# A Basis for Two-Loop Integrals

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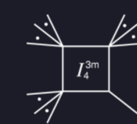
# Amplitudes in Gauge Theories

- Basic building block for physics predictions in QCD
- Powerful means of investigating structure of weak-coupling expansion — and strong coupling one! — in N = 4 SUSY
- Explicit calculations have lead to a lot of progress in discovering new symmetries (dual conformal symmetry) and new structures not manifest in the Lagrangian or on general grounds

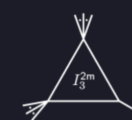
# On-Shell Methods

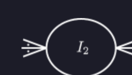
- Use only information from physical states
- Use properties of amplitudes as calculational tools
  - Factorization → on-shell recursion relations (Britto, Cachazo, Feng, Witten,…)
  - Unitarity → unitarity method (Bern, Dixon, Dunbar, DAK,…)
  - Underlying field theory → integral basis




- Formalism

Known integral basis:




$$\text{Ampl} = \sum_{j \in \text{Basis}} c_j \, \text{Int}_j + \text{Rational}$$

$c_j$ → Unitarity

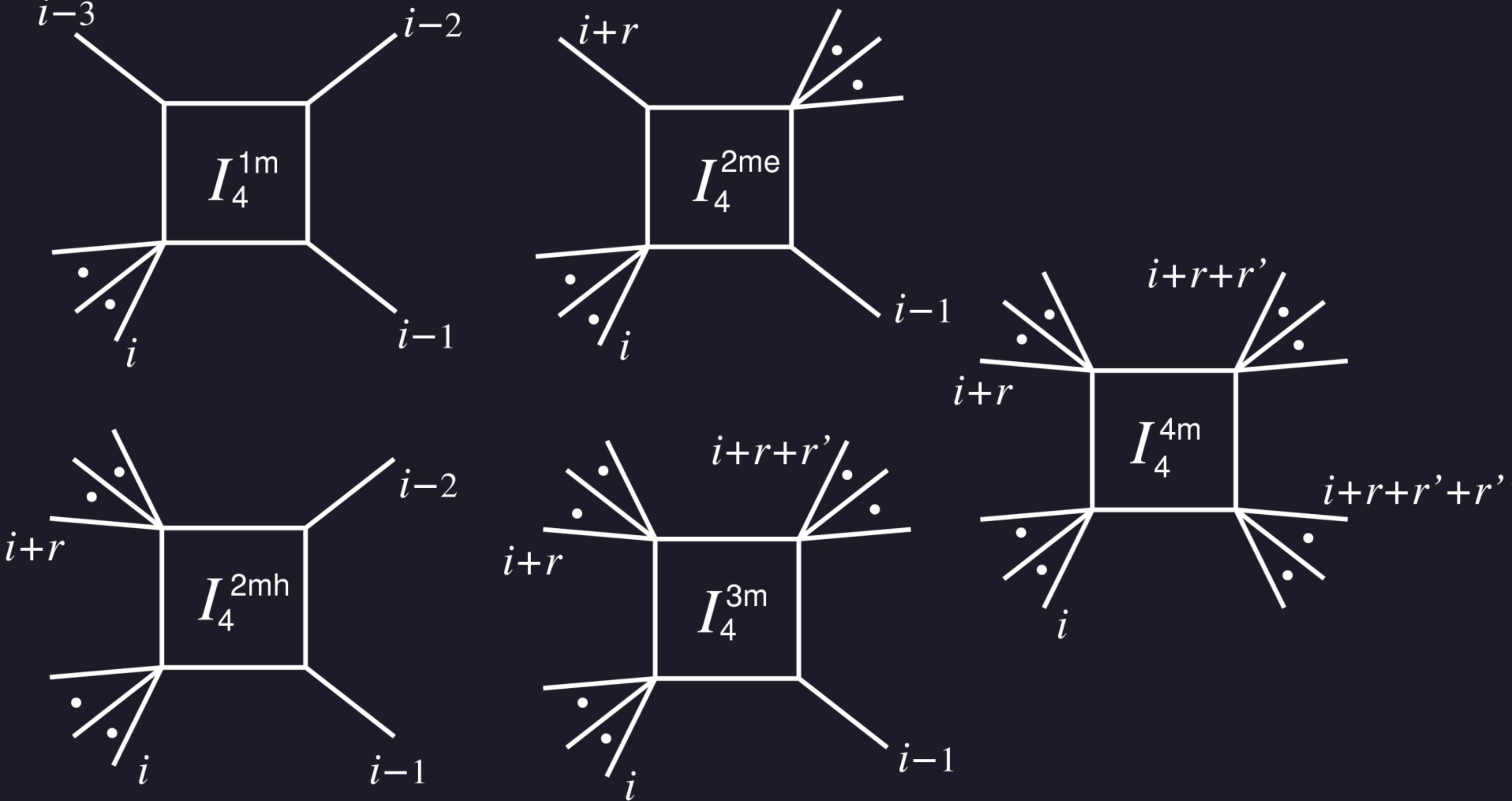
$\text{Int}_j$ → Known integral basis

Rational → On-shell Recursion; *D*-dimensional unitarity via ∫ mass

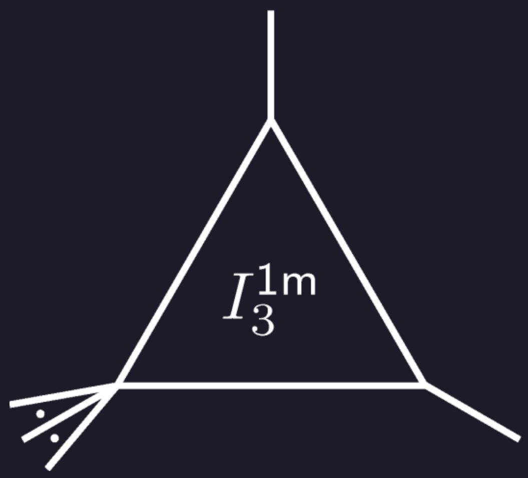
# Knowledge of Integrals

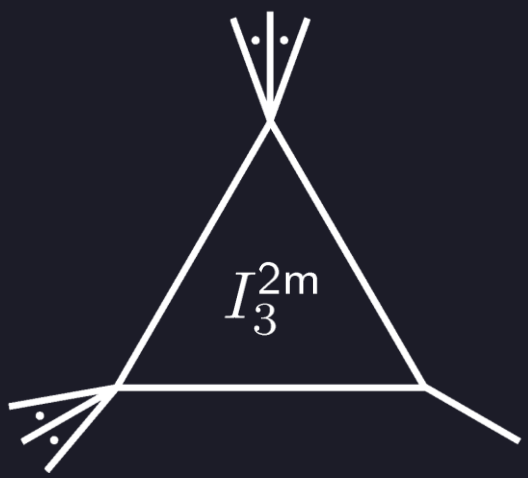
- Master Integrals for Feynman-diagram based calculations
  - Not essential in principle
  - Don't have to be algebraically independent
  - Required only to streamline calculations
- Process-dependent basis integrals
  - Needed for 'minimal generalized unitarity' calculations (just enough cuts to break apart process into trees)
  - Can be deduced as part of the unitarity calculation, on a process-by-process basis
- All-multiplicity results & Numerical applications of unitarity
  - Require a priori knowledge of algebraically independent integral basis

