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Mass gap and confinement in (2+1) dim Yang-Mills theory

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Why (2+1) dim?

a) Easier to analyze, guide to 4d case

b) Approximation to high-T 4D YM

mass gap in YM_3 ~ magnetic mass in high-T YM_4

$(e^2 = g^2 T)$

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Outline and important results

- Hamiltonian analysis
- Gauge invariant formulation
- Explicit calculation of integration measure in gauge invariant configuration space $\frac{\mathcal{A}}{\mathcal{G}}$
- Schrodinger equation
- origin of mass gap
- analytical calculation of vacuum wave function and string tension (excellent agreement with lattice)

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Hamiltonian

$A_0 = 0$ SU(N) YM

$$\mathcal{H} = \underbrace{\frac{e^2}{2}\int E_i^a E_i^a}_{T} + \underbrace{\frac{1}{2e^2}\int B^a B^a}_{V}$$

$e^2 \sim$ [mass]

Gauge invariant parametrization

$z = x_1 - i x_2$ $\bar{z} = x_1 + i x_2$

$A = \frac{1}{2}(A_1 + i A_2)$ $\bar{A} = \frac{1}{2}(A_1 - i A_2)$

$A = -\partial M M^{-1}$ $\bar{A} = M^{\dagger -1}\bar{\partial} M^\dagger$

M: (N×N) complex matrix $(SU(N)^{\mathbb{C}})$

Under gauge transformations:

$A_i \to g A_i g^{-1} - \partial_i g\, g^{-1}$

$M \to g M$ $M^\dagger \to M^\dagger g^{-1}$

$\boxed{H = M^\dagger M}$ gauge invariant hermitian field

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$M = U\rho$ (ρ → hermitian part; U ↑ unitary part)

$H = M^\dagger M = \rho^2 \in \frac{SL(N,\mathbb{C})}{SU(N)}$

Ambiguity in defining $M, M^\dagger$

$$\left.\begin{aligned} A &= -\partial M M^{-1} \\ \bar{A} &= M^{\dagger -1}\bar{\partial} M^\dagger \end{aligned}\right\} \text{invariant if} \quad \begin{aligned} M &\to M\bar{V}(\bar{z}) \\ M^\dagger &\to V(z) M^\dagger \end{aligned}$$

Invariance of theory under

$H \to V(z)\, H\, \bar{V}(\bar{z})$

"holomorphic invariance"

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H parametrize configuration space $\mathcal{C}$

$$\mathcal{C} = \frac{\text{gauge potentials}}{\text{gauge transformations}} = \frac{\mathcal{A}}{\mathcal{G}}$$

<u>Integration measure in $\mathcal{C}$</u>

$\mathcal{A}$: $ds^2 = \mathrm{tr}\int \delta A\, \delta \bar{A}$

$= \mathrm{tr}\int D(\delta M M^{-1})\, \bar{D}\, (M^{+-1}\delta M^{+})$

$D = \partial + [A,\ ]$ $\qquad \bar{D} = \bar{\partial} + [\bar{A},\ ]$

measure $= d\mu(\mathcal{A}) = \prod_{x,a} dA^a(x)\, d\bar{A}^a(x)$

$= \det(D\bar{D})\, \underbrace{d\mu(M,M^+)}_{\text{Haar measure for } SL(N,\mathbb{C})}$

$M = Up$

$$d\mu(M,M^+) = \underbrace{d\mu(H)}_{\substack{\text{gauge inv.}\\ \text{part}}}^{=\prod_x (\partial H H^{-1})^n} \quad \underbrace{d\mu(U)}_{\text{volume of } \mathcal{G}}^{=\prod_x (U^{-1}dU)^n}$$

$$\left(\begin{array}{l} H = e^{t^a \phi^a} \\ H^{-1}\delta H = \delta\phi^a\, r_{ab}\, t^b \end{array} \rightarrow d\mu(H) = \det r \prod_{x,a} d\phi^a \right)$$

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$d\mu(\mathcal{A}) = \det(D\bar{D})\; d\mu(H)\; \underbrace{d\mu(U)}_{\text{volume of gauge transf.}}$

$$\mathcal{C} = \frac{\mathcal{A}}{\mathcal{G}} : \quad d\mu(\mathcal{C}) = \frac{d\mu(\mathcal{A})}{d\mu(\mathcal{G})}$$

$$= \det(D\bar{D})\; d\mu(H)$$

Using gauge invariant regulator

$$\boxed{d\mu(\mathcal{C}) = e^{2C_A S(H)}\; d\mu(H)}$$

$C_A\, \delta^{ab} = f^{amn} f^{bmn}$; $\quad C_A = N$ for $SU(N)$

$S(H)$ = WZW action

$$S(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi}\int \mathrm{Tr}(H^{-1}dH)^3$$

