PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Quantum Fields beyond Perturbation Theory

KITP, January 27 – 31, 2014

Entanglement & C-theorems

R. Myers

Zamolodchikov c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i = 1,2,3,\cdots\}$ with beta-functions as "velocities"

- for unitary, Lorentz-inv. QFT's in two dimensions, there exists a positive-definite real function of the coupling constants $C(g)$:

1. monotonically decreasing along flows: $\frac{d}{dt}C(g) \leq 0$

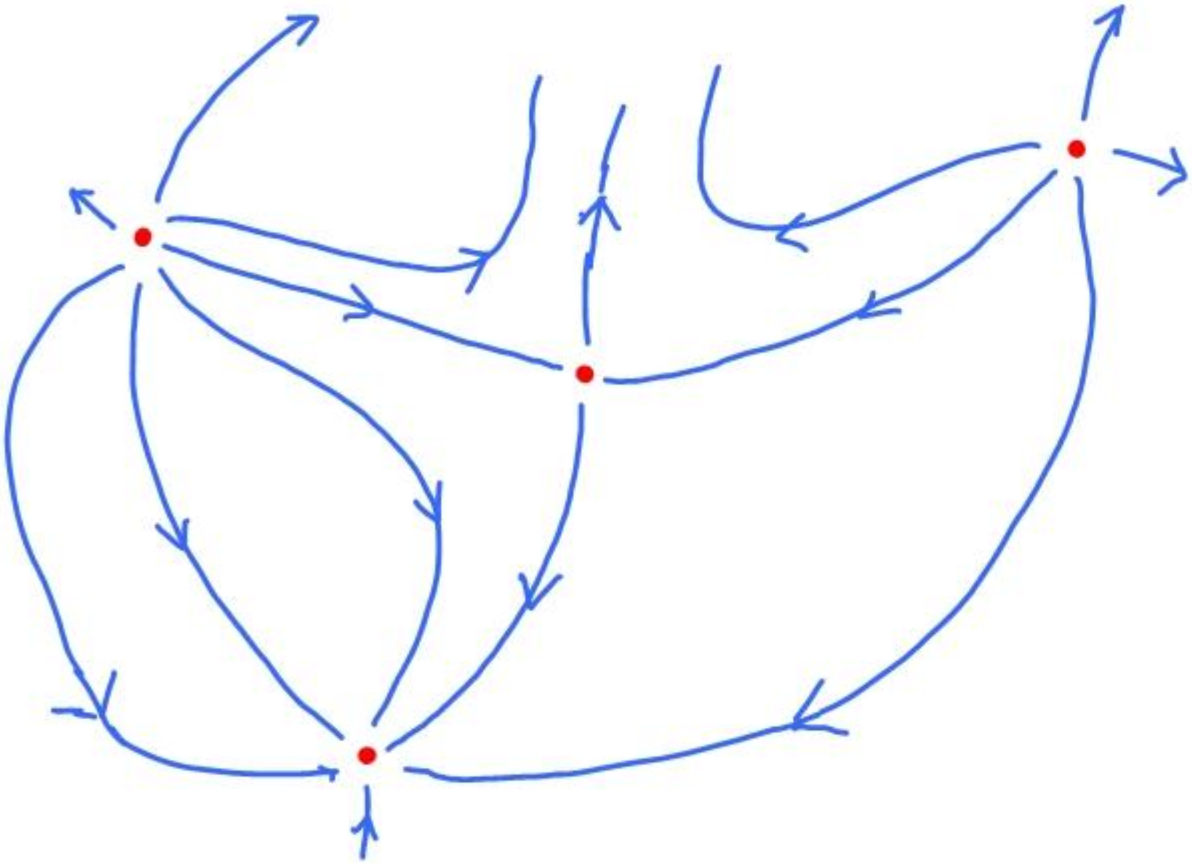
2. "stationary" at fixed points $g^i = (g^*)^i$:

$$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i}C(g) = 0$$

3. at fixed points, it equals central charge of corresponding CFT

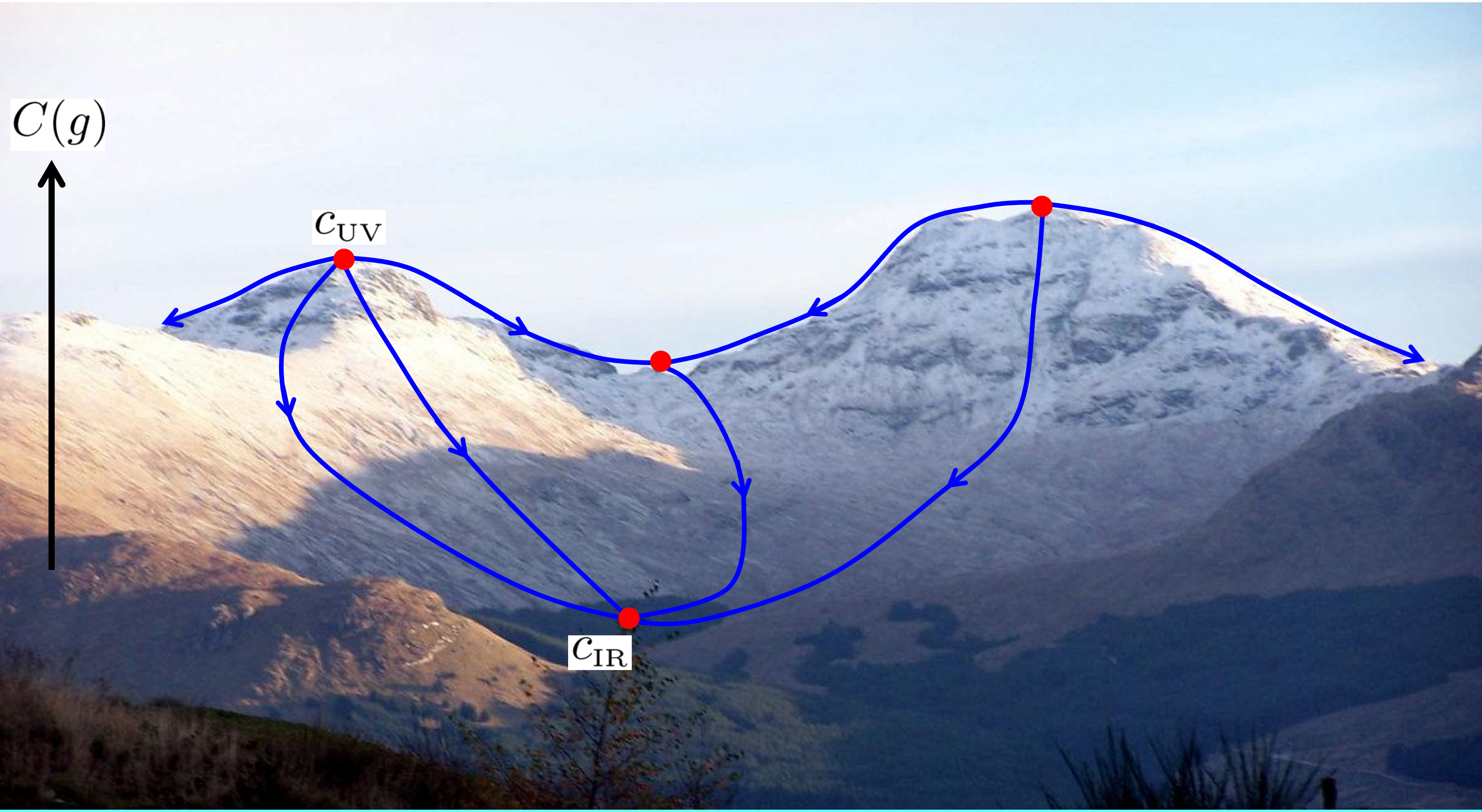
$$C(g^*) = c$$

Zamolodchikov's C-function adds a dimension to RG flows:



BECOMES

Zamolodchikov's C-function adds a dimension to RG flows:



Simple consequence for any RG flow in d=2: $c_{\mathrm{UV}} > c_{\mathrm{IR}}$

(Casini & Huerta '04)

Entanglement and c-theorem: Part 1

- c-theorem for d=2 RG flows can be established using unitarity, Lorentz invariance and **strong subadditivity inequality**:

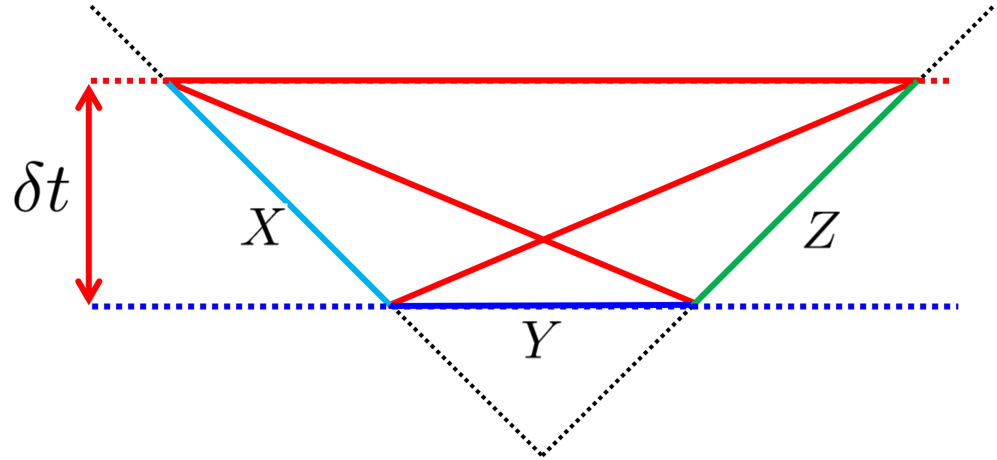
$$S(X\cup Y\cup Z)-S(X\cup Y)-S(Y\cup Z)+S(Y)\leq 0$$

- for d=2 CFT: $S(\ell)=\frac{c}{3}\log(\ell/\delta)+a_0$ (Holzhey, Larsen & Wilczek) (Calabrese & Cardy)

- define: $C(\ell)=3\,\ell\;\partial_\ell S(\ell)$ $\longrightarrow$ $C_{\rm CFT}(\ell)=c$

- with SSA and limit $\delta t\to 0$

$\longrightarrow$ $\partial_\ell C(\ell)\leq 0$



- hence $C(\ell)$ decreases monotonically and $c_{\rm UV}>c_{\rm IR}$
- note: no simple map between Zamo. and entropic C-functions

Entanglement and c-theorem: Part 2 (RM & Sinha `10)

- next connection to entanglement emerged for c-theorems in higher dimensions using holography

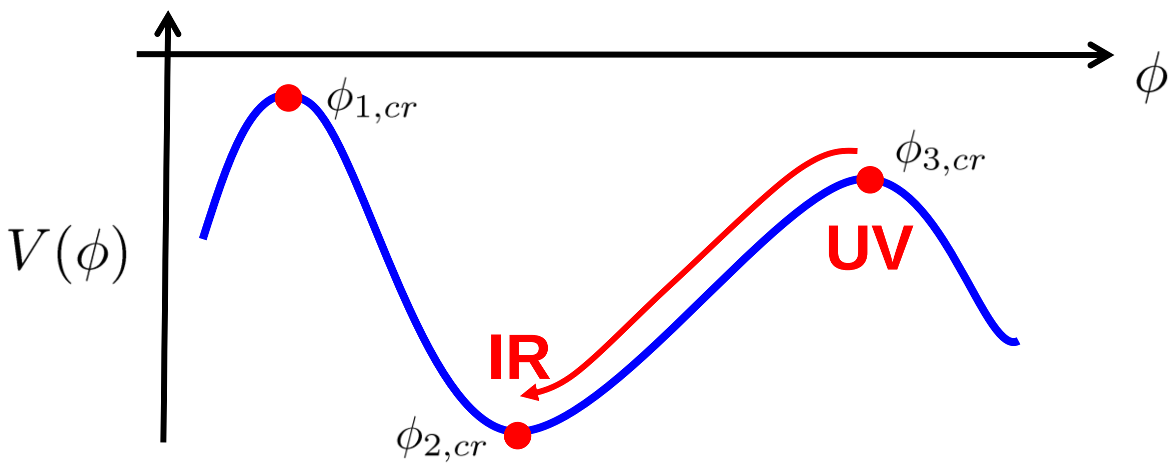
- first recall standard holographic RG flows (Girardello, Petrini, Porrati and Zaffaroni, `98) (Freedman, Gubser, Pilch & Warner, `99)

$$I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right]$$

- imagine potential has stationary points giving negative Λ

$$\longrightarrow \quad V(\phi_{i,cr}) = -\frac{d(d-1)}{L^2}\alpha_i^2$$

- hRG flow: solution starts at one stationary point at large radius and ends at another at small radius – connects CFT_{UV} to CFT_{IR}



