Quantum state and parameter estimation with continuous quantum measurements.

KITP - February, 2013.

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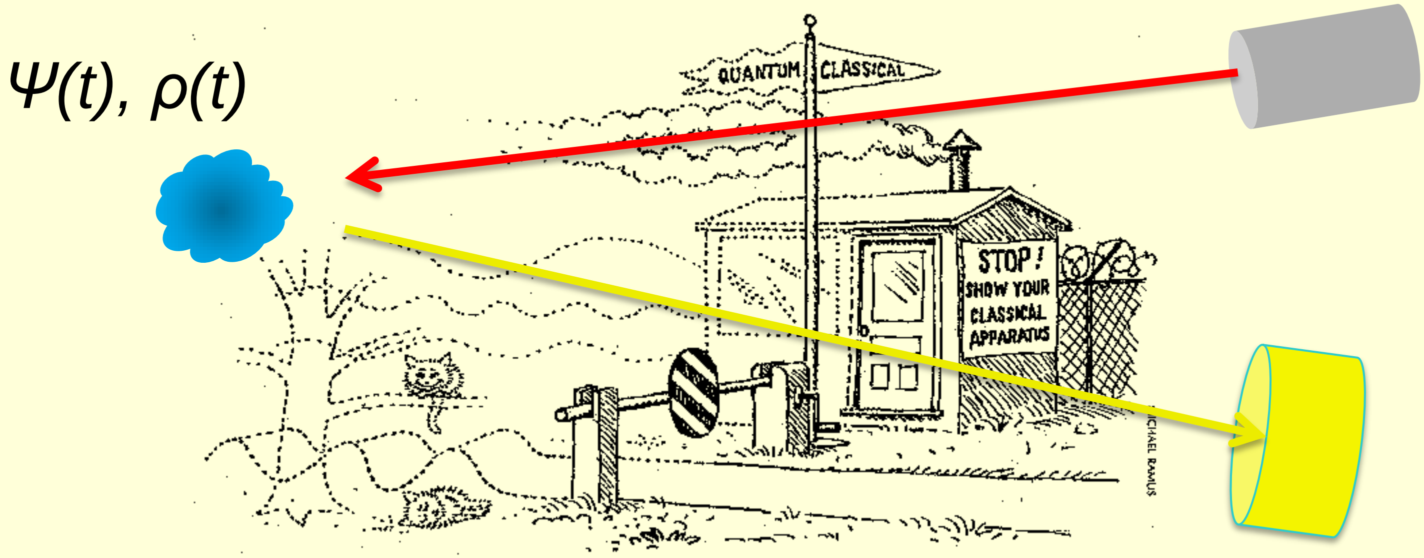
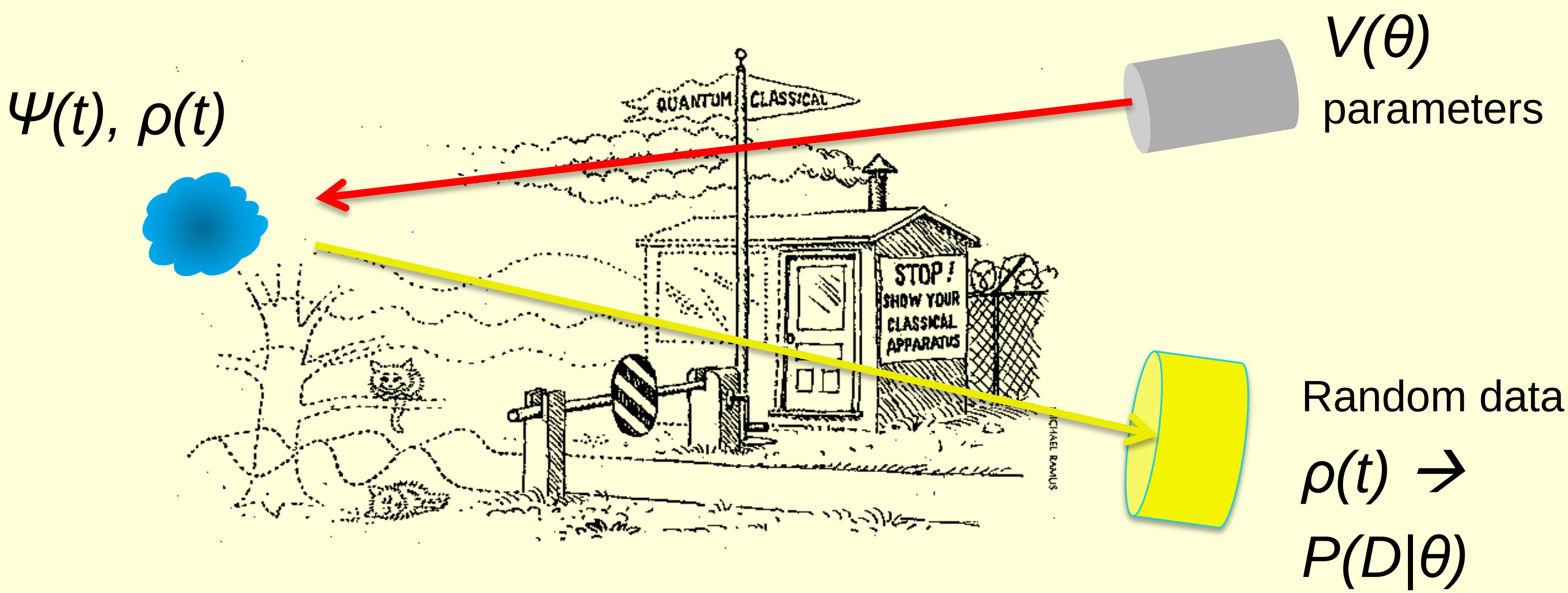
Driven quantum systems
… are also driven by their output



Measurements on a quantum system imply

- wave function collapse - back action - state reduction

Driven quantum systems
…. are also driven by their output



Measurements on a quantum system imply

- wave function collapse - back action - state reduction

Measurement data reveal unknown parameters

- general state reduction - $P(\theta|D)$.

