloid embedded in flat 5+1 dimensional Minkowski space

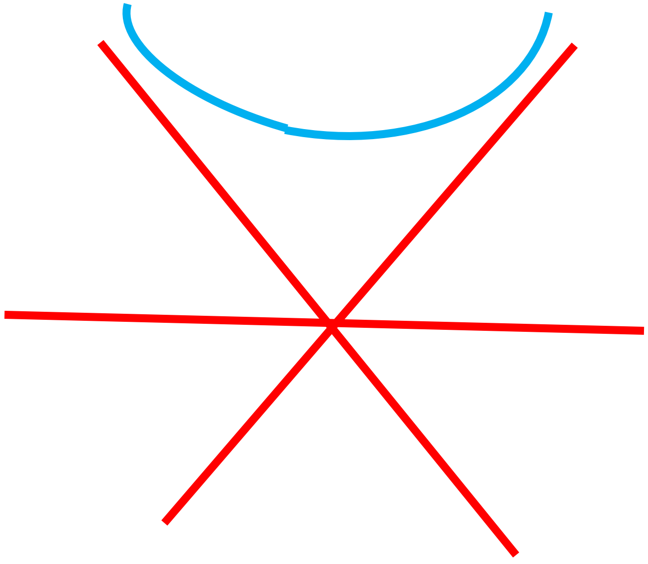
$$X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_0^2 = R^2$$



Symmetries of deSitter Space

Euclidean Anti-deSitter space can also be thought of as a hyperboloid embedded in flat 5+1 dimensional Minkowski space

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_0^2 = R^2$$



Introduction

Symmetries of Euclidean AntideSitter 4 dim. space are the same as those of a 3 dim. Euclidean CFT (AdS/CFT correspondence).

The fact that deSitter space and EAdS are related by analytic continuation will help organise our discussion of the symmetries.

Introduction

During inflation the geometry is not exactly deSitter space. Corrections can be measured in terms of slow roll parameters:

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \ll 1$$

$$\delta = \frac{H\ddot{H}}{\dot{H}^2} \ll 1$$

And are small.

Introduction

For our symmetry considerations to apply the scalar sector must also preserve the full SO(1,4) symmetry to good approximation.

Introduction

- The measured two point correlation provides good evidence for approximate scale invariance($n_s \simeq 0.96$)
- Requiring a time dependent metric which is translational and rotationally invariant and also approximately scale invariant leads to deSitter space which has the full conformal group of symmetries.

Introduction

dS space:

$$ds^2 = -dt^2 + e^{2Ht}(dx^i)^2$$

Introduction

- However a scalar theory which has scale invariance need not be invariant under the full conformal group.

Example:

$$S = \int \{ (\partial_t \phi)^2 - \left[(\partial_i \phi)^2) \right]^2 \}$$

- Our considerations will not apply to such cases. This includes DBI, Ghost Inflation etc. Some of these are anyways disfavoured by Planck Data.

Introduction

•Considerations apply to the canonical model of slow –roll inflation.

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi G}[R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)]$$

$$\phi = \bar{\phi} + \delta\phi$$

•Action for small perturbations to leading order:

$$S_{\delta\phi} = \int d^4 \sqrt{-g} \frac{1}{16\pi G}[-\frac{1}{2}(\nabla\delta\phi)^2]$$

Introduction

- This action only sees the background through the geometry. The symmetries of the geometry are therefore symmetries for the scalar perturbations as well.

(A more precise discussion of gauge fixing and scalar perturbations will follow).

Introduction

$$S - \int d^4x\sqrt{-g}\frac{1}{16\pi G}[R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \frac{1}{\Lambda^2}R^2 + \frac{1}{\Lambda^4}R^3 + \cdots]$$

The higher derivative terms involve the curvature and would be important if

$$H \sim \Lambda < M_{Pl}$$

This could happen for example if

$$\Lambda = M_{st} < M_{pl}$$

E.g. if

$$M_{st} \sim \frac{M_{Gut}^2}{M_{pl}} \sim 10^{14} Gev$$

Introduction

$$S - \int d^4x\sqrt{-g}\frac{1}{16\pi G}[R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \frac{1}{\Lambda^2}R^2 + \frac{1}{\Lambda^4}R^3 + \cdots]$$

Higher derivative terms involving scalars could have a small effect in determining the background if the scalar rolled slowly.

