Overview of Recent Progress on Renormalization Group Flows

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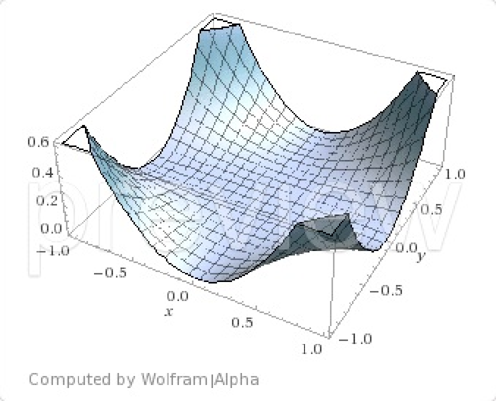
References

Several hundreds of **very recent** works directly or indirectly related to the subjects of this overview. I therefore apologize that the references given here do not do justice to many interesting works. Self references are omitted. Apologies to my collaborators in the audience!!

The idea that one can be forgetful about heavy degrees of freedom is very old. It is directly applicable in many realistic physical problems and it is extremely powerful.

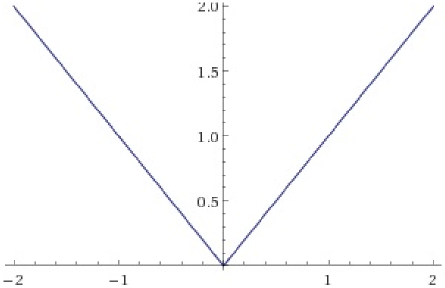
Introduction

Example: Quantum Mechanics with $V = \frac{1}{2}x^2y^2$, i.e. $2H = \dot{x}^2 + \dot{y}^2 + x^2y^2$. Is the spectrum continuous or discrete?



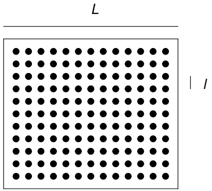
Introduction

Solution: There are flat directions at $y = 0$, $x = 0$ so the naive answer is that we have a continuous spectrum (like a free particle). However, consider large x. Then, y is a very heavy degree of freedom. It sits at the ground state with energy $E = \frac{1}{2}x$. So we can integrate out y and find the effective theory $2H = \dot{x}^2 + |x|$. This is a confining potential$\longrightarrow$ discrete spectrum.



Introduction

In the context of QFT, we put the system in a large box L with some cutoff l. Then we have $\sim (L/l)^d$ degrees of freedom.



We package them into functions defined in the box and we provide a Hamiltonian $H[f_i]$. Very roughly speaking, denoting the number of functions we need by K, the total number of degrees of freedom is $K(L/l)^d$.

Introduction

We can then ask many questions, such as

- Characterize situations when we have self similarity (i.e. we integrate out heavy degrees of freedom and end up with the same Hamiltonian, either after every step, or after finitely many steps).
- The overall number of degrees of freedom is reduced in the process of decreasing the cutoff l^{-1}, but what about K? Can we make this a well-posed problem in the continuum?

These questions appear to be very closely related.

General Comments on Theories without a Mass Scale

If we have a fixed point, it must not have a mass scale, for otherwise, sufficiently many re-scalings of l would change the Hamiltonian.

Suppose we have such a theory without a mass scale. In the simplest case this means that all the correlation functions are power laws. The naive symmetry group:

$$ISO(d) \rtimes \mathbb{R} \ .$$

Surprisingly, we often discover that the symmetry group is actually

$$SO(d+1,1)$$

So we have d unexpected conserved charges.

General Comments on Theories without a Mass Scale

The idea that this symmetry enhancement is a general phenomenon in QFT has been around for many decades (Migdal, Polyakov, Wilson, and others wrote about this already in the 70s).

It has been realized fairly early (although I am not sure when and by whom) that unitarity is a key ingredient in having these d extra generators.