Holographic c-theorems:

(Girardello, Petrini, Porrati and Zaffaroni, `98)
(Freedman, Gubser, Pilch & Warner, `99)

- consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + \cdots + dx_{d-1}^2) + dr^2$
- at stationary points, AdS_5 vacuum: $A(r) = r/\tilde{L}$ with $\tilde{L} = L/\alpha_i$
- for hRG flow solutions, define: $a_d(r) \equiv \dfrac{\pi^{d/2}}{\Gamma(d/2)\,(\ell_P A'(r))^{d-1}}$

$$a'(r) \;=\; -\frac{(d-1)\pi^{d/2}}{\Gamma(d/2)\,\ell_P^{d-1} A'(r)^d} A''(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)\,\ell_P^{d-1} A'(r)^d}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

Einstein equations ⤴ **null energy condition** ⤴

- at stationary points, $a(r) \to a^* = \pi^{d/2}/\Gamma(d/2)\,(\tilde{L}/\ell_P)^{d-1}$ and so

$$\boxed{a^*_{UV} \geq a^*_{IR}}$$

- using holographic trace anomaly: $a^* \propto$ central charges

(e.g., Henningson & Skenderis)

- above for even d; what about odd d?
- all central charges equal for Einstein gravity

(RM & Sinha `10)

“Improved” Holographic RG Flows:

- add higher curvature interactions to bulk gravity action

→ provides holographic field theories with, eg, $a \neq c$
so that we can clearly distinguish evidence of a-theorem

(Nojiri & Odintsov; Blau, Narain & Gava)

- construct “toy models” with fixed set of higher curvature terms (where we can maintain control of calculations)

What about the swampland?

- constrain gravitational couplings with consistency tests (positive fluxes; causality; unitarity) and use best judgement

- ultimately one needs to fully develop string theory for interesting holographic backgrounds!

- *“if certain general characteristics are true for all CFT’s, then holographic CFT’s will exhibit the same features”*

Toy model:

(RM & Robinsion; RM, Paulos & Sinha)

$$I = \frac{1}{2\ell_P^{d-1}} \int \mathrm{d}^{d+1}x \sqrt{-g} \left[\frac{d(d-1)}{L^2} \alpha + R + \frac{\lambda L^2}{(d-2)(d-3)} \mathcal{X}_4 - \frac{8(2d-1) \mu L^4}{(d-5)(d-2)(3d^2-21d+4)} \mathcal{Z}_{d+1} \right]$$

curvature squared

curvature cubed

- three dimensionless couplings: $L/\ell_P\,,\ \lambda\,,\ \mu$
- for holographic RG flows with general d, gravitational eom and null energy condition yield

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

where $a_d^* = \frac{\pi^{d/2} L^{d-1}}{\Gamma(d/2) f_\infty^{\frac{d-1}{2}} \ell_P^{d-1}} \left(1 - \frac{2(d-1)}{d-3} \lambda f_\infty - \frac{3(d-1)}{d-5} \mu f_\infty^2 \right)$

with $\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

What is a_d^* ??

$$a_d^* = \frac{\pi^{d/2} L^{d-1}}{\Gamma(d/2)\, f_\infty^{\frac{d-1}{2}}\, \ell_P^{d-1}} \left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty - \frac{3(d-1)}{d-5}\mu f_\infty^2\right)$$

with $\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

- a_d^* is NOT C_T , coefficient of leading singularity in

$$\langle\, T_{ab}(x)\, T_{cd}(0)\,\rangle = \frac{C_T}{x^{2d}}\, \mathcal{I}_{ab,cd}(x)$$

- a_d^* is NOT C_S , coefficient in entropy density: $s = C_S\, T^{d-1}$

What is a_d^* ??

$$a_d^* = \frac{\pi^{d/2} L^{d-1}}{\Gamma(d/2) f_\infty^{\frac{d-1}{2}} \ell_P^{d-1}} \left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty - \frac{3(d-1)}{d-5}\mu f_\infty^2\right)$$

with $\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

- trace anomaly for CFT's with even d: (Deser & Schwimmer)

$$\langle\, T_\mu{}^\mu \,\rangle = \sum B_i (\text{Weyl invariant})_i - 2(-)^{d/2} A\, (\text{Euler density})_d$$