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \ll 1$$

Introduction

Higher derivative terms quadratic in inflaton would be important for perturbations. These would respect approximately the conformal symmetry.

Introduction

In short, our analysis would apply to models where higher derivatives become important, as long as the conformal symmetry group is approximately preserved.

Introduction

Summary of assumptions:

1) The conformal group SO(1,4) is an approximate symmetry during inflation.
   The slow roll parameters are small:

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \ll 1$$

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \ll 1$$

$$\delta = \frac{H\ddot{H}}{\dot{H}^2} \ll 1$$

Introduction

Summary of assumptions:

2) One scalar field

3) Initial state: Bunch Davies vacuum

General Symmetry Considerations

- Symmetry considerations are often usefully stated in terms of the wave function.

- If the state is symmetric the wave function is invariant, up to a possible phase, under the symmetry transformation.

Symmetry Considerations

- In particle physics the wave function is not natural. Rather we compute the S-matrix.

- In cosmology on the other hand, the wave function is natural since we are interested in expectation values at some instance of time.

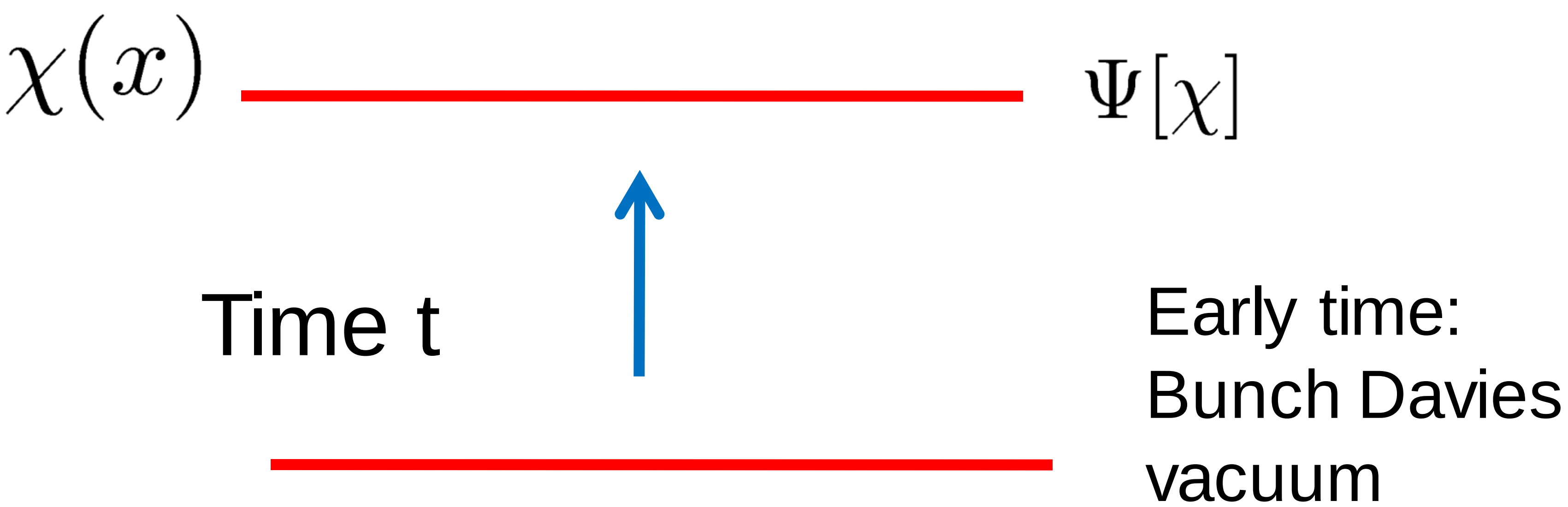
Symmetry Considerations

- This difference will be important. E.g., contact terms are important in these expectation values and contribute to observational effects as importantly as the non-contact terms.

- These terms are not usually important for S-matrix elements.

Symmetry Considerations

We will be interested in the wave function at late times, when the modes of interest have exited the horizon and stopped time evolving.