Quantum metrology

Strategies to

Prepare optimal states of quantum probes

cooling, alignment, squeezing, entanglement, …

Extract maximum amount of information

spectroscopy, filters, Bayes rule, adaption …

Establish general results

Cramer-Rao bounds, Fisher information, …

Goals

high precision/sensitivity measurements, feedback control, quantum functional devices, …

Caves
Braunstein
Milburn
Mabuchi
Wiseman
Gambetta
Lloyd
Maccone
Tsang
Smerzi
Belavkin
James
Plenio
…

Outline

Metrology with continuous quantum measurements.

Part I

A toy model for ”normal” and ”optimum” use of a fluorescence signal.

”Fishing with Fisher for the signal in the noise”

Part II

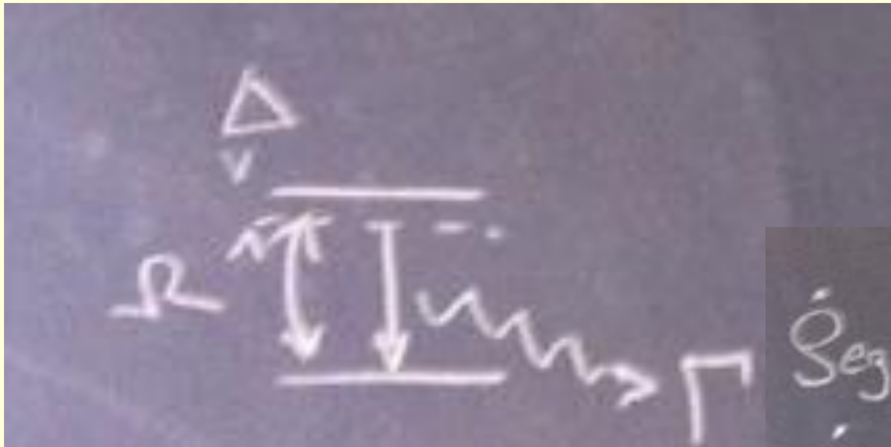
General Stochastic Master Equation theory.

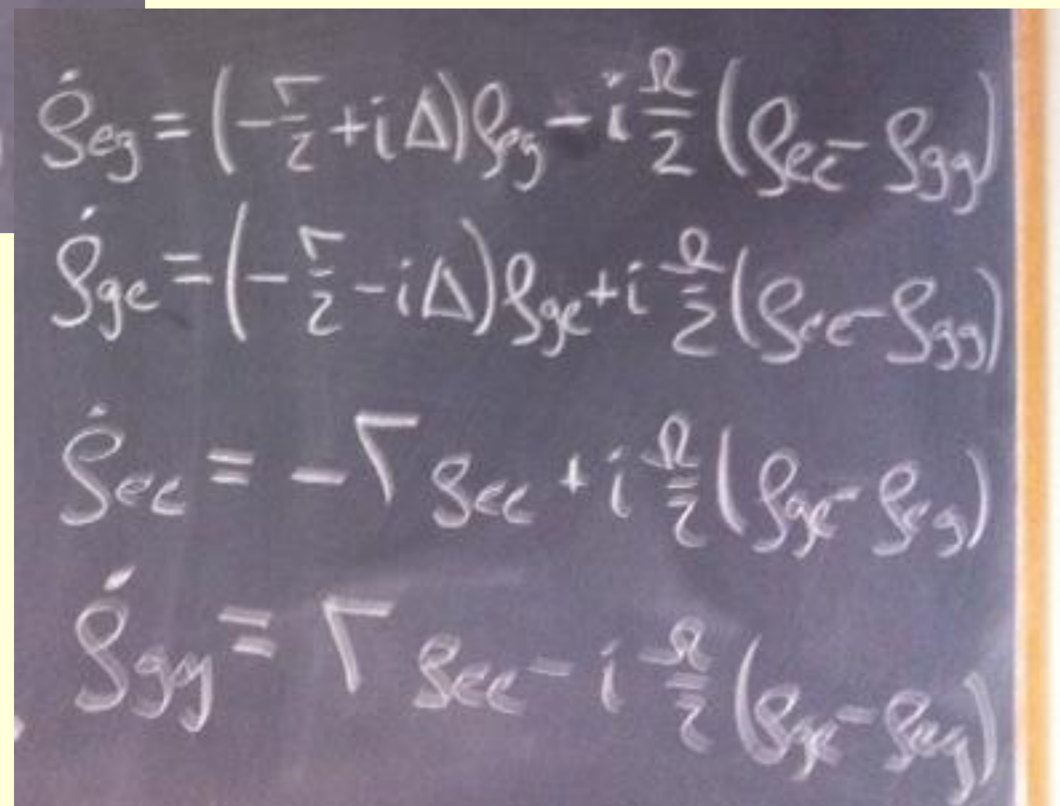
The ”complex landscape” in this presentation is

the infinite set of possible data records.

The ”message”: the experiment walks the path, we just have to look!