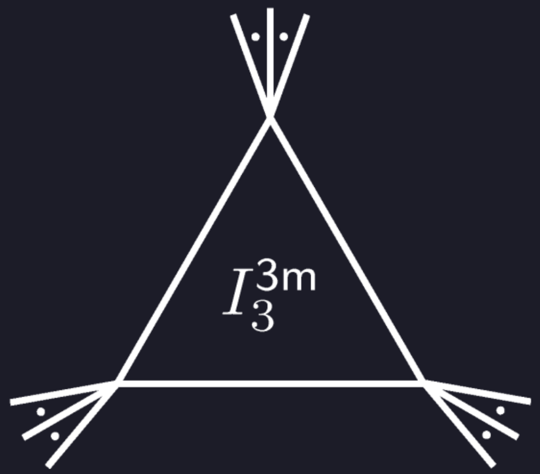
# Integral Basis at One Loop

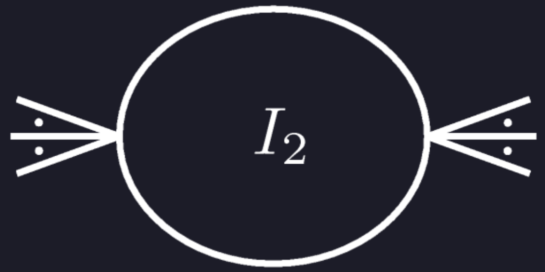



- Boxes…










- Triangles and bubbles

# Higher Loops

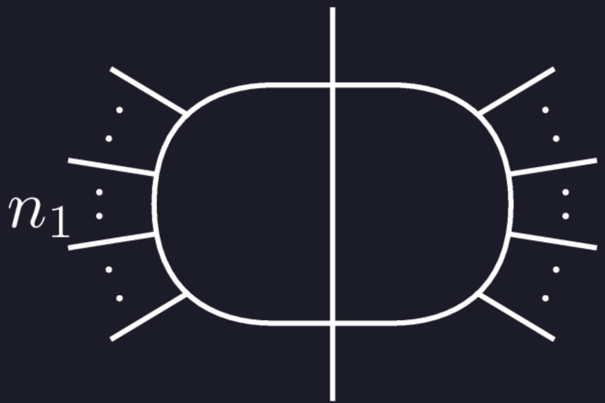
- How do we generalize this to higher loops?

- Work with dimensionally-regulated integrals
  - Ultraviolet regulator
  - Infrared regulator
  - Means of computing rational terms
  - External momenta, polarization vectors, and spinors are strictly four-dimensional
- Two kinds of integral bases
  - To all orders in $\varepsilon$ ("*D*-dimensional basis")
  - Ignoring terms of $\mathcal{O}(\varepsilon)$ ("Regulated four-dimensional basis")

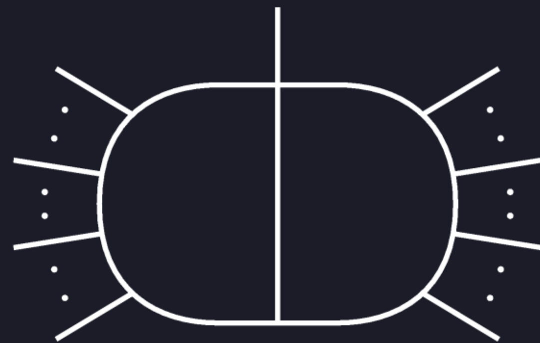
# Planar Two-Loop Integrals

- Focus on planar integrals at two loops
- Massless internal lines
- Massless or massive external lines

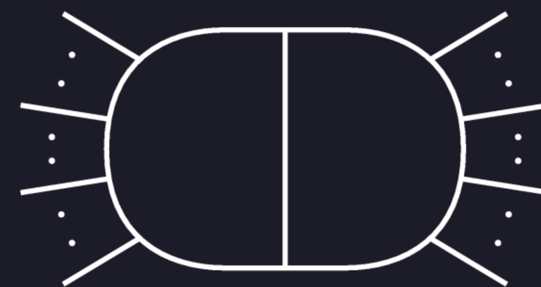
# Cast of Characters




$P_{n_1,n_2}$

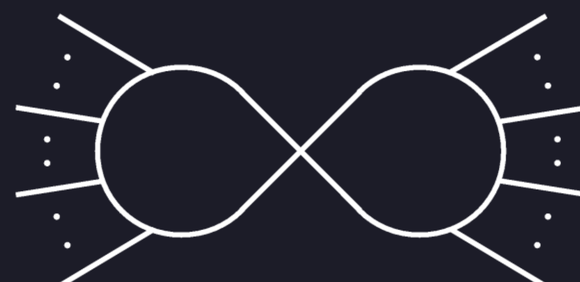


$P^{*}_{n_1,n_2}$



$P^{**}_{n_1,n_2}$

$I_{n_1,n_2}$



$I^{*}_{n_1,n_2}$

# Inventory of Weapons

- Tensor reduction

Brown and Feynman (1952); Passarino and Veltman (1979)