Symmetry Considerations

For a system which is close to Gaussian we can expand schematically as follows:

$$\Psi[\chi] = exp(-\frac{1}{2}\int d^3x d^3y \chi(x)\chi(y) < O(x)O(y) > + \frac{1}{6}\int d^3x d^3y d^3z \chi(x)\chi(y)\chi(z) < O(x)O(y)O(z) > + \cdots)$$

$< (O(x)O(y) >, < O(x)O(y)O(z) >$ for now are just coefficients which determine the wave function.

Symmetry Considerations

- Symmetry considerations lead to conditions on the coefficient functions.

- We will find that the coefficient functions under the conformal group will behave like appropriate correlators in a Conformal field theory.

Symmetry Considerations

- This will allow us to recast our study of symmetries into question of the constraints imposed in a conformal field theory on appropriate correlators.

- This is the central idea behind the analysis.

Symmetry Considerations

For example, suppose the perturbation χ is a scalar which we denote here as $\delta\phi$.

Under the scaling isometry

$$t \to t + \frac{1}{H}\ln(\lambda), x^i \to \frac{x^i}{\lambda}$$

$$\delta\phi(x,t) \to \delta\phi'(x,t) = \delta\phi(\frac{x}{\lambda}, t + \frac{1}{H}\log(\lambda))$$

Symmetry Considerations

If $\delta\phi$ is time independent

$$\delta\phi(x) \rightarrow \delta\phi'(x) = \delta\phi(\tfrac{x}{\lambda})$$

For the wave function to be invariant

$$\Psi[\delta\phi(x)] = \Psi[\delta\phi(\tfrac{x}{\lambda})]$$

Symmetry Considerations

- In $\psi[\delta\phi]$ every additional power of $\delta\phi$ is accompanied by O(x) and an integral over x with measure d^3x.

- So schematically speaking this means $\int d^3x \delta\phi(\frac{x}{\lambda})O(x)$ must be invariant.

- By a change of variables

$$\int d^3x \delta\phi(\frac{x}{\lambda})O(x) = \int d^3x \delta\phi(x)O(\lambda x)\lambda^3$$

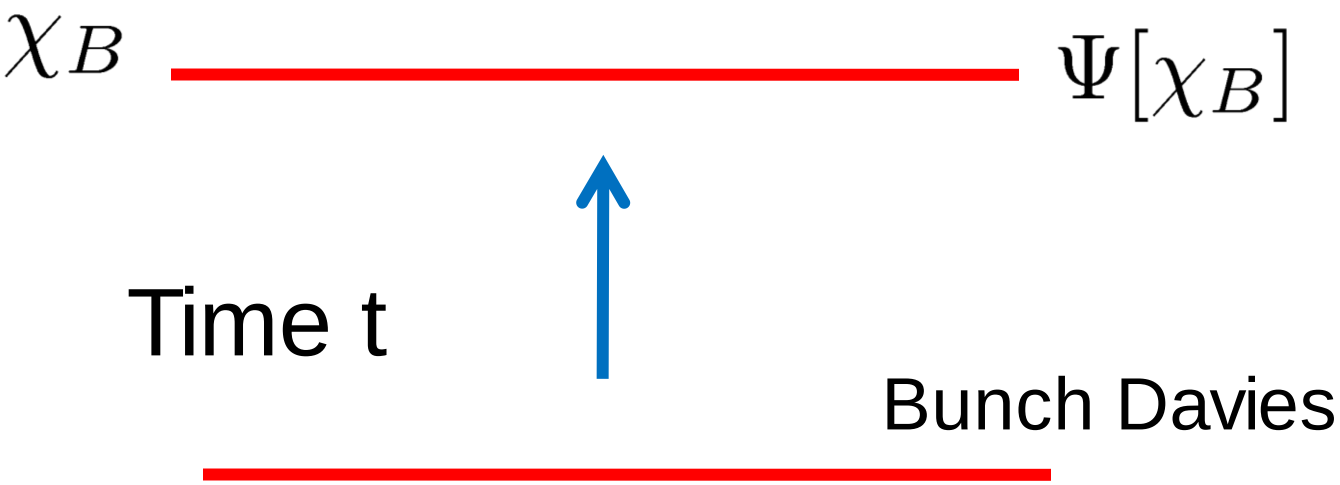
Symmetry Considerations

- Since this is true for arbitrary $\delta\phi$ we learn that coefficient functions must be invariant under the transformation $O(x) \to \lambda^3 O(\lambda x)$

- That is O(x) must behave like an operator of a CFT with dimension 3
- (marginal operator in 3 dim.).