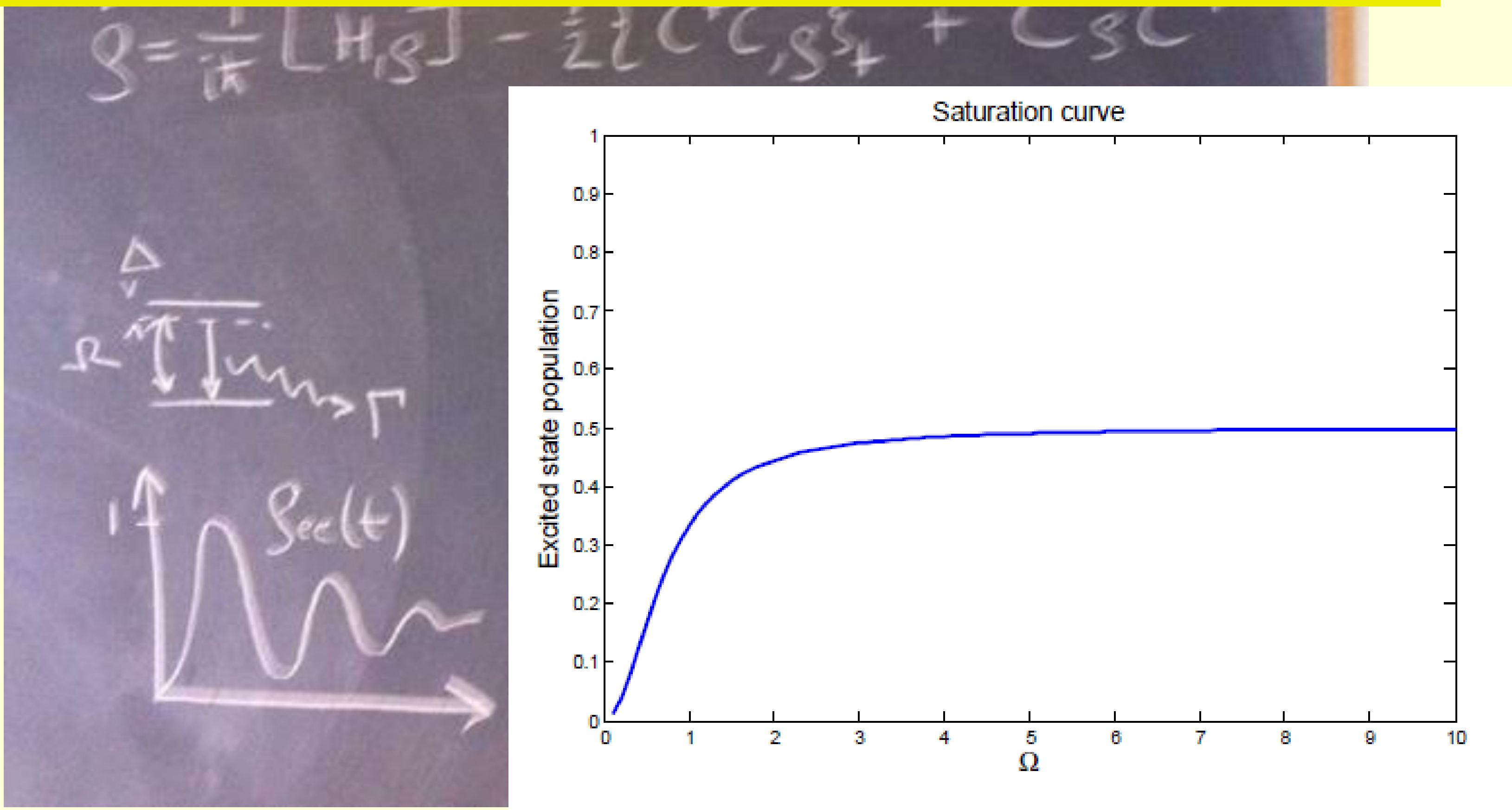
Toy model: laser driven two level atom



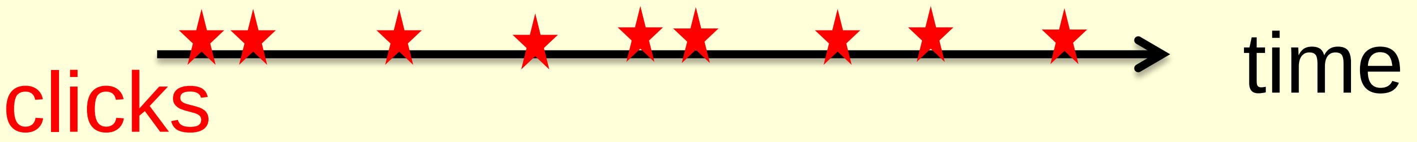


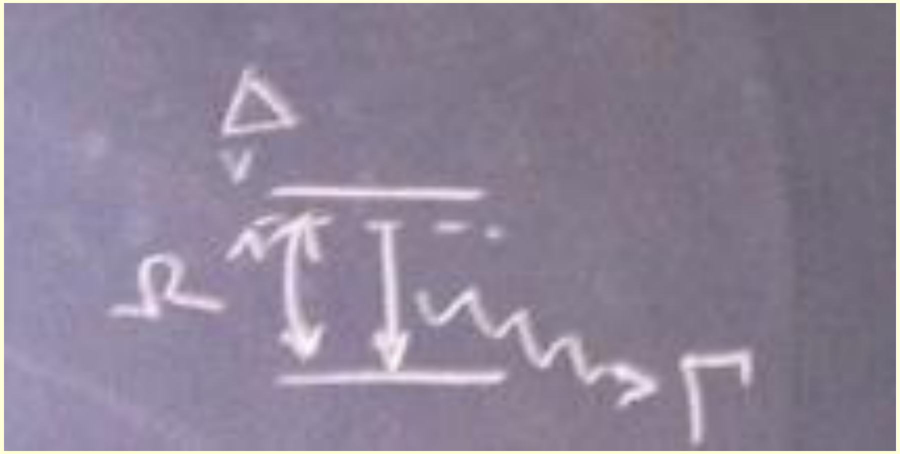
Toy model: laser driven two level atom

Steady state excited state population is a function of the model parameters



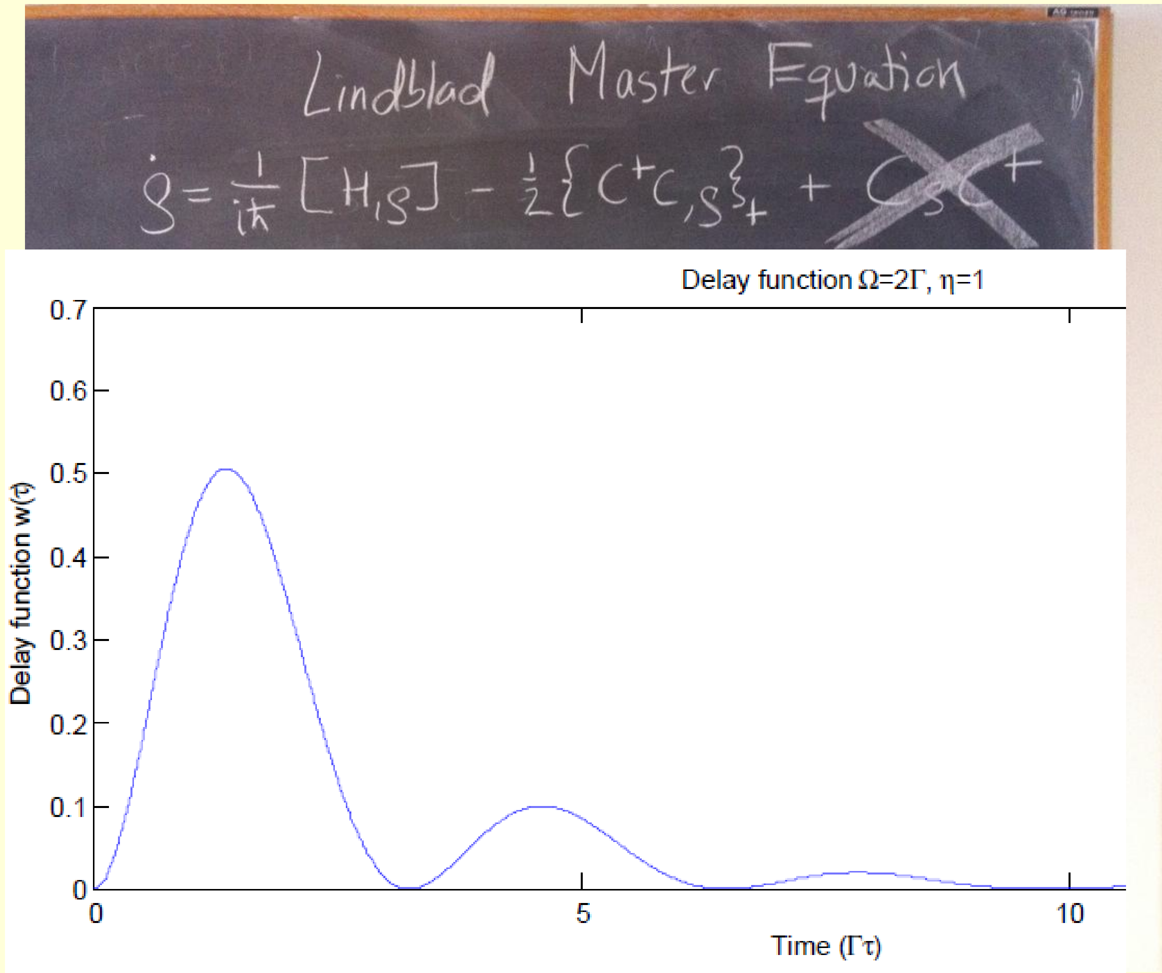
Toy model: laser driven two level atom