- can verify that above precisely reproduces central charge

$$\boxed{a_d^* = A}$$

(Henningson & Skenderis; Nojiri & Odintsov; Blau, Narain & Gava; Imbimbo, Schwimmer, Theisen & Yankielowicz)

- holographic c-theorem: $(a_d^*)_{UV} \geq (a_d^*)_{IR}$

→ agrees with Cardy's conjecture (1988)

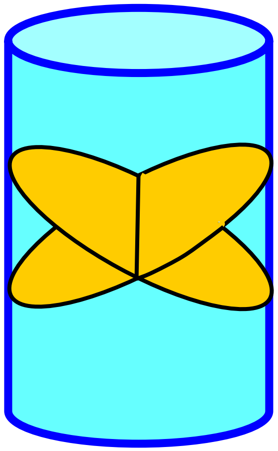
What about odd d?

(Casini, Huerta & RM; RM & Sinha)

Holographic Entanglement Entropy:

- S_{EE} for CFT in d-dim. flat space and choose S^{d-2} with radius R
- conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and $T = 1/2\pi R$
- holographic dictionary: thermal bath in CFT = black hole in AdS

$$S_{EE} = S_{thermal} = S_{horizon}$$



- desired "black hole" is a hyperbolic foliation of AdS
- bulk coordinate transformation implements desired conformal transformation on boundary
- apply Wald's formula (for any gravity theory) for horizon entropy

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \log(2R/\delta) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots \quad \text{for odd d}$$

Entropic C-theorem conjecture:

(RM & Sinha)

- identify central charge with universal contribution in entanglement entropy of ground state of CFT across sphere S^{d-2} of radius R:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1} 4\, a_d^* \log(2R/\delta) & \text{for even } d \\ (-)^{\frac{d-1}{2}} 2\pi\, a_d^* & \text{for odd } d \end{cases}$$

- for RG flows connecting two fixed points

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

⟶ unified framework to consider c-theorem for **odd** or even *d*

⟶ connect to Cardy's conjecture: $a_d^* = A$ for any CFT in even *d*

F-theorem:

(Jafferis, Klebanov, Pufu & Safdi)

- examine partition function for broad classes of 3-dimensional quantum field theories on three-sphere (SUSY gauge theories, perturbed CFT's & O(N) models)
- in all examples, F= – log Z(S^3)>0 and decreases along RG flows

$\longrightarrow$ **conjecture:** $F_{UV} \;>\; F_{IR}$

- also naturally generalizes to higher odd *d*

- **coincides with entropic c-theorem** (Casini, Huerta & RM)
- focusing on renormalized or universal contributions, eg,

$$F_3 = -\,\log Z\big|_{finite} = -S_{univ} = 2\pi\, a_3^*\,.$$

- generalizes to general odd d:

$$F_d = -\,\log Z\big|_{finite} = -S_{univ} = (-)^{\frac{d+1}{2}}\, 2\pi\, a_d^*\,.$$

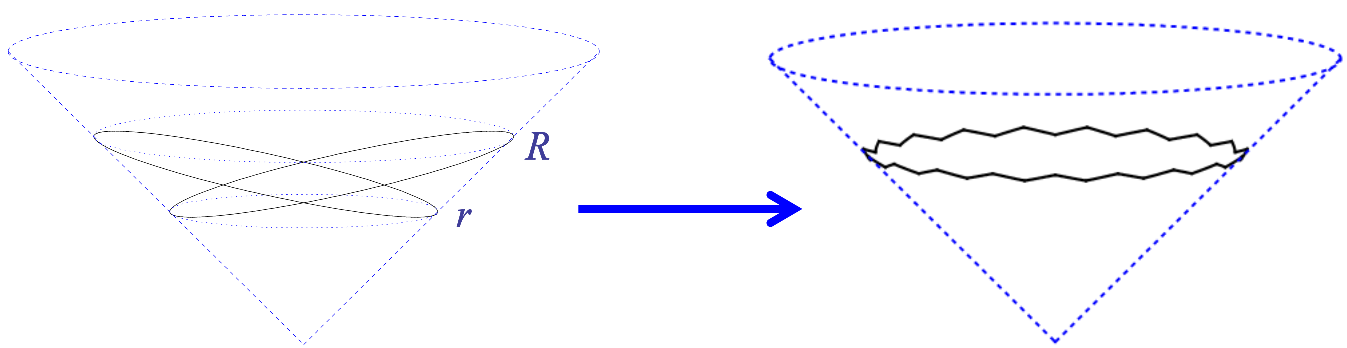
(Casini & Huerta '12)