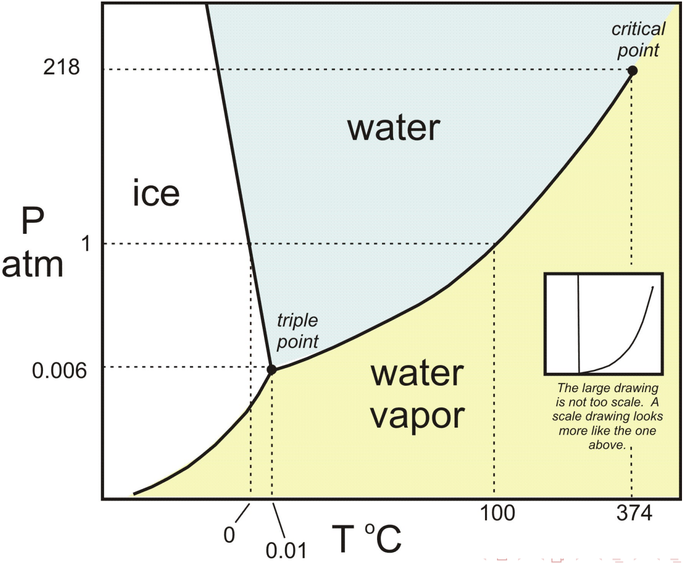
General Comments on Theories without a Mass Scale

The additional d generators are the special conformal transformations.

They are extremely important. They allow to fix three-point functions in terms of finitely many coefficients and they lead to many other constraints, such as inequalities among anomalous dimensions.

There is experimental, numerical (bootstrap, Monte Carlo), and theoretical evidence that we have the $SO(d, 2)$ enhanced symmetry in unitary theories.

General Comments on Theories without a Mass Scale



Monotonicity and Asymptotics in $d = 2$

One can define the number of degrees of freedom, c, as the coefficient of the two-point function $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ at the conformal fixed point. Zamolodchikov showed that for any flow $CFT_{uv} \to CFT_{ir}$ we have $c_{uv} > c_{ir}$.

Polchinski has shown that, under favorable assumptions, scale invariance implies conformal invariance. The two results use rather similar techniques and appear to be closely related.

Monotonicity and Asymptotics in $d = 3$

We can compute $Z_{\mathbb{S}^3}$:

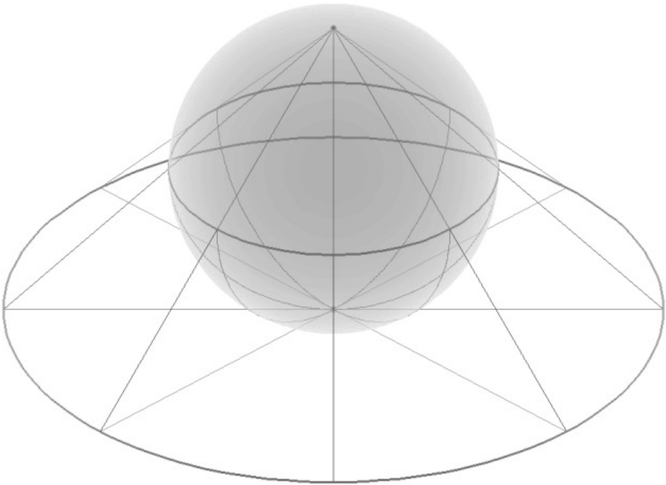
$$-\log Z_{\mathbb{S}^3} = \Lambda_{UV}^3 + \Lambda_{UV} + f$$

The cubic divergence is associated to $\int d^3x\sqrt{g}$ and the linear one to $\int d^3xR$. The only conceivable counter-term for f is $\int d^3x \left(\omega \wedge d\omega + \omega^3\right)$ but this actually vanishes on the sphere.

Hence, f is a physical observable.

Monotonicity and Asymptotics in $d = 3$

This computation makes sense because there is a canonical way to place a CFT_3 on $\mathbb{S}^3$, via the stereographic mapping.



Monotonicity and Asymptotics in $d = 3$

Recent work has shown that it is an extremely interesting observable!

- f is nonzero even in topological field theories, like in CS, where for $U(1)_k$ we have $f = \log k$.
- It seems that we always have $f > 0$, even in dynamical theories. Proof?
- f can be computed by localization of $\mathcal{N} = 2$ theories on $\mathbb{S}^3$ [Kapustin-Willet-Yaakov, Jafferis]. This leads to a lot of "data" about f and its behavior in different RG flows.

If we have $CFT_{uv} \to CFT_{ir}$ we have $f_{uv} > f_{ir}$, as was first conjectured by [Myers-Sinha, Jafferis-Klebanov-Pufu-Safdi] based on Holography and $\mathcal{N} = 2$ SUSY theories.