Symmetry Considerations

The wave function can be obtained by doing a path integral in dS space:



$$\Psi[\chi_B] = \int^{\chi_b} D\phi e^{iS}$$

Symmetry Considerations

- The conformal symmetries are isometries and the path integral is invariant under coordinate transformations.

- Also the Bunch Davies boundary conditions are conformally invariant.

If $\chi_B \to \chi'_B$ we learn that $\Psi[\chi_B] = \Psi[\chi'_B]$

Symmetry Considerations

- Bunch Davies boundary conditions are conformally invariant because a massless scalar in flat space in Poincare invariant vacuum preserves conformal invariance.

- Alternatively path integral can be done by taking (conformal time) to imaginary values $\eta \to \eta(1 - i\epsilon), \epsilon > 0$ and taking field to vanish as $\eta \to -\infty$

Symmetry Considerations

Then since the fields transform homogeneously under the conformal transformations we see that the Bunch Davies boundary conditions preserve these symmetries.

More on perturbations

Rotational invariance along spatial directions can be used to characterise perturbations.

There is one tensor (spin 2) and one scalar perturbation (spin 0).

More on Perturbations

Metric in ADM-like form:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$h_{ij} = a^2[(1 + 2\zeta)\delta_{ij} + \gamma_{ij}]$$

Choose a gauge where $\partial_i \gamma_{ij} = \gamma_{ii} = 0$

γ_{ij} Tensor perturbation.

Scalar Perturbation

Inflaton : $\phi = \bar{\phi} + \delta\phi$

Gauge 1: $\delta\phi = 0, \zeta$

Gauge 2: $\zeta = 0, \delta\phi$

Scalar Perturbation

To go between the two gauges we do a time reparametrisation

$$t \to t - \frac{\zeta}{H}$$

$$\delta\phi = -\frac{\dot{\phi}}{H}\zeta$$

We will find it convenient to do work in gauge 2 for calculating the correlations, then convert to gauge 1 after the end of inflation.

Scalar Perturbation

- Inverting the relation gives

$$\zeta = -\frac{1}{\sqrt{2\epsilon}}\delta\phi$$

- The limit where the slow roll parameters go to zero can therefore be sometimes confusing in terms of the ζ variable.

Scalar Perturbation

- This is useful because it allows us to think about the conformally invariant limit in a straightforward manner.

- While also allowing us to follow what happens after the modes cross the horizon in a simple way, since ζ
- is conserved.

Symmetry Considerations

Working in gauge 2 the wave function:

$$\begin{aligned}\psi[\delta\phi,\gamma_{ij}] \quad = \quad & exp[\frac{M_{pl}^2}{H^2}(-\frac{1}{2}\int d^3x d^3y \delta\phi(\vec{x})\delta\phi(\vec{y})\langle O(\vec{x})O(\vec{y})\rangle \\ & -\frac{1}{2}\int d^3x d^3y \gamma_{ij}(\vec{x})\gamma_{kl}(\vec{y})\langle T^{ij}(\vec{x})T^{kl}(\vec{y})\rangle \\ & -\frac{1}{4}\int d^3x d^3y d^3z \delta\phi(\vec{x})\delta\phi(\vec{y})\gamma_{ij}(\vec{z})\langle O(\vec{x})O(\vec{y})T^{ij}(\vec{z})\rangle + \cdots)].\end{aligned}$$

The coefficient functions will transform correctly under conformal transformations if $O(x), T_{ij}(x)$ transform like a scalar operator of dimension 3 and the stress energy tensor of a CFT respectively.

The stress tensor of a CFT is conserved. This property follows for T_{ij} from invariance of ψ under coordinate transformations

$$x^i \to x^i + v^i(x)$$

The ward identities due to stress energy conservation are also therefore met.

General Considerations

One important point is that we will work in momentum space.

Often calculations in CFT are done in coordinate space.

Fourier transforming position space answers is not straightforward. (This is also related to the issue of contact terms).