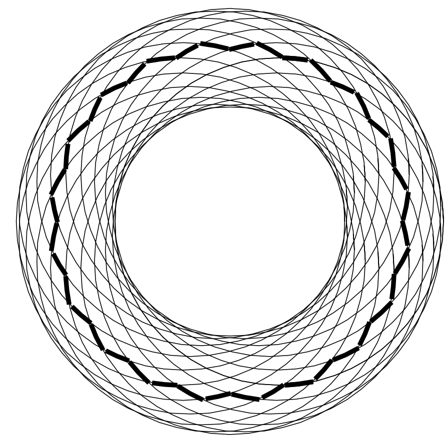
Entanglement proof of F-theorem:

- F-theorem for d=3 RG flows established using unitarity, Lorentz invariance and **strong subadditivity**

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)$$

- geometry more complex than d=2: consider many circles intersecting on **null** cone





- no corner contribution from intersection in null plane
- define: $C(R) = RS'(R) - S(R)$
- for d=3 CFT: $S(R) = c_0\, R - 2\pi a_3 \longrightarrow C_{\text{CFT}}(R) = 2\pi a_3$
- with SSA and "continuum" limit $\longrightarrow \partial_R C(R) \leq 0$
- hence $C(R)$ decreases monotonically and $[a_3]_{\text{UV}} > [a_3]_{\text{IR}}$

Why is constant term in S_{EE} universal? (Schwimmer & Theisen)

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^*\, \log(2R/\delta) & \textbf{for even } d \\ (-)^{\frac{d-1}{2}}\; 2\pi\, a_d^* & \textbf{for odd } d \end{cases}$$

Why is constant term in S_{EE} universal? (Schwimmer & Theisen)

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^*\, \log(2R/\delta) & \textbf{for even } d \\ (-)^{\frac{d-1}{2}}\, 2\pi\, a_d^* & \textbf{for odd } d \end{cases}$$

“Renormalized” Entanglement Entropy: (Liu & Mezei)

- divergences determined by local geometry of entangling surface with covariant regulator, eg,

$$S = c_0(\mu_i\delta)\frac{R^{d-2}}{\delta^{d-2}} + c_2(\mu_i\delta)\frac{R^{d-4}}{\delta^{d-4}} + \cdots + (-)^{\frac{d-1}{2}}\, 2\pi\, a_d(\mu_i\delta) + O(\delta/R)$$