Monotonicity and Asymptotics in $d = 3$

- $f_{uv} > f_{ir}$ includes topological degrees of freedom, not just dynamical ones! this is different from $d = 2$ (and $d = 4$). Are there examples where topological dofs morph into dynamical ones?! An exponentially large CS term could morph into $O(1)$ propagating dofs.
- A pure $U(1)$ gauge theory has "infinitely many" degrees of freedom in the UV. Proof: it can flow to the topological theory $U(1)_k$ for all $k \in \mathbb{Z}$, and the latter have $f = \log k$.
- Localizing on a squashed sphere, $\mathbb{S}^3_b$, one can extract exact information about correlation functions like $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ in $\mathcal{N} = 2$ SCFTs on $\mathbb{R}^4$. Can we re-derive these results directly by some clever analysis in flat space (maybe along the lines of [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees]....)?

Monotonicity and Asymptotics in $d = 3$

Not all unitarity scale-invariant theories are conformal. An $\mathbb{R}$ gauge theory is a counter-example. The infinitely many degrees of freedom we associated to a free photon are related to its nonconformality in the UV.

This counter-example to scale$\rightarrow$conformal is special for two reasons

- It is free.
- On $\mathbb{R}^3$ it is physically indistinguishable from a free scalar in 3d. The latter is conformal. Hence, the non-conformality of a free photon is only a "formality."

Monotonicity and Asymptotics in $d = 4$

The number of degrees of freedom of a CFT is defined by the Weyl anomaly a, $T^{\mu}_{\mu} = aE_4 + ...$, and for $CFT_{uv} \to CFT_{ir}$ it satisfies $a_{uv} > a_{ir}$. It does not count topological dofs.

There has been a lot of recent work on the problem of scale/conformal invariance. Let us note that, as in $d = 3$, there is a simple free counter-example: the two-form gauge potential with a noncompact gauge group.

In $d = 4$ concrete results about the problem of scale/conformal invariance exist, and they explain why this counter-example has to be regarded as a "formality."

We will delve more deeply into the problem of scale vs conformal invariance, especially in $d = 4$.

Monotonicity and Asymptotics in $d > 4$

For $d = 5$ the natural conjecture is again that the finite constant in $Z_{\mathbb{S}^5}$ behaves monotonically under RG flows.

For $d = 6$ the natural conjecture is that the a-anomaly is monotonic.

In both cases there is no proof, although nice progress in $d = 6$ was done by [Elvang-Freedman-Hung-Kiermaier-Myers-Theisen].

Not much is known about the asymptotics of RG flows, although it is worth mentioning that some believe no nontrivial local CFTs exist for $d > 6$.

A Question

In $d = 2$, the free energy density, energy-momentum two-point function, sphere partition function, all coincide and behave monotonically.

In higher dimensions, very intuitive measures of the number of degrees of freedom such as the free energy density (which is often used in cosmology) are not generally monotonic. One can even find perturbative counter-examples.

Instead, one finds unconventional quantities like f, a etc. Why do they have to exist?

Monotonicity & Entanglement

Consider $\mathbb{S}^{d-2}$, and compute the EE associated to it, S_{EE}. Then for even d it contains the Weyl a-anomaly and for odd d it contains the finite piece in the partition function over S^d.

It is therefore an exciting idea that perhaps EE can provide an overarching principle that would explain monotonicity in general QFT_d.

We don't know whether that's going to be the case.

Monotonicity & Entanglement

Positive

- Monotonicity does remind of the second law, which is all about entropy.
- An argument for the c-theorem in $d = 2$ and an argument for the f-theorem in $d = 3$ was given by Casini-Huerta. This is related to the so-called strong subadditivity inequality.
- It comes out naturally from holography when one studies simple toy models for holographic RG flows [Myers-Sinha].

Monotonicity & Entanglement

Negative

- It is not clear how the EE is defined non-perturbatively (the analytic continuation to $n \to 1$ may or may not exist...). One can argue that there are no continuum counter-terms for the finite/log piece, but it is not clear how the continuum quantity is defined in the first place.
- There are some arguments that the current line of attack initiated by Casini-Huerta would not work for $d > 3$ [Liu-Mezei].

Monotonicity & Entanglement

- The c-functions constructed in $d = 2, d = 3$ via the EE are non-stationary and therefore "unphysical" [e.g. Klebanov-Nishioka-Pufu-Safdi]. To understand the evolution at very short distances, one first needs to choose a vacuum! By contrast, in the field-theoretic derivations of the c,a-theorems, one gets perfectly stationary interpolating functions, and one can compute them in perturbation theory around the UV.

Asymptotics

Let us now discuss recent progress regarding the asymptotics of QFT.