General Considerations

For example:

$$\begin{aligned}\partial_{x^i}\langle T_{ij}(\vec{x})O(\vec{y_1})O(\vec{y_2})\rangle &= [\partial_{x^j}\delta^3(\vec{x}-\vec{y_1})]\langle O(\vec{y_1})O(\vec{y_2})\rangle \\ &+[\partial_{x^j}\delta^3(\vec{x}-\vec{y_2})]\langle O(\vec{y_1})O(\vec{y_2})\rangle .\end{aligned}$$

The Specific Correlator

Specific Correlator

$$< \delta\phi(\vec{k}_1)\delta\phi(\vec{k}_2)\gamma_{ij}(\vec{k}_3) >$$

Related to

$$< O(\vec{k}_1)O(\vec{k}_2)T_{ij}(\vec{k}_3) >$$

The Specific Correlator

I) Using Translational and Rotational Invariance

$$
\begin{aligned}
\langle O(\vec{k}_1)O(\vec{k}_2)T_{ij}(\vec{k}_3)\rangle \quad &= \quad [k_{1i}k_{1j}f_1(k_1,k_2,k_3) + k_{2i}k_{2j}f_1(k_2,k_1,k_3) \\
&\quad +(k_{1i}k_{2j} + k_{2i}k_{1j})f_2(k_1,k_2,k_3) \\
&\quad +\delta_{ij}f_3(k_1,k_2,k_3)](2\pi)^3\delta^3(\sum_i \vec{k}_i).
\end{aligned}
$$

The Specific Correlator

I) Using Translational and Rotational Invariance

$$\begin{aligned}\langle O(\vec{k}_1)O(\vec{k}_2)T_{ij}(\vec{k}_3)\rangle \quad = \quad & [k_{1i}k_{1j}f_1(k_1,k_2,k_3) + k_{2i}k_{2j}f_1(k_2,k_1,k_3) \\ & +(k_{1i}k_{2j} + k_{2i}k_{1j})f_2(k_1,k_2,k_3) \\ & +\delta_{ij}f_3(k_1,k_2,k_3)](2\pi)^3\delta^3(\sum_i \vec{k}_i).\end{aligned}$$

The Specific Correlator

From scale invariance f_1, f_2, f_3 must have dimension 1.

The Specific Correlator

II) Additional constraints take the form

$$< \delta O(\vec{k}_1) O(\vec{k}_2) T_{ij}(\vec{k}_3) > + < O(\vec{k}_1) \delta O(\vec{k}_2) T_{ij}(\vec{k}_3) > \\ + < O(\vec{k}_1) O(\vec{k}_2) \delta T_{ij}(\vec{k}_3) > \;=\; 0$$

For the special conformal transformations these result in 3 coupled second order differential equations.

Specific Correlator

For special conformal transformations

$$\begin{aligned}
\delta O(\vec{k}) &= -\tilde{D} O(\vec{k}) \\
\delta T_{ij}(\vec{k}) &= 2\tilde{M}_i^l T_{lj} + 2\tilde{M}_j^l T_{il} - \tilde{D} T_{ij}, \\
\tilde{M}_i^l &\equiv b^l \partial_{k^i} - b^i \partial_{k^l}, \\
\tilde{D} &\equiv \vec{b} \cdot \vec{k} \partial_{k^i} \partial_{k^i} - 2k_j \partial_{k_j} (\vec{b} \cdot \vec{\partial}_k).
\end{aligned} \tag{1}$$

Specific Correlator

For special conformal transformations

$$\delta O(\boldsymbol{k}) = -\bar{D} O(\boldsymbol{k}),$$

$$\delta T_{ij}(\boldsymbol{k}) = 2\bar{M}_i^{\,l} T_{lj} + 2\bar{M}_j^{\,l} T_{il} - \bar{D} T_{ij},$$

$$\bar{M}_i^{\,l} \equiv b^l \partial_{k^i} - b^i \partial_{k^l},$$

$$\bar{D} \equiv (\boldsymbol{b}\cdot\boldsymbol{k})\partial_{k^i}\partial_{k^i} - 2k_j \partial_{k_j}(\boldsymbol{b}\cdot\boldsymbol{\partial}_{\boldsymbol{k}}).$$

The Specific Correlator

III) Finally we also impose the ward identities due to stress energy conservation.

$$\langle O(\vec{k}_1)O(\vec{k}_2)T_{ij}(\vec{k}_3)\rangle = M_{ij}(\vec{k}_1, \vec{k}_2, \vec{k}_3)(2\pi)^3\delta(\sum_i \vec{k}_i).$$

$$M_{ij}k_3^j = -k_1^3 k_1^j - k_2^3 k_2^j$$

The specific correlator

The general solution can be expressed in terms of 4 functions.