- Gram determinants $G\begin{pmatrix} p_1, \cdots, p_l \\ q_1, \cdots, q_l \end{pmatrix} \equiv \det_{i,j \in l \times l} (2 p_i \cdot q_j)$

  vanish when $p_i$ or $q_i$ are linearly dependent

- Integration by parts (IBP)

Tkachov and Chetyrkin (1981); Laporta (2001); Anastasiou and Lazopoulos (2004); A. Smirnov (2008)

$$\int \frac{d^D \ell_1}{(2\pi)^D} \frac{d^D \ell_2}{(2\pi)^D} \frac{\partial}{\partial \ell_j^\mu} \frac{v^\mu}{D(\ell_1, \ell_2, \{K_i\})} = 0$$

$$v = \ell_1, \ell_2, k_i$$

# Roadmap

- Basis is finite
  - Abstract proof by A. Smirnov and Petuchov (2010)
  - Constructive proof using tensor reduction & Gram determinants

- *D*-dimensional basis
  - Tensor reduction & IBP

- Regulated four-dimensional basis
  - Gram determinants

Computing in gauge theory $\Rightarrow$ determines class of integrals we need to consider: up to $n_1+n_2+2$ powers of loop momenta distributed between $\ell_1$ and $\ell_2$ (twice as high a limit in gravity)

All external momenta & vectors are taken to be four-dimensional

Expand any vector dotted into loop momenta in terms of four external momenta

$$v^\mu = \frac{1}{G\begin{pmatrix} b_1,b_2,b_3,b_4 \\ b_1,b_2,b_3,b_4 \end{pmatrix}} \left[ G\begin{pmatrix} v,b_2,b_3,b_4 \\ b_1,b_2,b_3,b_4 \end{pmatrix} b_1^\mu + G\begin{pmatrix} b_1,v,b_3,b_4 \\ b_1,b_2,b_3,b_4 \end{pmatrix} b_2^\mu + G\begin{pmatrix} b_1,b_2,v,b_4 \\ b_1,b_2,b_3,b_4 \end{pmatrix} b_3^\mu + G\begin{pmatrix} b_1,b_2,b_3,v \\ b_1,b_2,b_3,b_4 \end{pmatrix} b_4^\mu \right]$$

# Review of One Loop Integrals

- Starting set:

  For $n$-point amplitude: up to $n$-point integrals with up to $n$ powers of the loop momentum, dotted into external momenta, polarization vectors, or spinor strings

  Step 1. Re-express all external vectors in terms of four chosen external momenta ($n \geq 5$), or use Lorentz invariance to obtain a similar form with $n-1$ external momenta ($n \leq 4$)

Step 2. Re-express $\ell \cdot b_i$ as a linear combination of propagator denominators & external invariants — always possible at one loop.

$$\ell \cdot K_j = \frac{1}{2}\left[(\ell - K_{1\cdots(j-1)})^2 - (\ell - K_{1\cdots j})^2 + K_{1\cdots j}^2 - K_{1\cdots(j-1)}^2\right]$$

$$I_n[(\ell \cdot v)^n] \longrightarrow I_{n-1}[(\ell \cdot v)^{n-1}] \;\oplus I_n[(\ell \cdot v)^{n-1}]$$

⇒ Scalar integrals (trivial numerator) with up to *n* external legs

Step 3. Note that $$G\begin{pmatrix} \ell, 1, 2, 3, 4 \\ 5, 1, 2, 3, 4 \end{pmatrix} = 0\,;$$

insert this into the numerator of an *n*-point integral ($n \geq 6$) to obtain an equation relating it to six ($n-1$)-point integrals

⇒ Scalar integrals (trivial numerator) with up to five external legs

This is the *D*-dimensional basis

Step 4. Note that $G\begin{pmatrix} \ell, 1, 2, 3, 4 \\ \ell, 1, 2, 3, 4 \end{pmatrix}$

is of $\mathcal{O}(\varepsilon)$ and is vanishes in all regions that give rise to soft and collinear singularities; insert this into the numerator of a five-point integral to obtain a relation relating it to five box integrals, up to terms of $\mathcal{O}(\varepsilon)$.

