Topological (quantum) synchronization of coupled van der Pol oscillators

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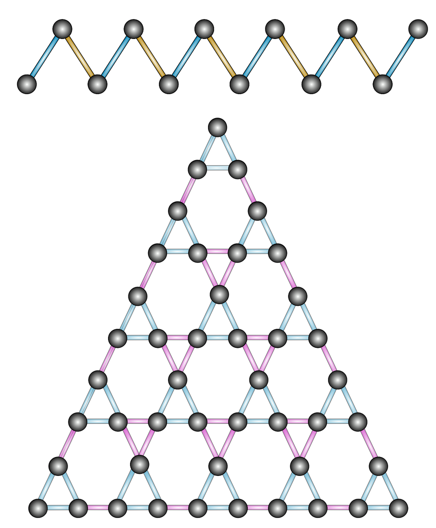
Motivation

Topological insulators exhibit a remarkable robustness against certain imperfections in closed systems [1]. However, it is far from being completely understood whether this feature carries over to nonlinear systems and under nonequilibrium conditions. Here, we explore in the classical and quantum regime whether topological protection can be exploited to enhance the robustness of synchronization, a hallmark of collective behavior, where interactions lead to the adjustment of rhythms [2].

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010), [2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambridge, 2001).

Topological lattice of quantum van der Pol oscillators

Tight binding Hamiltonian



We study a lattice of $j = 1,\ldots,N$ sites, each consisting of a harmonic oscillator with identical frequencies ω_0. The tight-binding Hamiltonian is given by

$$H = H_0 + H_{\mathrm{top}} = \hbar \sum_j \omega_0 a_j^\dagger a_j + \hbar \sum_j \sum_{j' \neq j} \lambda_{jj'} (a_j^\dagger a_{j'} + a_{j'}^\dagger a_j)$$

where $a_j^\dagger (a_j)$ denote bosonic creation (annihilation) operators. This general form of the system Hamiltonian allows us to realize different topological lattices via $\lambda_{jj'}$.

Nonlinear open quantum system

A quantum harmonic oscillator subject to one-phonon gain with rate κ_1 and two-phonon loss with rate κ_2 represents the quantum analogue of the classical van der Pol oscillator. In an open quantum system approach, the dynamics of the system density matrix $\varrho(t)$ is then given by the master equation [3]

$$\dot{\varrho} = -\frac{\mathrm{i}}{\hbar}\left[H, \varrho\right] + \sum_j \left\{ \kappa_1 \mathscr{D}\left[a_j^\dagger\right]\varrho + \kappa_2 \mathscr{D}\left[a_j^2\right]\varrho \right\},$$

where we use the notation $\mathscr{D}\left[O\right]\varrho = O\varrho O^\dagger - \frac{1}{2}\left\{O^\dagger O, \varrho\right\}$.

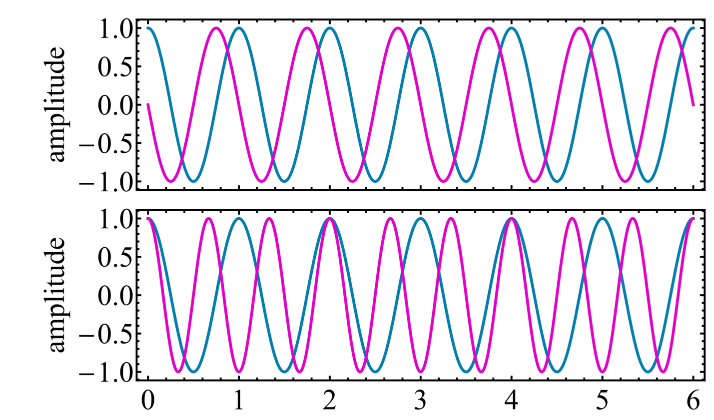
[3] Tony E. Lee and H. R. Sadeghpour, Phys. Rev. Lett. **111**, 234101 (2013).