$$< O(k_1)O(k_2)T_{ij}(k_3) > e^{s,ij} = -2(2\pi)^3 C\delta^3(\sum_i \vec{k}_i)e^{s,ij}k_{1i}k_{2j}S$$

$e^{s,ij}$ transverse traceless wrt $\vec{k}_3$

C : constant fixed by normalisation of <OO> two point function

Specific Correlator

General Solution

$$S = \sum_{n_1,n_2=\pm 1} m_{n_1 n_2}\Big(\quad -n_1 n_2 \frac{k_2 k_3 k_1}{(n_1 k_1 + n_2 k_2 + k_3)^2} + n_1 k_1 + n_2 k_2 + k_3$$

$$-\frac{n_1 n_2 k_1 k_2 + n_2 k_3 k_2 + n_1 k_1 k_3}{n_1 k_1 + n_2 k_2 + k_3}\Big).$$

$$\sum_{n_1,n_2} m_{n_1,n_2} n_1^3 \quad = \quad 1,$$

$$\sum_{n_1,n_2} m_{n_1,n_2} n_2^3 \quad = \quad 1.$$

Specific Correlator

One now needs to consider special limits.

1) $$k_3 \ll k_1, k_2$$

2) $$k_1 \ll k_2, k_3$$

These can be thought of using the operator product expansion in CFT or, alternatively, purely in bulk terms (Maldacena consistency conditions).

Specific Correlator

For example the first limit corresponds to calculating the two point scalar perturbation in the presence of an essentially constant metric γ_{ij}

These two limits allow only one term in S leading to a unique answer.

Specific Correlator

Final answer:

$$< \zeta(k_1)\zeta(k_2)\gamma_s(k_3) >= (2\pi)^3\delta(\sum_i \vec{k}_i)\frac{8}{\Pi_i(2k_i^3)}P_S P_T e^{sij}k_{1i}k_{2j}S$$

$$S = (k_1 + k_2 + k_3) - \frac{\sum_{i>j} k_i k_j}{(k_1 + k_2 + k_3)} - \frac{k_1 k_2 k_3}{(k_1 + k_2 + k_3)^2}$$

$$< \zeta(k_1)\zeta(k_2) >= (2\pi)^3\delta^3(\sum(\vec{k}_1 + \vec{k}_2)P_s\frac{1}{k_1^3}$$

$$< \gamma_s(\vec{k}_1)\gamma_{s'}(\vec{k}_2) >= (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2)P_T\frac{\delta_{s,s'}}{k_1^3}$$

Specific Correlator

1) Functional Form completely fixed by symmetries.

2) Normalisation fixed in terms of the two point functions of the scalar and tensor perturbations.

Specific Correlator

- Agrees with the canonical slow roll model (Maldacena).

- Result is small. Roughly like having $f_{NL} \sim \epsilon$ (but with a tensor mode!)

Specific Correlator: Comments

$$S = (k_1 + k_2 + k_3) - \frac{\sum_{i>j} k_i k_j}{(k_1 + k_2 + k_3)} - \frac{k_1 k_2 k_3}{(k_1 + k_2 + k_3)^2}$$

Analytic in some momenta.
In position space would correspond to a contact term which contains a delta function in some coordinates.

Specific Correlator

The correlator has been calculated in position space for a general 3 dim. CFT in 1994 by Osborne and Petkou for non-coincident points.

However result does not have a well defined fourier transform. Regularising the integral is tied to also the issue of contact terms.

Other Correlators

- Other correlators: ongoing study.

- Especially $< \zeta\zeta\zeta >$

  Small $f_{NL} \sim O(\epsilon)$

(in approximately conformal invariant case).

Tied to the fact that in a CFT

<OOO> for an exactly

marginal operator vanishes

Other Correlators

- 4-point correlator $< \zeta\zeta\zeta\zeta >$

 Non vanishing in exactly conformally invariant limit.

- But small $\tau_{NL}, g_{NL} \sim \epsilon$

- Not unique (two cross ratios)

Summary

- Conformal Invariance can be a powerful constraint.