$\Rightarrow$ Scalar integrals (trivial numerator) with up to four external legs

This is the regulated four-dimensional basis

# Planar Two-Loop Integrals

- Reduce to a finite basis
- Step 1. Tensor reduction: consider $P_{n_1,n_2}[\ell_1\cdot v_1\ \ell_1\cdot v_2\ldots\ \ell_1\cdot v_n]$ $(n_1 \geq 4)$; reexpress $v_1$ in terms of first four momenta (attached to $\ell_1$ loop); write factor as a difference of propagator denominators and external invariants,

$$\ell \cdot K_j = \frac{1}{2}\left[(\ell - K_{1\cdots(j-1)})^2 - (\ell - K_{1\cdots j})^2 + K_{1\cdots j}^2 - K_{1\cdots(j-1)}^2\right]$$

- Step 2. Reduce scalar integrals with $(n_1 > 4)$, using the Gram determinant

$$G\begin{pmatrix} \ell_1, 1, 2, 3, 4 \\ 5, 1, 2, 3, 4 \end{pmatrix} = 0$$

- Basis contains integrals

$$P^{\natural,*,**}_{n_1\leq 4, n_2\leq n_1}$$

with scalar numerators, reducible or irreducible numerators — new feature at two loops, for example $\ell_1\cdot k_4$ in the double box

$\Rightarrow$ Explicit finite basis

- Step 3. Tensor reductions: remove reducible numerators as at one loop

- Step 4. Use integration by parts to arrive at a set of truly-irreducible numerators: generically, $(\ell_1\cdot k_4)^2$ is reducible by IBP even when $\ell_1\cdot k_4$ isn't

# Honing Our Weapons

- Standard IBP application introduces lots of unwanted integrals, and results in huge systems of equations

- Doubled propagators $\frac{\partial}{\partial \ell_\mu} \frac{1}{(\ell - K)^2} = 2 \frac{(\ell - K)^\mu}{\left[(\ell - K)^2\right]^2}$

- Would like to avoid them $\int \frac{d^D \ell_1}{(2\pi)^D} \frac{d^D \ell_2}{(2\pi)^D} \frac{\partial}{\partial \ell_j^\mu} \frac{v^\mu}{D(\ell_1, \ell_2, \{K_i\})} = 0$

- First idea: choose $v$ such that $v \cdot (\ell - K) = 0$

$$v^\mu = G\begin{pmatrix} \mu, \ell_1, \ell_2, 6, 7, 8 \\ \ell_2, \ell_1, 1, 2, 3, 4 \end{pmatrix}$$

- But this is too constraining

- It is sufficient to require that $v \cdot (\ell - K) \propto (\ell - K)^2$

- Need to require this for all propagators simultaneously:

$$[\sigma_{j1} v_1 + \sigma_{j2} v_2] \cdot (\sigma_{j1} \ell_1 + \sigma_{j2} \ell_2 - K_j) + u_j \, (\sigma_{j1} \ell_1 + \sigma_{j2} \ell_2 - K_j)^2 = 0$$

- $\sigma = 0,1$; $u_j$ arbitrary polynomial in the symbols

$$\{\ell_1^2, \ell_2^2, \ell_1 \cdot \ell_2, \ell_1 \cdot b_i, \ell_2 \cdot b_i, s_{12}\}$$

  with ratios of external invariants treated as parameters

- How can we find such vectors?