General Comments on Theories without a Mass Scale

Suppose that

$$T^\mu_\mu = \partial^\nu V_\nu$$

for some **local** operator V_ν. Then the theory is scale invariant and we have the conserved current

$$S_\mu = x^\nu T_{\mu\nu} - V_\mu \ .$$

To prove that a **unitary** scale invariant theory is conformal, one needs to show that

$$T^\mu_\mu = \Box L$$

for some **local** L. (Then we can improve to $T^\mu_\mu = 0$.)

Solution for $d = 2$

There is a nice argument solving the problem in $d = 2$ [Polchinski, 1988].

$d = 2$ is exceptionally simple because the scaling dimension of L is zero. So we just need to prove that in unitary scale invariant theories

$$T^{\mu}_{\mu} = 0 \ .$$

Strategy: Show that the two-point function $\langle T^{\mu}_{\mu}(x) T^{\mu}_{\mu}(0) \rangle = 0$ at $x \neq 0$.

Solution for $d = 2$

$$\langle T_{\mu\nu}(q) T_{\rho\sigma}(-q)\rangle = B(q^2)\tilde{q}_\mu \tilde{q}_\nu \tilde{q}_\rho \tilde{q}_\sigma \ ,$$

with $\tilde{q}_\mu = \epsilon_{\mu\nu} q^\nu$. This is the most general decomposition satisfying conservation and permutation symmetry. In a scale invariant theory we must take by dimensional analysis

$$B(q^2) = \frac{1}{q^2} \ .$$

Then,

$$\langle T^\mu_\mu(q) T^\rho_\rho(-q)\rangle \sim q^2 \ .$$

This is a contact term, thus, $T^\mu_\mu = 0$.

The Difficulty of the Problem for $d > 2$

There is no hope to repeat an argument of this kind in $d > 2$ because it is not true that unitarity and scale invariance imply that $T^\mu_\mu = 0$. Indeed, in many examples one finds a nontrivial L:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2$$

leads to $T^\mu_\mu = \frac{2-d}{4}\Box(\phi^2)$, i.e. $L = \frac{2-d}{4}\phi^2$. This is of course a conformal theory and $T^\mu_\mu = 0$ after an improvement.

The Difficulty of the Problem for $d > 2$

Take

$$\phi \simeq \phi + c \ , \qquad \text{for all } c$$

This is consistent because the set of operators where ϕ appears only with derivatives is closed under the OPE. It is local because we have the EM tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial \phi)^2 \ .$$

It is not conformal because the improvement $\sim \left(\partial_\mu \partial_\nu - \partial^2 \eta_{\mu\nu}\right) \phi^2$ is not an allowed operator.

In flat space this theory is indistinguishable from the ordinary scalar, it has consistent separated points correlation functions, OPE, consistent anomalies etc.

The Difficulty of the Problem for $d > 2$

So this theory is not conformal, but there is no local measurement on $\mathbb{R}^d$ that can distinguish it from a conformal theory.

There are no known scale invariant unitary theories which are distinguishable on $\mathbb{R}^d$ from conformal theories.

Perturbation theory at $d = 4$

In perturbation theory we have a clear list of candidates for L and V_μ and we need to check if the equations $T^\mu_\mu = \partial^\mu V_\mu$, $T^\mu_\mu = \Box L$ are satisfied. This has been checked very explicitly in many 4d models [Grinstein-Fortin-Stergiou] and a beautiful general argument (again in 4d) was offered by [Luty-Polchinski-Rattazzi] as well as [Osborn] and [Grinstein-Fortin-Stergiou].

Holography and SUSY

The problem also simplifies when there is a weakly-coupled holographic dual [Nakayama]. There is some evidence that all unitary solutions to $10d/11d$ Einstein equations with fluxes that are scale invariant are also conformal invariant. If that can be shown in some generality for $d > 2$ that would be fantastic.

Some simplification also takes place in SUSY theories, see for example [Antoniadis-Buican, Zheng, Nakayama, Fortin-Grinstein-Stergiou]

Outline of the Argument

Idea: since we need to prove that $T = \Box L$, let us try to establish the following necessary condition

$$\langle VAC | T^{\mu}_{\mu}(p_1) \ldots T^{\mu}_{\mu}(p_n) | Anything \rangle_{connected} = 0 \; , \qquad p_i^2 = 0 \; ,$$

and see where this takes us. Let us call this the "vanishing theorem." Of course, we assume unitarity – otherwise there are many counter-examples, which do not obey the vanishing theorem. Hence, the vanishing theorem is a nontrivial necessary condition.