Blatt, Ertmer, Zoller, Hall; PRA 1986

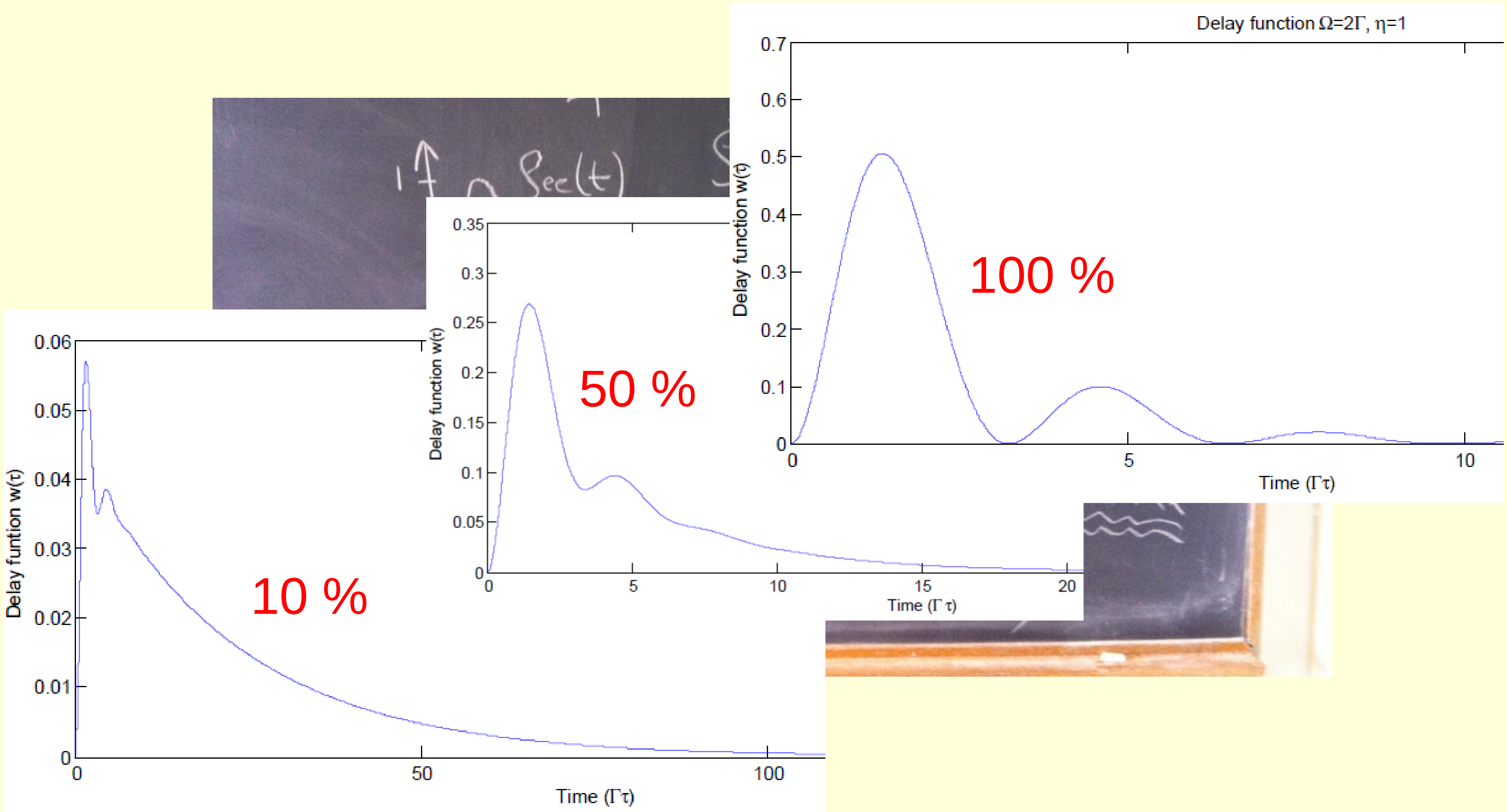
Toy model: laser driven two level atom



Toy model: laser driven two level atom

time

missing clicks



Toy model: laser driven two level atom

Fisher Information

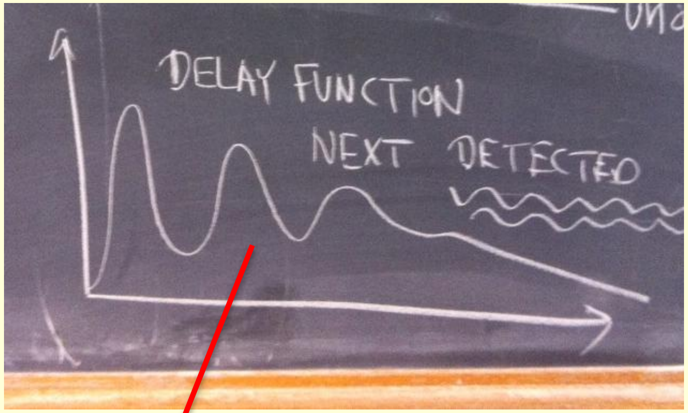
→ p(detection record|values)