# Solving for IBP-Generating Vectors

- Use Gröbner bases — Buchberger (1965)
  - Generalization of independence to multivariate non-linear setting
  - Polynomials in the independent invariants treated as symbols $\{\ell_1^2, \ell_1 \cdot \ell_2, \ell_2^2, \ell_1 \cdot b_i, \ell_2 \cdot b_i, s_{12}\}$
  - Choose ordering (DRL/ToP)
- We need a generalization of the usual Gröbner basis to vectors ("tuples" in mathematician-speak) of polynomials
  - Define using basis tuples $\boldsymbol{e}_i$ times polynomials
  - Define leading monomial of two vectors: depends on ordering
  - Define least common multiple of two vectorial monomials: $\mathrm{lcm}(\boldsymbol{e}_i, \boldsymbol{e}_j) \equiv 0$ if $i \neq j$
  - Certain shortcuts in Buchberger reduction algorithm cannot be used
  - Still "textbook" material, for appropriate choice of textbook

# Solving for IBP-Generating Vectors

- Write a general form for the vectors $v_{1,2}$ vectors

$$v_i^\mu = c_i^{(\ell_1)} \ell_1^\mu + c_i^{(\ell_2)} \ell_2^\mu + \sum_{b \in B} c_i^{(b)} b^\mu$$

- $c_i$ are functions of independent invariants

Organize equations

$$\tilde{c} = \left( c_1^{(\ell_1)} \cdots c_1^{(b_4)} c_2^{(\ell_1)} \cdots c_2^{(b_4)} u_1 \cdots u_n \right)$$

$$\tilde{c} E = 0$$

→ rows correspond to coeffs

→ columns correspond to propagators

# A Wrinkle

- We aren't interested in the most general solution to the equations
- Suppose we have a solution $v$
  - Any multiple of the solution is also a solution
  - But a multiple by a polynomial reducible over the denominator factors is useless: it won't give us an independent equation for the original integral

    suppose $I\big[\partial_1 \cdot (v_1 W) + \partial_2 \cdot (v_2 W)\big] = 0$ where $W$ are propagators

    consider
    $$I\big[p\big(\partial_1 \cdot (v_1 W) + \partial_2 \cdot (v_2 W)\big)\big] + I\big[W(v_1 \cdot \partial_1 p + v_2 \cdot \partial_2 p)\big] = 0$$
- Only interested in solutions which are not fully reducible with respect to the denominator factors
- Need to simplify solutions coming out of the Gröbner machinery

- Construct a Gröbner basis for rows of $E$, call it $G$
- $C$ is cofactor matrix: $G = CE$
- Because $G$ is a Gröbner basis, $E = QG$

- Find all $s$ such that $sG = 0 \Rightarrow$ matrix $S$ ("syzygies")
- Then the set of solutions is given by the rows of $SC$ along with the rows of $I-QC$
- Find linearly-independent solutions, requiring independence after reducibility (can do this numerically)

$\Rightarrow$ IBP-generating vectors $v_{1,2}$

- Double box $P^{**}_{2,2}$ example

$$
\begin{aligned}
v_{1;1} &= -2(k_4\cdot\ell_1 + \ell_1^2)k_1^\mu - \ell_1^2 k_2^\mu + (2k_1\cdot\ell_1 - \ell_1^2)k_4^\mu + (2k_1\cdot\ell_1 - 2k_3\cdot\ell_1 - s_{12})\ell_1^\mu\,, \\
v_{1;2} &= 2(\ell_2^2 - k_4\cdot\ell_2)k_1^\mu + \ell_2^2 k_2^\mu + (2k_1\cdot\ell_2 + \ell_2^2)k_4^\mu + (2k_3\cdot\ell_2 - 2k_1\cdot\ell_2 - s_{12})\ell_2^\mu
\end{aligned}
$$

$$\frac{\partial}{\partial \ell_1^\mu} \frac{v_{1;1}}{\ell_1^2(\ell_1-k_1)^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_2-k_4)^2(\ell_2-K_{34})^2} =$$

$$\frac{1}{\ell_1^2(\ell_1-k_1)^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_2-k_4)^2(\ell_2-K_{34})^2}$$