Synchronization of classical and quantum systems

Classical frequency synchronization

Synchronization of classical oscillatory systems can take many different forms. Here, we refer to two oscillatory systems being synchronized if they oscillate with the same frequency at a fixed phase difference:

$$\frac{d}{dt}\Delta\varphi_{jj'} = \frac{d}{dt}\left(\varphi_j - \varphi_{j'}\right) = 0.$$



Quantum synchronization

Generalizing the classical notion of synchronization to the quantum regime is challenging as phase space trajectories become ill defined concepts. Here, we use a measure to quantify synchronization of two quantum systems based on their dimensionless quadratures [4]

$$S_c(j, j') = \left\langle (X_{2j-1} - X_{2j'-1})^2 + (X_{2j} - X_{2j'})^2 \right\rangle^{-1} \leq 1,$$

where $X_{2j-1} = (a_j + a_j^\dagger)/\sqrt{2}$ and $X_{2j} = -\mathrm{i}(a_j - a_j^\dagger)/\sqrt{2}$, which extends the classical concept of 'error' into the quantum domain, i.e., the smaller the variance of the quadrature differences is, the larger is the synchronization measure, indicating how equivalent the dynamics of two quantum systems are.

[4] Mari *et al.*, Phys. Rev. Lett. **111**, 103605 (2013).

Classical mean field model

Classical equations of motion can be obtained for the expectation values $\alpha_j = \langle a_j \rangle$ by performing a mean field approximation. The governing equation of the complex-valued mean field amplitudes $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ is then given by

$$\dot{\boldsymbol{\alpha}} = -\frac{\mathrm{i}}{\hbar}\left(\underline{\mathrm{H}}_0 + \underline{\mathrm{H}}_{\mathrm{top}}\right)\boldsymbol{\alpha} + \frac{\kappa_1}{2}\boldsymbol{\alpha} - \kappa_2\left(\boldsymbol{\alpha} \odot \boldsymbol{\alpha}^* \odot \boldsymbol{\alpha}\right),$$

where $\odot$ denotes the Hadamard product defined as $(\boldsymbol{\alpha} \odot \boldsymbol{\alpha}^* \odot \boldsymbol{\alpha})_n = \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n^* \cdot \boldsymbol{\alpha}_n$