$$F(q) = -\sum_{\rho_1,\ldots\rho_M} p(\{\rho_s\};q)\frac{\partial^2 \log p(\{\rho_s\};q)}{\partial q^2}$$

Delay times are uncorrelated.

Distribution of delay times:

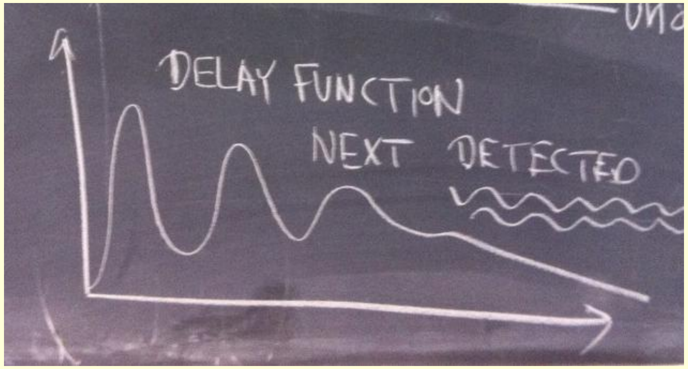
Histogram with Poisson noise

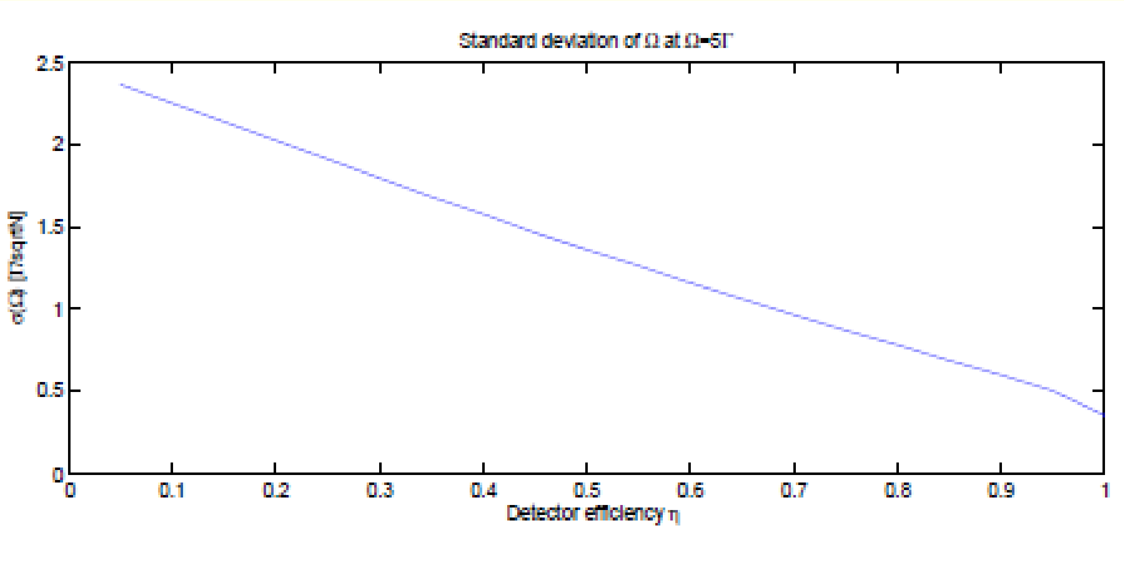


$$F(q) = 4\int \mathrm{d}x \left[\frac{\partial|\Phi(x;q)|}{\partial q}\right]^2 \leftarrow \sqrt{}$$