$$\times \Big( D(2k_1\cdot\ell_1 - 2k_3\cdot\ell_1 - s_{12}) - (8k_1\cdot\ell_1 - 8k_3\cdot\ell_1 - 4s_{12} + s_{14})$$

$$+ \frac{4}{(\ell_1+\ell_2)^2}(2k_1\cdot\ell_2 k_4\cdot\ell_1 - 2k_1\cdot\ell_1 k_4\cdot\ell_2 + k_1\cdot\ell_2\ell_1^2 - k_3\cdot\ell_2\ell_1^2$$

$$+2k_1\cdot\ell_1\ell_2^2 + k_2\cdot\ell_1\ell_2^2 + k_4\cdot\ell_1\ell_2^2 + \ell_1^2 s_{12}/2 - \ell_2^2 s_{12}/2)\Big)$$

$$\frac{\partial}{\partial \ell_2^\mu} \frac{v_{1;2}}{\ell_1^2(\ell_1-k_1)^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_2-k_4)^2(\ell_2-K_{34})^2} =$$

$$\frac{1}{\ell_1^2(\ell_1-k_1)^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_2-k_4)^2(\ell_2-K_{34})^2}$$

$$\times \Big( D(2k_1\cdot\ell_2 - 2k_3\cdot\ell_2 + s_{12}) + (8k_1\cdot\ell_2 - 8k_3\cdot\ell_2 + 4s_{12} + s_{14})$$

$$- \frac{4}{(\ell_1+\ell_2)^2}(2k_1\cdot\ell_2 k_4\cdot\ell_1 - 2k_1\cdot\ell_1 k_4\cdot\ell_2 + k_1\cdot\ell_2\ell_1^2 - k_3\cdot\ell_2\ell_1^2$$

$$+2k_1\cdot\ell_1\ell_2^2 + k_2\cdot\ell_1\ell_2^2 + k_4\cdot\ell_1\ell_2^2 + \ell_1^2 s_{12}/2 - \ell_2^2 s_{12}/2)\Big)$$

- Can also use Gröbner bases to solve inhomogeneous equations

$$\tilde{c}E = f$$

- If *f* is reducible over the Gröbner basis of E,

$$f = q_f G$$

  there is a solution; otherwise not.
- If *f* has free coefficients, we get a system of auxiliary equations requiring reducibility of *f*

- The solution is $\tilde{c} = q_f C$

# Planar Two Loop Integrals, cont.

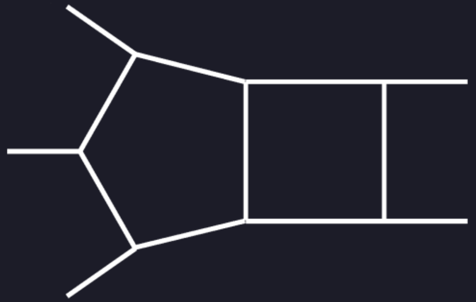
- Find and apply IBP-generating vectors to each of the integrals in the basis list: generate IBP equations with $pv_{1,2}$ where $p$ is irreducible

- Vectors will depend on number and pattern of external masses

- Number of truly-irreducible integrals ('master integrals') also depends on number and pattern of external masses

# Examples

- Massless, one-mass, diagonal two-mass, long-side two-mass double boxes $P^{**}_{2,2}$: two integrals
- Short-side two-mass, three-mass double boxes: three integrals
- Four-mass double box: four integrals
- Massless pentabox $P^{**}_{3,2}$: three integrals



All integrals with $n_2 \leq n_1 \leq 4$, that is with up to 11 propagators

$\Rightarrow$ This is the $D$-dimensional basis

- Consider expressions of the form

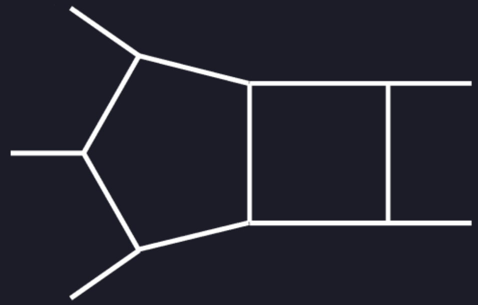
$$G\begin{pmatrix} \ell_1, b_1, b_2, b_3, b_4 \\ \ell_1, b_1, b_2, b_3, b_4 \end{pmatrix} G\begin{pmatrix} \ell_2, b'_1, b'_2, b'_3, b'_4 \\ \ell_2, b'_1, b'_2, b'_3, b'_4 \end{pmatrix}; \quad G\begin{pmatrix} \ell_1, b_1, b_2, b_3, b_4 \\ \ell_2, b'_1, b'_2, b'_3, b'_4 \end{pmatrix};$$