Classical synchronization in the SSH chain

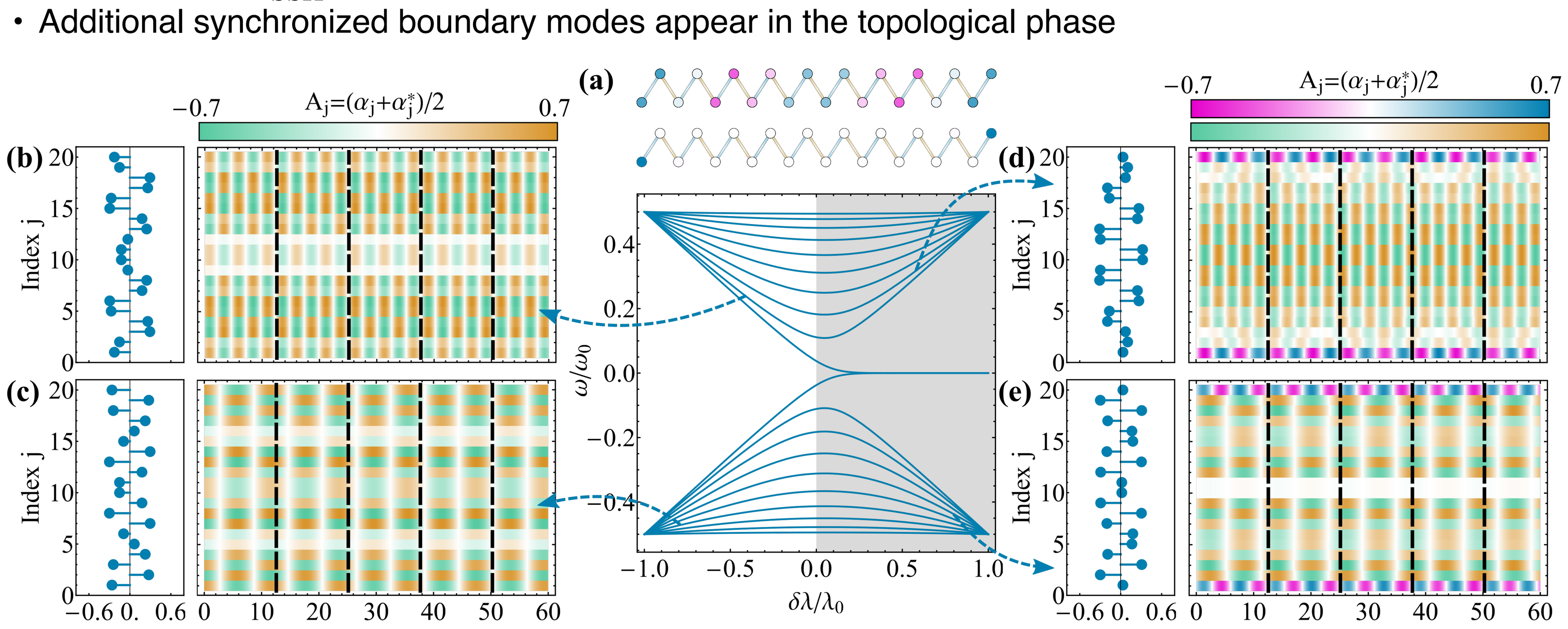
The SSH model is a one-dimensional dimerized lattice with staggered nearest-neighbor hopping and time reversal, particle-hole and chiral symmetry. The Hamiltonian is given by

$$H_{\mathrm{SSH}} = \hbar \sum_j \lambda_j (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j),$$

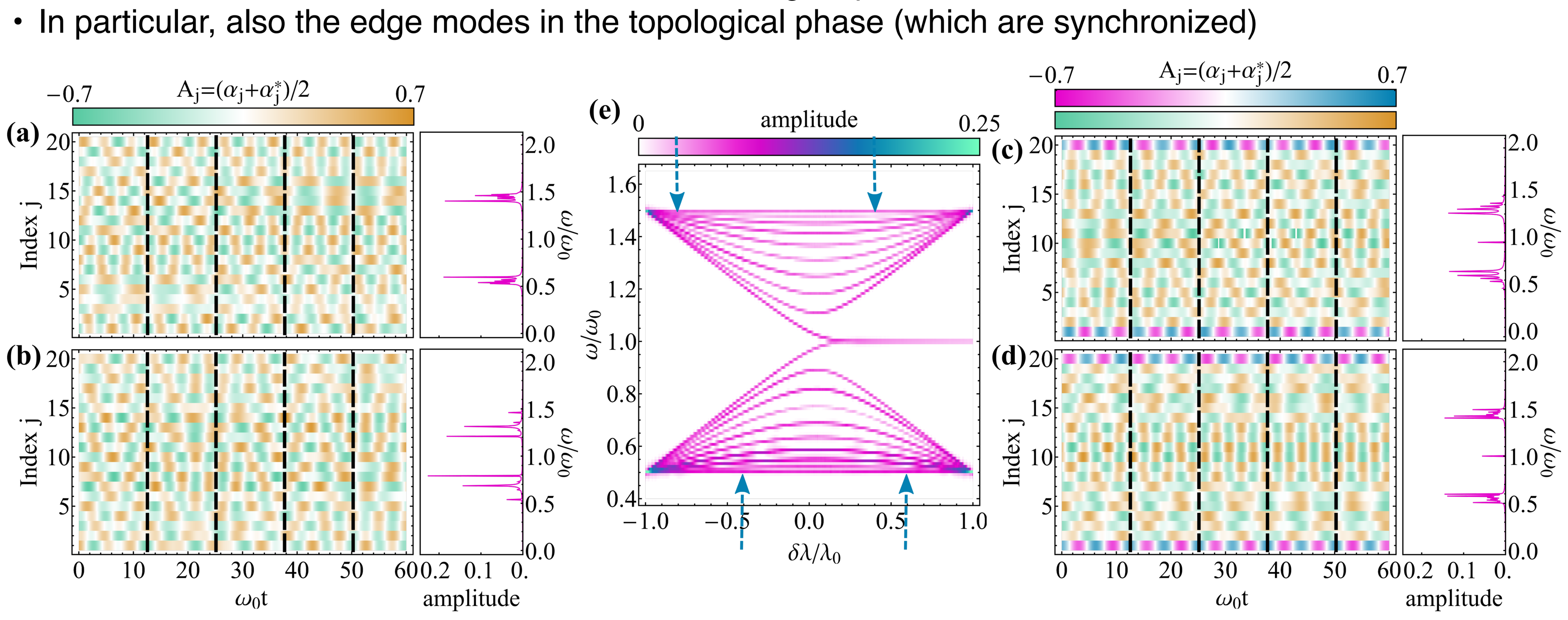
where $\lambda_j = \lambda_1$ if j is odd and $\lambda_j = \lambda_2$ otherwise.