Toy model: laser driven two level atom
Fisher Information

$$F(q) = 4 \int \mathrm{d}x \left[\frac{\partial |\Phi(x;q)|}{\partial q} \right]^2$$





1/√F

error per click

Around *Ω=5Γ*

General theory

H. Mabuchi, Quant. Semiclass. Opt. **8**, 1103, (1996);

J. Gambetta and H. M Wiseman, Phys. Rev. A 64, 042105 (2001).

Søren Gammelmark, KM: arXiv:1212.5700

Data *D*, parameter(s) θ

Bayes probability rule:

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)} \qquad P(D) = \int d\theta P(D|\theta)P(\theta)$$

Likelihood (no need for normalization)

$$L(D|\theta) = P(D|\theta)/P_0(D) \qquad l(D|\theta) = \log L(D|\theta)$$

Likelihood of measurement data
Random state evolution

$$d\rho_t = \left[-i\,[H,\rho_t] - \frac{1}{2}\left\{c^\dagger c,\rho_t\right\} + \mathrm{Tr}(c^\dagger c\rho_t)\rho_t\right] dt + \left[\frac{c\rho_t c^\dagger}{\mathrm{Tr}(c^\dagger c\rho_t)} - \rho_t\right] dN_t,$$

← 0 or 1

$$\rho|m = \frac{\Omega(m)\rho\Omega^\dagger(m)}{\mathrm{Tr}(\Omega^\dagger(m)\Omega(m)\rho)}$$

$$p(m) = \mathrm{Tr}(\Omega^\dagger(m)\Omega(m)\rho)$$

$$\int dm\,\Omega^\dagger(m)\Omega(m) = \mathbb{1}$$

Probability for the measurement data

Likelihood of measurement data
Random state evolution

$$d\rho_t = \left[-i\,[H,\rho_t] - \frac{1}{2}\left\{c^\dagger c, \rho_t\right\} + \mathrm{Tr}(c^\dagger c \rho_t)\rho_t\right] dt + \left[\frac{c\rho_t c^\dagger}{\mathrm{Tr}(c^\dagger c \rho_t)} - \rho_t\right] dN_t,$$

← 0 or 1

Likelihood of measurement data
Random state evolution

$$d\rho_t = \left[-i\,[H,\rho_t] - \frac{1}{2}\left\{c^\dagger c, \rho_t\right\} + \mathrm{Tr}(\cancel{c^\dagger c \rho_t})\rho_t \right] dt + \left[\frac{c\rho_t c^\dagger}{\mathrm{Tr}(\cancel{c^\dagger c \rho_t})} - \rho_t \right] dN_t,$$

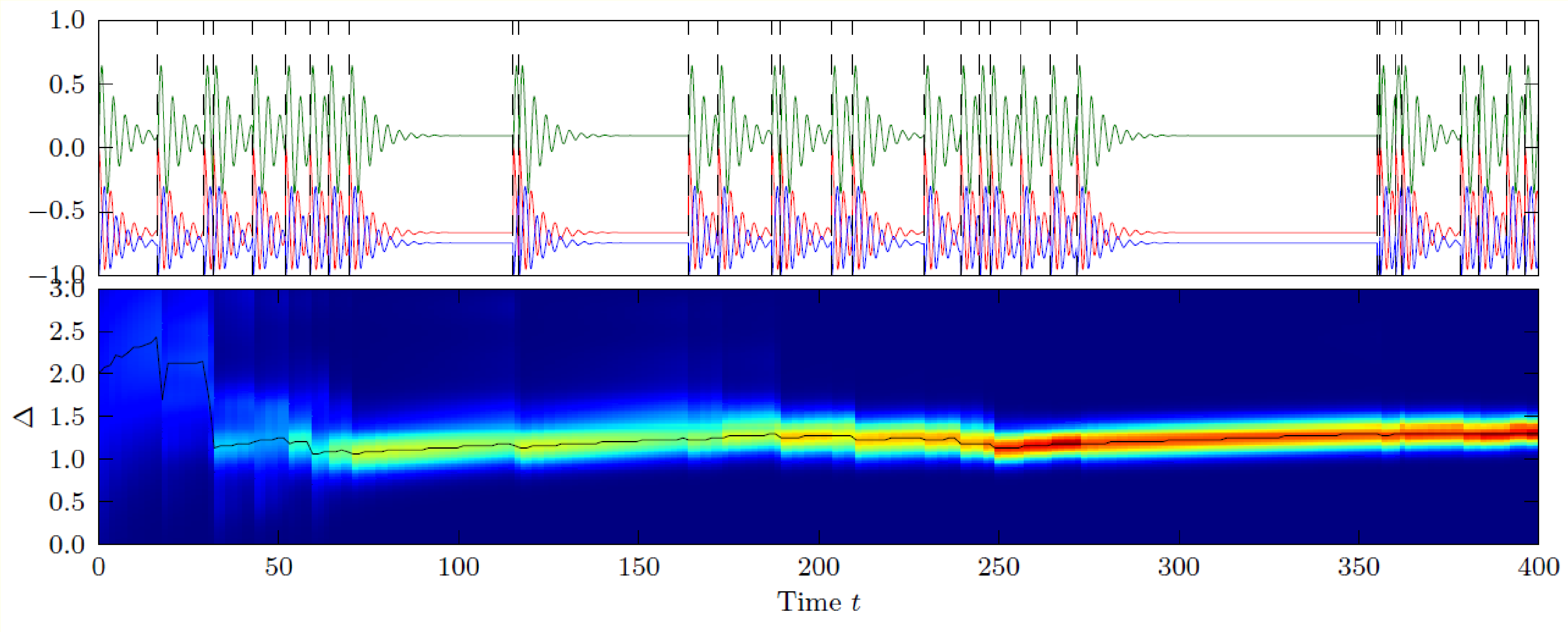
← 0 or 1

$$dL_t = (\lambda - \mathrm{Tr}(c^\dagger c \rho_t))L_t dt + dN_t \left[\lambda^{-1}\,\mathrm{Tr}(c^\dagger c \rho_t) - 1\right] L_t$$

Likelihood determined for different θ

→ their relative probability given the data measured.