$$G\begin{pmatrix} \ell_1, \ell_2, b_1, b_2, b_3 \\ \ell_1, \ell_2, b''_1, b''_2, b''_3 \end{pmatrix}; \quad G\begin{pmatrix} \ell_1, \ell_2, b_1, b_2, b_3, b_4 \\ \ell_1, \ell_2, b'_1, b'_2, b'_3, b'_4 \end{pmatrix}$$

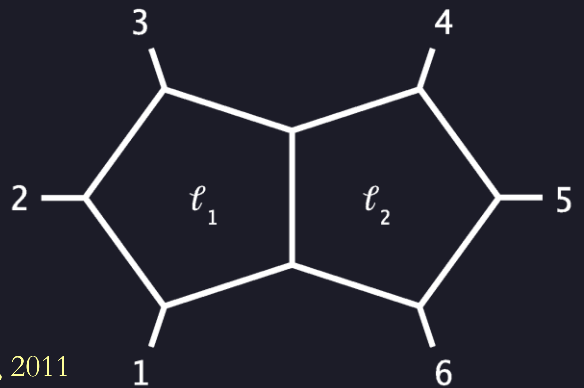
- Vanish when loop momenta are four-dimensional, vanish in all regions generating soft and collinear singularities $\Rightarrow$ give integrals of $\mathcal{O}(\varepsilon)$

- No new equations for double boxes $P^{**}_{2,2}$

- Reduce three integrals for the pentabox $P^{**}_{3,2}$ to one



- Reduce all double pentagons $P^{**}_{3,3}$ to simpler integrals



- Indeed, we can eliminate all integrals beyond the pentabox $P^{**}_{3,2}$ , that is all integrals with more than eight propagators

- $\Rightarrow$ This is the regulated four-dimensional basis, dropping terms which are ultimately of $\mathcal{O}(\varepsilon)$ in amplitudes

# Connection to Generalized Unitarity

Cut propagators to isolate coefficients in an amplitude (which are then products of the surviving trees)

Maximal unitarity: cut as many propagators as possible

But how many can we cut?

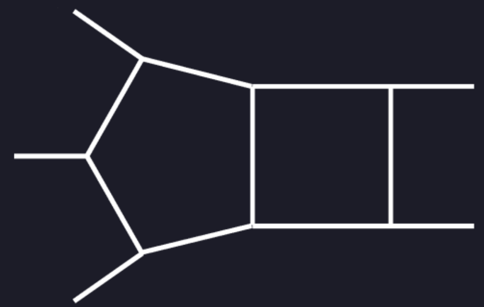
In four dimensions, up to four per loop (four degrees of freedom)

In D dimensions, $4\,L + L\,(L+1)/2$

- Only integrals with $4\,L + L(L+1)/2$ or fewer propagators can enter $D$-dimensional basis
- Structure of surviving irreducible numerators is trickier: is the bound

$$\#\text{irreducible numerator integrals} + \#\text{ propagators} \leq 4\,L + L\,(L+1)/2$$

  satisfied?
- Conjecture: four-dimensional basis behaves like basis for massive integrals $\Rightarrow$ only integrals with up to $4\,L$ propagators enter



# Summary

- Basis is finite
- Refined version of IBP approach to simplifying irreducible numerators
- Additional equations reduce basis when $\mathcal{O}(\varepsilon)$ terms are dropped
- Conjecture: latter basis contains only integrals with 4 L or fewer propagators, only latter have irreducible numerators.