Outline of the Argument

Note: [Luty, Polchinski, Rattazzi] estabslihed the case of $n = 2$, i.e.

$$\langle VAC|T^\mu_\mu(p_1)T^\mu_\mu(p_2)|Anything\rangle_{connected} = 0 \ , \qquad p_1^2 = p_2^2 = 0 \ .$$

A Proof of the Vanishing Theorem

We couple any SFT to a background metric. Then we can consider the generating functional $W[g_{\mu\nu}]$. The UV divergences are characterized by

$$\int d^4x\sqrt{g}\left(\Lambda + \mathfrak{a}R + \mathfrak{b}R^2 + \mathfrak{c}W^2\right) \ ,$$

Consider metrics of the type

$$g_{\mu\nu} = (1+\Psi)^2\eta_{\mu\nu}$$

with $\partial^2\Psi = 0$ then neither of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ contribute.

A Proof of the Vanishing Theorem

Thus $W[\Psi]$ is well defined up to a momentum-independent piece.

We define

$$A_n(p_1, ..., p_{2n}) = \frac{\delta^n W[\Psi]}{\delta\Psi(p_1)\delta\Psi(p_2)...\delta\Psi(p_{2n})}$$

and we will choose all the momenta to be null, $p_i^2 = 0$.

A Proof of the Vanishing Theorem

Let us start from $n = 2$. We can prepare forward kinematics $p_3 = -p_1$ and $p_4 = -p_2$. We have the dispersion relation

$$A_4(s) = \frac{1}{\pi} \int ds' \frac{ImA_4(s')}{s - s'} + subtractions \ , \qquad s = (p_1 + p_2)^2 \ .$$

By dimensional analysis, $ImA_4 = \kappa s^2$. We immediately see that $ImA_4 = 0$. Had it not been zero, we would have needed a subtraction which goes like s^2.

A similar argument proceeds for all the amplitudes A_{2n}, in other words, in forward kinematics

$$ImA_{2n} = 0$$

.

A Proof of the Vanishing Theorem

Now we use unitarity, more precisely, the optical theorem.

All the contributions to ImA_4 are positive definite since there is just one cut (s-channel and t-channel, depending on whether $s > 0$ or $s < 0$).
Hence,

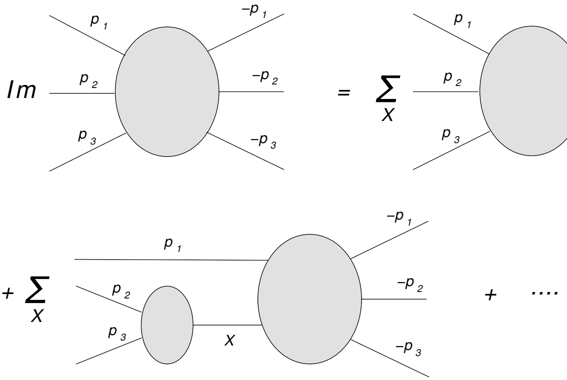
$$\langle T_\mu^\mu(p_1) T_\mu^\mu(p_2) | Anything \rangle = 0 \ , \qquad p_1^2 = p_2^2 = 0$$

A Proof of the Vanishing Theorem

Starting from $n = 3$, the situation is tougher.

- There are many cuts.
- Many of them are generally non-positive.

A Proof of the Vanishing Theorem



A Proof of the Vanishing Theorem

However, after some work one can show inductively that the non-positive cuts are absent. Thus,

$$\langle T^{\mu}_{\mu}(p_1)T^{\mu}_{\mu}(p_2)...T^{\mu}_{\mu}(p_n)|Anything\rangle = 0\ , \qquad p_1^2 = p_2^2 = .. = p_n^2 = 0$$

We have thus proved our nontrivial necessary condition.

A Proof of the Vanishing Theorem

The fact that $T^\mu_\mu(p_1)T^\mu_\mu(p_2)...T^\mu_\mu(p_n) = 0$ for all n on the light cone is very suggestive. Indeed, if one could say that this product is analytic in momentum, the vanishing on the light cone would imply the existence of some local L such that $T^\mu_\mu = \Box L$, simply by Taylor expanding around the light cone.