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Inner product for physical states

$$\langle 1|2\rangle = \int d\mu(H)\; e^{2c_A S(H)}\; \Psi_1^*(H)\,\Psi_2(H)$$

$$\left\{\begin{matrix}\text{matrix elements}\\ \text{of}\\ (2+1)\text{d YM}\end{matrix}\right\} = \left\{\begin{matrix}\text{correlators of}\\ \text{hermitian WZW}\\ \text{(CFT)}\end{matrix}\right\}$$

- correlators of $e^{(k+2c_A)S}$ hermitian WZW

only integrable reps (spin $\leq \frac{k}{2}$) have finite correlators

(nonintegrable correlators $= \infty$)

since $k=0$, only primary operator $\mathbb{1}$

$\Rightarrow$ Finite norm wavefunctions Ψ are functions of

$$\boxed{J = \frac{c_A}{\pi}\,\partial H H^{-1}}$$

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- Wilson loop expressed in terms of J

$$W(C) = \mathrm{Tr}\, P\, e^{-\oint_C (A\,dz + \bar{A}\,d\bar{z})}$$

$$= \mathrm{Tr}\; P\; e^{-\frac{\pi}{c_A}\oint J}$$

$$A = -\partial M M^{-1} = -M^{\dagger -1}\,\partial H H^{-1} M^{\dagger} + M^{\dagger -1}\partial M^{\dagger}$$

$$\bar{A} = \quad M^{\dagger -1}\,\bar{\partial} M^{\dagger}$$

$(A, \bar{A})$ = complex gauge transform of $(-\partial H H^{-1},\, 0)$

Gauge invariant states generated by J

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Hamiltonian in terms of J

$$V = \frac{1}{2e^2}\int B^2 = \frac{\pi}{mC_A}\int \bar\partial J_a \; \bar\partial J_a$$

$$\boxed{m = \frac{e^2 C_A}{2\pi}}$$

$$T\,\Psi(J) = -\frac{e^2}{2}\int \frac{\delta^2 \Psi}{\delta A(x)\,\delta\bar A(x)}$$

$$= -\frac{e^2}{2}\int \frac{\delta^2 J(u)}{\delta A(x)\,\delta\bar A(x)}\,\frac{\delta\Psi}{\delta J(u)} + \frac{\delta J(u)}{\delta A(x)}\,\frac{\delta J(v)}{\delta\bar A(x)}\,\frac{\delta^2\Psi}{\delta J(u)\,\delta J(v)}$$

$$= m\left[\int J_a \frac{\delta}{\delta J_a} + \int \Omega_{ab}(x,y)\,\frac{\delta}{\delta J^a(x)}\,\frac{\delta}{\delta J^b(y)}\right]\Psi$$

$$\Omega_{ab}(x,y) = \left[\frac{C_A}{\pi^2}\,\delta_{ab}\,\partial_y - i f_{abc} J_c(y)\right]\frac{1}{\pi(x-y)}$$

$$\boxed{T\,J_a(x) = m\,J_a(x)}$$

(nonperturbative "gluons" with dynamical mass)

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"Holomorphic" invariance:

$$H \to V(z)\,H\,\bar V(\bar z)$$

J not invariant

$$\bar\partial J \to V\,\bar\partial J\,V^{-1}$$

Invariant 2J- state (lowest glueball)

$$\Psi_2 = \int f(x,y)\,\mathrm{tr}\; :\bar\partial J_a(x)\,\left[H(x,\bar y)\,H^{-1}(y,\bar y)\right]_{ab}\,\bar\partial J_b(y):$$

$x \to y$

$$T\; :\bar\partial J_a(x)\,\bar\partial J_a(x): \;= 2m\; :\bar\partial J_a(x)\,\bar\partial J_a(x):$$

$$\boxed{T\;:V: \;=\; 2m\;:V:}$$

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Vacuum wavefunction

Consider only T (strong coupling)