Topological boundary mode synchronization

- Eigenstate of $\underline{\mathrm{H}}_{\mathrm{SSH}}$ as initial state translates directly to dynamics of the oscillators
- Additional synchronized boundary modes appear in the topological phase

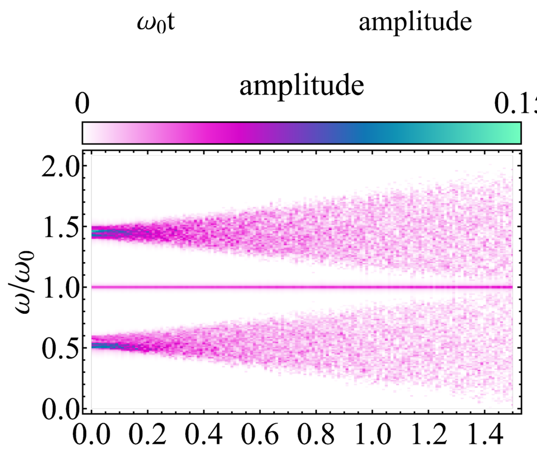


- Random initial conditions allow to reconstruct the eigenspectrum
- In particular, also the edge modes in the topological phase (which are synchronized)



Topological robustness to local perturbations

Topological insulators exhibit extremely robust surface states since no local perturbation can change their global topology. To test whether this extraordinary feature still persists, we apply random disorder $\lambda_j \rightarrow \lambda_j + \delta\lambda_j^{\mathrm{r}}$ with $\delta\lambda_j^{\mathrm{r}} \sim \mathscr{U}(-r, r)$. While the frequencies within the upper and lower band spread, the edge mode persists even for large disorder strengths