Let us see how to say this precisely:

A Proof of the Vanishing Theorem

Consider the effective field theory coupling Ψ (the conformal factor of the metric) to the SFT

$$\mathcal{S} = \int d^4x (\partial\Psi)^2 + \mathcal{S}_{SFT} + \frac{1}{M}\int d^4x \Psi T^\mu_\mu + \cdots$$

where the $\cdots$ are determined by diff invariance.

To leading order in $energy/M$, the S-matrix for Ψ scattering into SFT states is governed by our vanishing correlation functions $\langle T^\mu_\mu(p_1) T^\mu_\mu(p_2) ... T^\mu_\mu(p_n) | Anything \rangle = 0$.

A sufficient condition

Clearly, if the SFT is a CFT and $T_\mu^\mu = \Box L$, then the coupling $\frac{1}{M}\int d^4x \Psi T_\mu^\mu = \frac{1}{M}\int d^4x \Box\Psi L$ vanishes on-shell and can be removed by a local change of variables, consistent with the trivial S-matrix.

But we can also argue for the converse: a trivial S-matrix means the theories are decoupled. Hence, there is a local L such that $T_\mu^\mu = \Box L$.

An S-matrix Digression

Let us take two theories A and B. Suppose there is a local change of variables connecting A and B. Then $S_A = S_B$.

Does it follow from $H_A \simeq H_B$ and $S_A \simeq S_B$ that there is a local change of variables connecting A and B? The answer is negative. For example, the kink-field duality, electric-magnetic duality...

However, here we just have a *small perturbation* of an existing model with trivial S-matrix. If such a small perturbation does not affect the S-matrix, then the perturbation must vanish on-shell and the change of variables needs to be local.

It is like saying that the S-matrix characterizes the physical theory modulo topological degrees of freedom that don't play any role in $\mathbb{R}^4$.

A sufficient condition

Let us explain how the 2-form fits into this. We cannot solve $T_\mu^\mu = \Box L$. However, since the theory is physically indistinguishable in $\mathbb{R}^4$ from a conformal theory, the S-matrix is insensitive to this subtle zero mode that is absent. So the vanishing theorem is obeyed.

Conclusion

Our precise conclusion is that unitary scale invariant theories are either conformal or indistinguishable from conformal theories on $\mathbb{R}^4$. This means that, for all practical purposes, scale invariance and unitarity imply conformality.

So when Rob said that scale invariance "more or less" implies conformal invariance in four dimensions, this is presumably what he meant.

A List of a Few Tangible Challenges

- Perturbative proof that in $d = 3$ scale invariance implies conformal invariance in CS+matter theories.
- $f > 0$ (or "Z" < 1). If this is to measure dofs, better be true.
- Explore 3d RG flows and check whether we *really need* to assign nonzero number of dofs to topological sectors.
- In $\mathcal{N} = 2$ SUSY theories various quantities other than a seem always monotonic (e.g. c). Why?
- Hofman-Maldacena showed that $\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$. Examples show that all interacting theories sit in a *strictly smaller* window. Why?
- A connection between scale/conformal invariance and EE?
- A holographic understanding why scale invariance implies conformal invariance.

Discussion: Renormalization Group For non-Equilibrium Systems?

In thermal equilibrium one pumps energy and energy gets dissipated via the fluctuation-dissipation theorem. There is equilibrium at any given frequency ω. The RG formalism applies and leads to many celebrated results.

Imagine: we pump energy at some scale ω_{pump} but we arrange the system to dissipate the energy only at some ω_{diss}.

- In 2d turbulence we have $\omega_{pump} \ll \omega_{diss}$. The energy is inserted by some small scale stirring and it is dissipated by very large eddies. This is commonplace in the atmosphere.
- In 3d turbulence we have $\omega_{pump} \gg \omega_{diss}$. We stir on a very large scale and the eddies become smaller and smaller until they are tiny and dissipate.

This is extreme non-equilibrium.

Discussion: Renormalization Group For non-Equilibrium Systems?

Discussion: Renormalization Group For non-Equilibrium Systems?

In both cases we have a small number, either $\omega_{pump}/\omega_{diss}$ or its inverse, such that one can hope for universality.The energy scales in between ω_{pump} and ω_{diss} are called the "inertial range."

The inertial range displays scale invariance of a sort (perhaps spontaneously broken) and great degree of universality.

For example, in 2d, one finds critical exponents that are completely independent of the details of the stirring force at short distance – i.e. one forgets about the UV!!

In 3d there is apparently a slightly lesser degree of universality, and also the flow of information is from the IR to the UV...