$T\Psi_0 = 0 \Rightarrow \Psi_0 = 1$ normalizable

$$\langle \Psi_0 | \Psi_0 \rangle = \int e^{2C_A S} d\mu(H)$$ WZW partition function

$\sim$ volume of $\mathcal{C}$

= "finite" $(=\infty$ for $C_A = 0)$

Include potential V

$(T+V)\Psi = 0 \Rightarrow \Psi = e^P$

For $k \ll e^2 \sim m$ treat V perturbatively,

P expanded in powers $\frac{1}{m}$

$$P = -\frac{\pi}{m^2 C_A} \mathrm{Tr} \int :\bar{\partial} J \, \bar{\partial} J:$$
$$-\left(\frac{\pi}{m^2 C_A}\right)^2 \mathrm{Tr} \int :\bar{\partial} J (D\bar{\partial}) \bar{\partial} J + \frac{1}{3} \bar{\partial} J [J, \bar{\partial}^2 J]:$$
$$+ \ldots$$

where $D = \frac{C_A}{\pi}\partial - [J, \;]$

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Sum up all (infinite many) "2J", "3J"... terms

$$\Psi = e^P$$

$$P = -\frac{2}{e^2} \left[\left(\frac{\pi}{C_A}\right)^2 \int \bar{\partial} J_a \left[\frac{1}{m + \sqrt{m^2 - \nabla^2}} \right] \bar{\partial} J_a \right.$$
$$\left. + f_{abc} \int f^{(3)}(x,y,z) \, J_a(x) J_b(y) J_c(z) \right]$$

$$f^{(3)}(\vec{k},\vec{p},\vec{q}) = (2\pi)^3 \delta(\vec{k}+\vec{p}+\vec{q}) \left(\frac{\pi}{2C_A}\right)^3 \times \frac{(E_k - m)(E_p - m)}{E_k + E_p + E_q} \frac{\bar{k} - \bar{p}}{kp}$$

$$E_k = \sqrt{m^2 + \vec{k}^2}$$

"2J" $\gg$ "3J" $\quad k \gg m$, $k \ll m$

1) "3J" term nonlocal in B

2) smooth interpolation between short and long distance regimes

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$$\Psi = e^{P}$$

$$\sim e^{-\frac{1}{2e^2}\int B \frac{1}{\sqrt{-\nabla^2}} B} \quad \text{for } k > m$$

(perturbative limit)

$$\sim e^{-\frac{1}{4e^2 m}\int B^2} \quad \text{for } k \ll m$$

Consider leading order term $k \ll m$

$$\langle \psi | \mathcal{O} | \psi \rangle = \int \underbrace{e^{-\frac{1}{2me^2}\int B^2}}_{\substack{\text{2d (Euclidean) YM with} \\ g^2 = me^2}} \mathcal{O}$$

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String tension

Wilson loop

$$\langle W_F(C) \rangle = e^{-e^4 \frac{c_A c_F}{4\pi} \mathcal{A}(C)}$$

$$= e^{-\sigma \mathcal{A}(C)}$$

σ: sting tension $= e^4 \frac{N^2-1}{8\pi}$ for $SU(N)$

Comparison with lattice (Teper)

$\frac{\sqrt{\sigma}}{e^2}$:

	lattice	ours	
SU(2)	0.335	0.345	3%
SU(3)	0.553	0.564	2%
SU(4)	0.759	0.772	2%
SU(5)	0.966	0.977	1%
SU(6)	1.167	1.180	1%

$N \to \infty$ extrapolation

$\sqrt{\sigma} = e^2 N \times 0.1976 \qquad \times 0.1995 \qquad < 1\%$

For higher representations we find

Casimir scaling : $\frac{\sigma_R}{\sigma} = \frac{C_R}{C_F}$

(in agreement with lattice)

Group \ Rep	Fundamental	k=2 antisym	k=3 antisym	k=2 sym	k=3 sym	k=3 mixed
SU(2)	0.345 0.335	N/A	N/A			N/A
SU(3)	0.564 0.553	N/A	N/A			
SU(4)	0.772 0.759	0.891 0.883	N/A	1.196 1.110 *		
SU(5)	0.977 0.966					
SU(6)	1.180 1.167	1.493 1.484	1.583 1.569	1.784 1.727	2.318 2.251	1.985 1.921
$N \to \infty$	$N \times 0.1995$ $N \times 0.1976$					

difference $\lesssim 3\%$ (*)