Effective quantum model

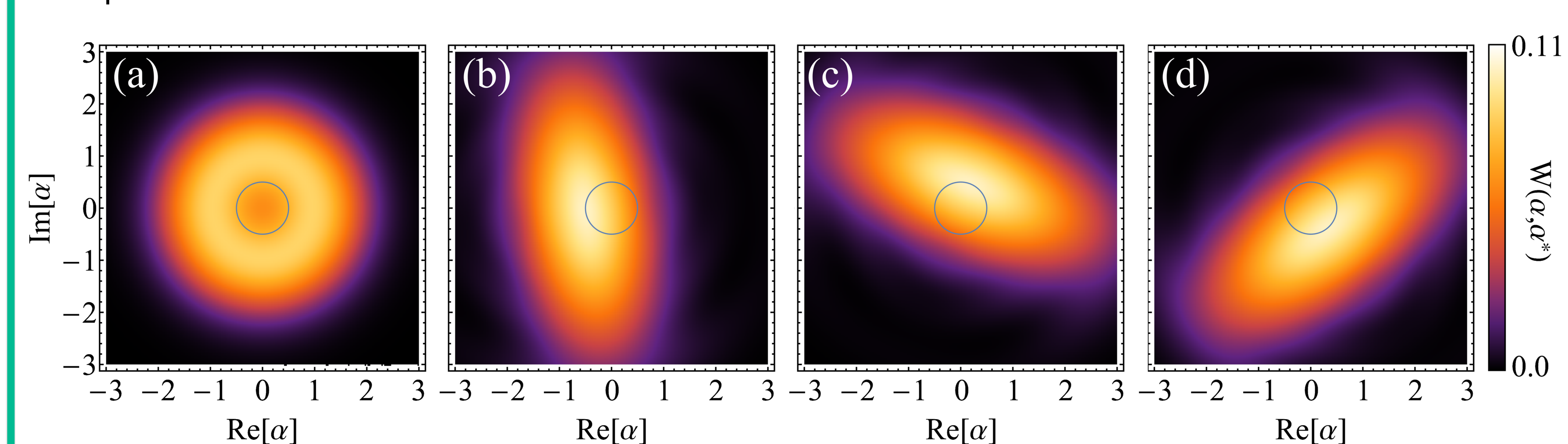
To solve the full open quantum system is a non-trivial task due to the large number of interacting oscillators and the involved nonlinearities in the dissipators. We therefore define the density matrix in the displaced frame as $\varrho_\alpha(t) = D^\dagger[\boldsymbol{\alpha}(t)]\varrho(t)D[\boldsymbol{\alpha}(t)]$ [5]. By neglecting terms of order $\mathscr{O}(a_j^3)$, we obtain an effective master equation of Lindblad form,

$$\dot{\varrho}_\alpha(t) = -\frac{\mathrm{i}}{\hbar}\left[H_\alpha(t), \varrho_\alpha(t)\right] + \sum_j \left\{ \kappa_1 \mathscr{D}\left[a_j^\dagger\right]\varrho_\alpha(t) + 4\kappa_2 \left|\alpha_j(t)\right|^2 \mathscr{D}\left[a_j\right]\varrho_\alpha(t) \right\}$$

with effective time-dependent (squeezing) Hamiltonian

$$H_\alpha(t) = H - \mathrm{i}\hbar\frac{\kappa_2}{2}\sum_j \left\{ \left[\alpha_j(t)\right]^2 (a_j^\dagger)^2 - \left[\alpha_j^*(t)\right]^2 a_j^2 \right\}.$$

The time-dependent mean field amplitudes appear in the effective Hamiltonian and the time-dependent dissipation rates.



Covariance matrix

As the Hamiltonian is quadratic, the dynamics of the effective quantum model is fully described by the covariance matrix of the quadratures $\underline{\mathrm{C}}_{mn}(t) = \mathrm{Tr}[\varrho_\alpha(t)\{\mathrm{X_m}, \mathrm{X_n}\}/2]$ with equation of motion

$$\underline{\dot{\mathrm{C}}}(t) = \underline{\mathrm{B}}(t)\,\underline{\mathrm{C}}(t) + \underline{\mathrm{C}}(t)\,\underline{\mathrm{B}}^{\mathrm{T}}(t) + \underline{\mathrm{D}}(t),$$