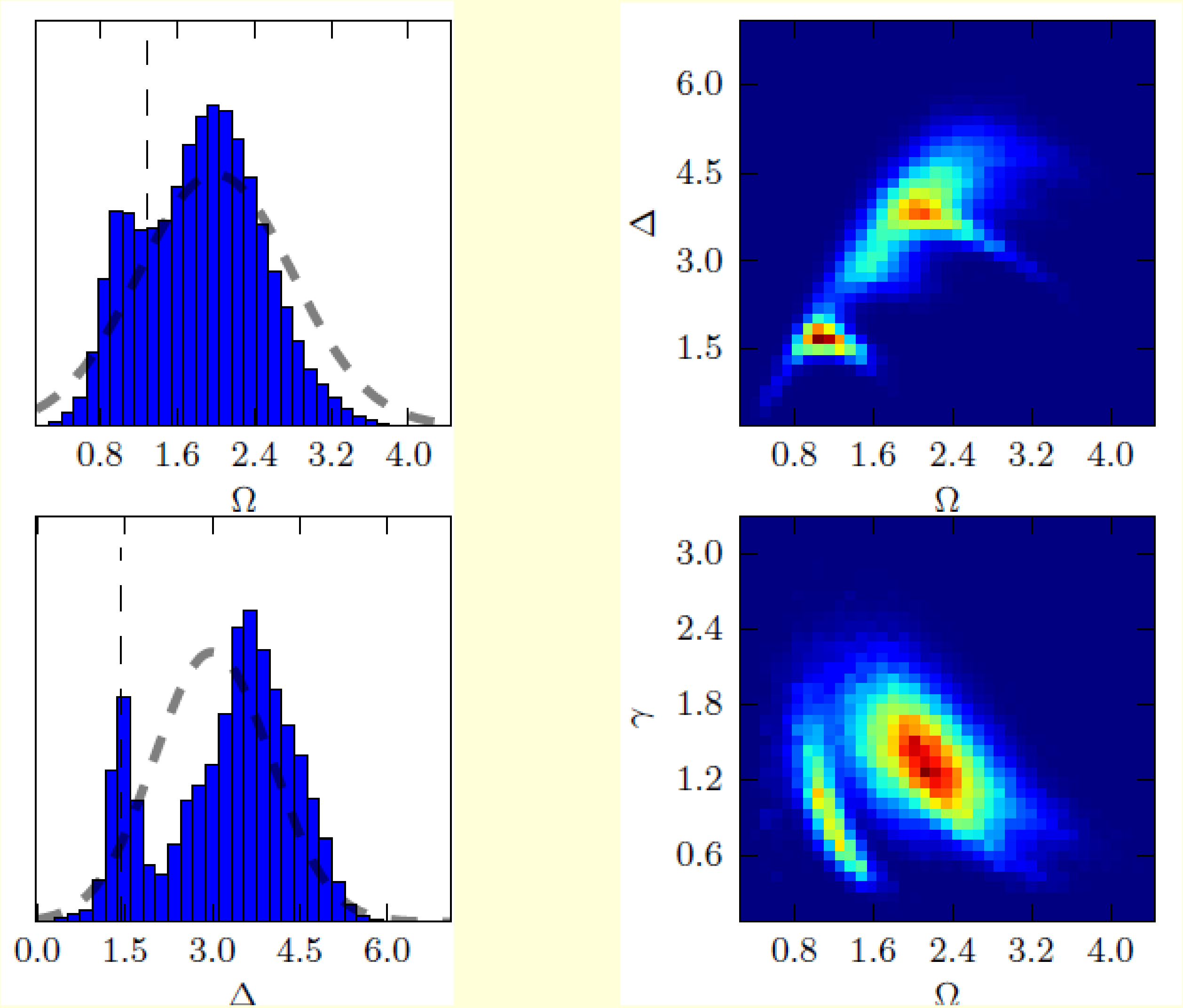
Data (click) record, find *Δ*



Here, we do not use our insight or analytical knowledge about the anti-bunching and delay function.

It is *automatically* taken into account in the SME.

Unknown *Δ, Ω and γ*



Asymptotic behavior
Fisher information

$$L(D|\theta) = P(D|\theta)/P_0(D)$$

$$l(D|\theta) = \log L(D|\theta)$$

$$\begin{aligned} I_{ij} &= \mathbb{E}\left[\frac{\partial \log L(D|\theta)}{\partial \theta_i}\frac{\partial \log L(D|\theta)}{\partial \theta_j}\right], \\ &= \mathbb{E}\left[L(D|\theta)^{-2}\frac{\partial L(D|\theta)}{\partial \theta_i}\frac{\partial L(D|\theta)}{\partial \theta_j}\right] \end{aligned}$$