- The correlator discussed here can serve as a good model independent probe for approximate conformal invariance during inflation.

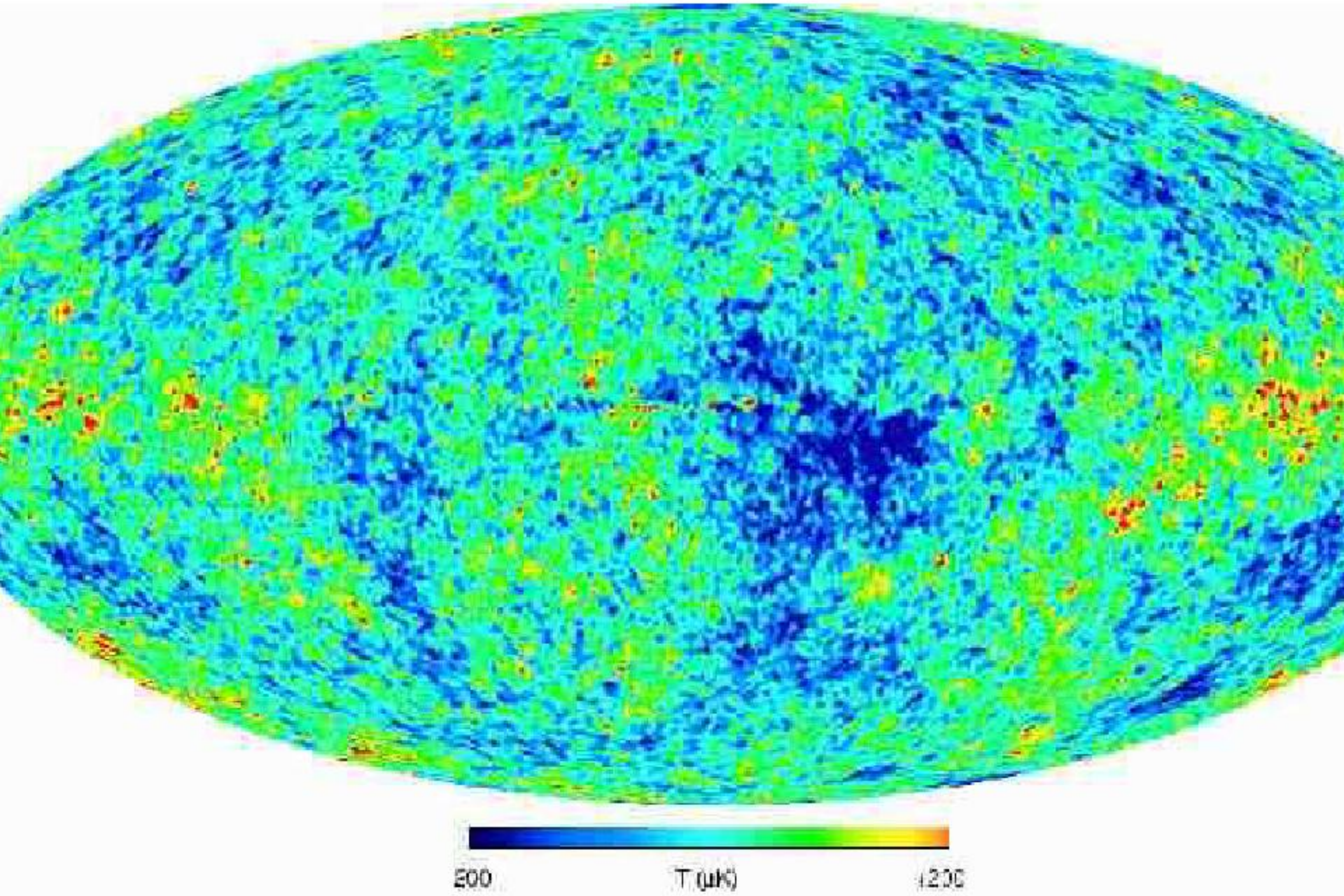
Summary

- The smallness of the non-gaussianity when conformal invariance is a good approximate symmetry is a damper. But it is consistent with Planck.

- Perhaps future observations, on large scale structure etc can measure such a small non gaussianity.

Summary

Perhaps future observations will discover a larger non gaussianity. This would teach us that conformal invariance was not a good symmetry of the inflationary universe.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