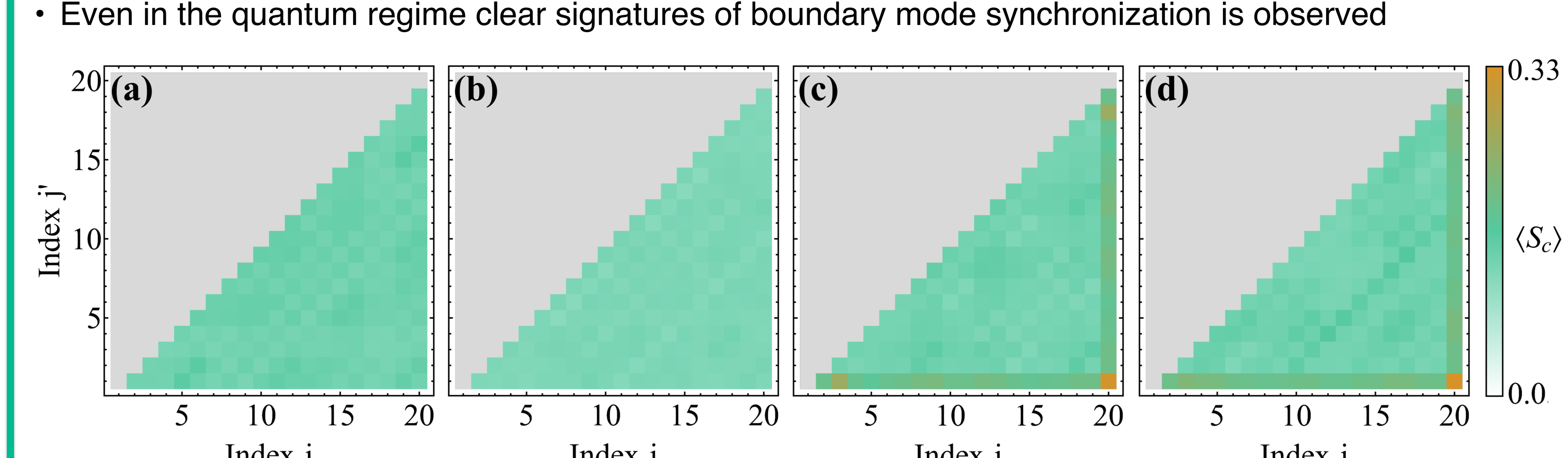
where $\underline{\mathrm{B}}(t)$ and $\underline{\mathrm{D}}(t)$ are determined through the master equation. The synchronization measure is easily accessible through the entries of this covariance matrix.

[5] Lörch *et al.*, Phys. Rev. X **4**, 011015 (2014).

Quantum synchronization in the SSH chain

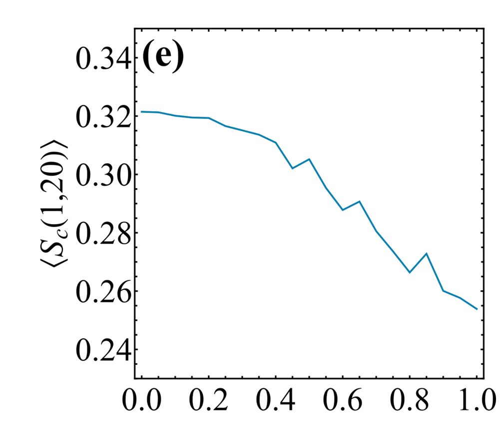
Topological boundary mode synchronization

- Even in the quantum regime clear signatures of boundary mode synchronization is observed



Topological robustness to local perturbations

The topologically protected synchronization of edge modes observed for the classical mean field amplitudes also persists for the quantum model over a long range of (symmetry-preserving) random disorder $\lambda_j \rightarrow \lambda_j + \delta\lambda_j^{\mathrm{r}}$ with $\delta\lambda_j^{\mathrm{r}} \sim \mathscr{U}(-r, r)$. This shows that the effect goes beyond a simple classical mean field description.



Conclusions

- Coupled (quantum) van der Pol oscillators may reflect the features of an underlying topological lattice even though the system is highly nonlinear and far away from equilibrium
- The topological character manifests as synchronized boundary modes, which are robust against random initial conditions and (symmetry-preserving) local perturbations.

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