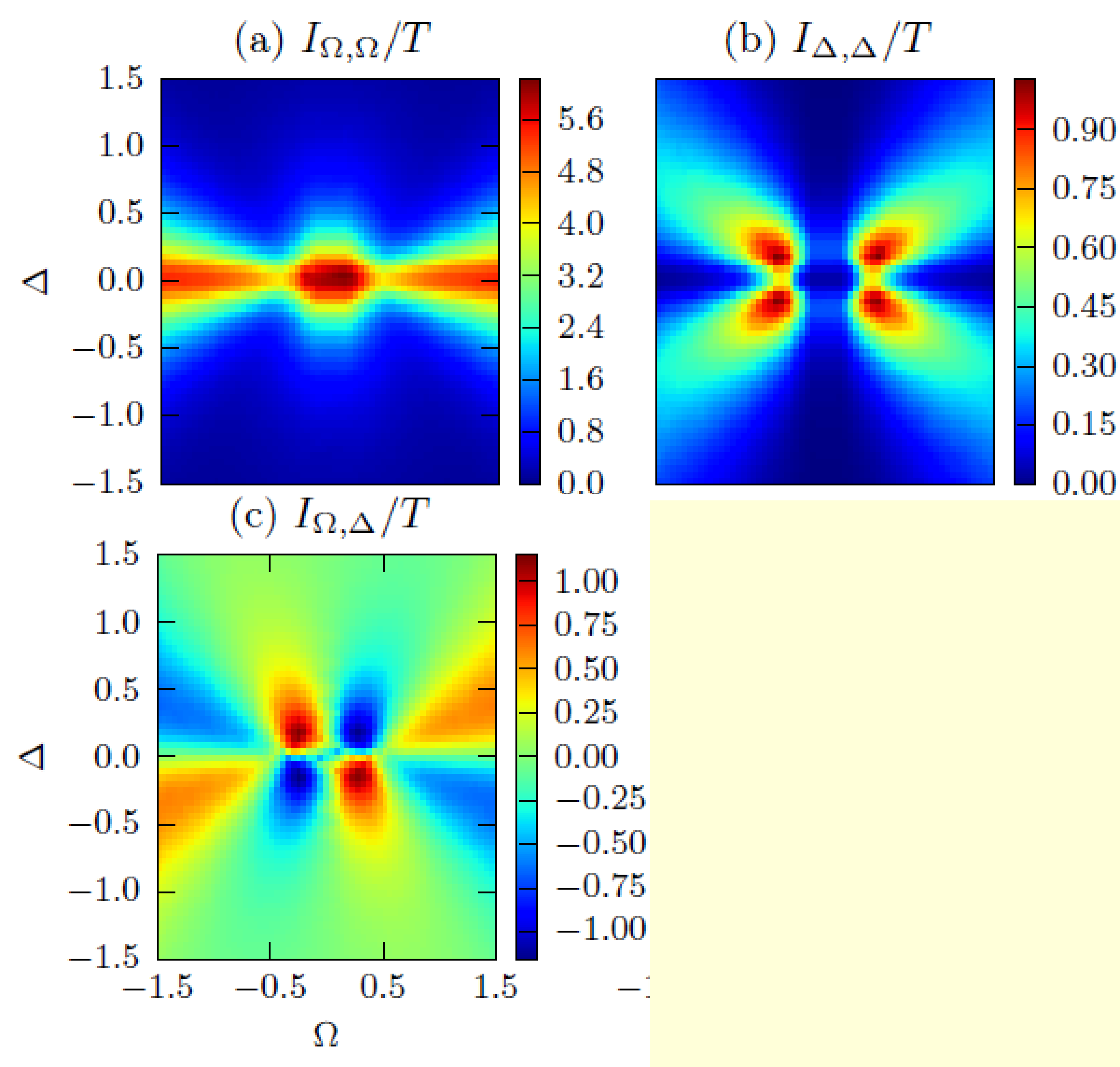
Sample the Fisher information

$$I_{ij} = \mathbb{E}\left[\frac{\partial \log L(D|\theta)}{\partial \theta_i}\frac{\partial \log L(D|\theta)}{\partial \theta_j}\right],$$

Auxiliary density matrices: $\mathrm{Tr}(\rho_t^i)\,\mathrm{Tr}(\rho_t^j)$

$$\begin{aligned} d\rho_t^i = &\left[-i\left[H,\rho_t^i\right] - \frac{1}{2}\left\{c^\dagger c,\rho_t^i\right\} + \mathrm{Tr}(c^\dagger c\rho_t^i)\rho_t^i\right]dt \\ &+ \left[-i\left[\partial_i H,\rho_t\right] - \frac{1}{2}\left\{\partial_i(c^\dagger c),\rho_t^i\right\}\right]dt \\ &+ dN_t(c\rho_t^i c^\dagger + (\partial_i c)\rho_t c^\dagger + c\rho_t(\partial_i c^\dagger) - \rho_t^i), \end{aligned}$$

Fisher information reveals sensitivity of detection method:



Summary/outlook

- Quantum optical systems ”survive” that we perform measurements on them.
- Basic theory is stochastic (measurement conditioned) master equation.

- An observed record + the conditioned quantum state constitute a *quantum trajectory* (Carmichael).

- The quantum trajectory carries its own likelihood function with it, and by associating the likelihood with different candidate values for unkown parameters, these can be estimated.

- The method exhausts the information present in the full data record !
- Optimization over measurement strategies, adaption, … .

Likelihood of measurement data
Count vs. Homodyne detection

$$d\rho_t = \left[-i\left[H,\rho_t\right] - \frac{1}{2}\left\{c^\dagger c, \rho_t\right\} + \mathrm{Tr}(c^\dagger c\rho_t)\rho_t\right] dt + \left[\frac{c\rho_t c^\dagger}{\mathrm{Tr}(c^\dagger c\rho_t)} - \rho_t\right] dN_t,$$

← 0 or 1

$$d\rho_t = \left[-i\left[H,\rho\right] - \left\{c^\dagger c, \rho\right\}/2 + c\rho c^\dagger\right] dt + (\mathcal{M}(\rho_t) - \mathrm{Tr}(\mathcal{M}(\rho_t))\rho_t)(dY_t - \mathrm{Tr}(\mathcal{M}(\rho_t))dt)$$

$$dY_t = \mathrm{Tr}(\mathcal{M}(\rho_t))dt + dW_t$$