- can isolate finite term with appropriate manipulations, eg,

d=3: $\mathcal{S}_3(R) = RS'(R) - S(R)$ ⟵ c-function of Casini & Huerta

d=4: $\mathcal{S}_4(R) = R^2 S''(R) - RS'(R)$

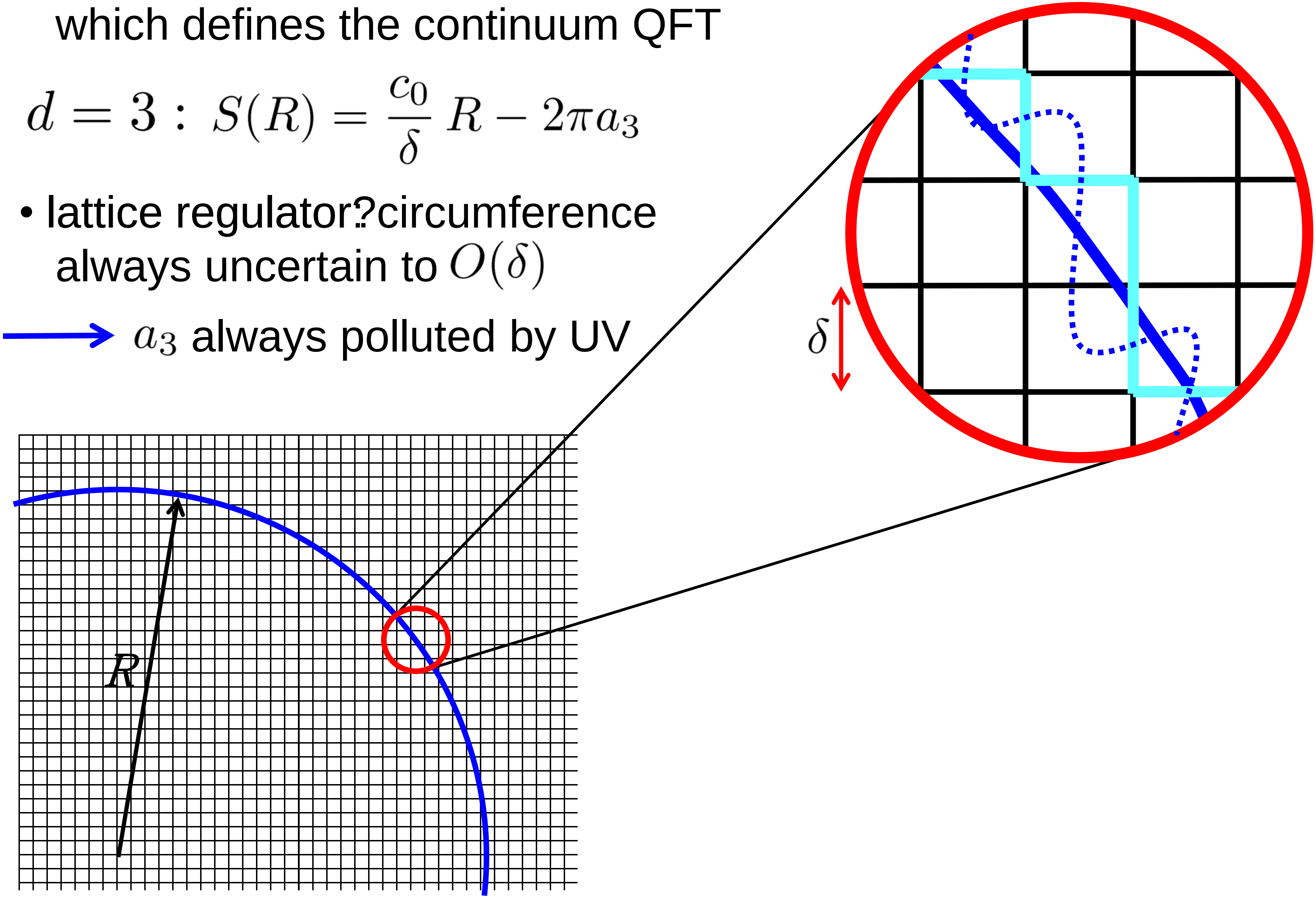
- unfortunately, holographic experiments indicate $\mathcal{S}_d(R)$ are **not** good C-functions for *d*>3
- approach demands special class of regulators: “covariant”

- if a_3 is physical, we should be able to use any regularization which defines the continuum QFT

$$d = 3: \; S(R) = \frac{c_0}{\delta}\, R - 2\pi a_3$$

- lattice regulator?circumference always uncertain to $O(\delta)$

→ a_3 always polluted by UV

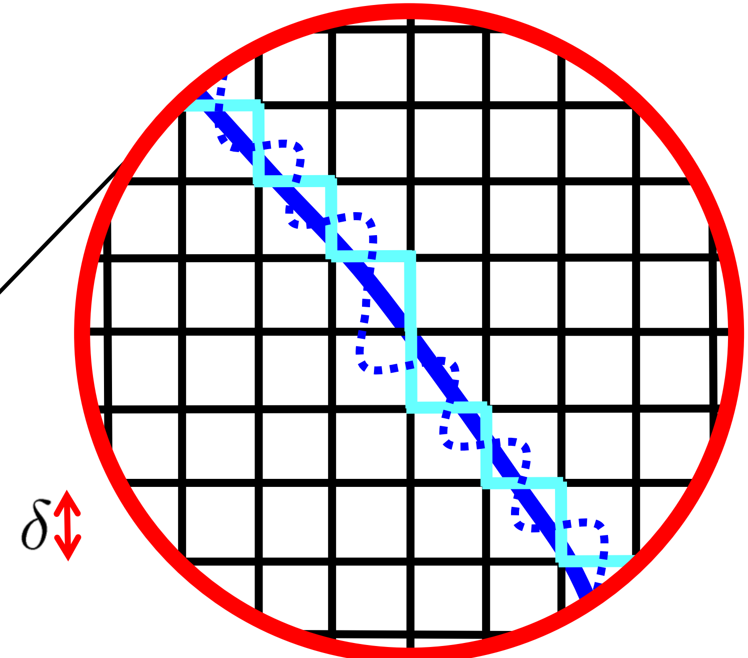


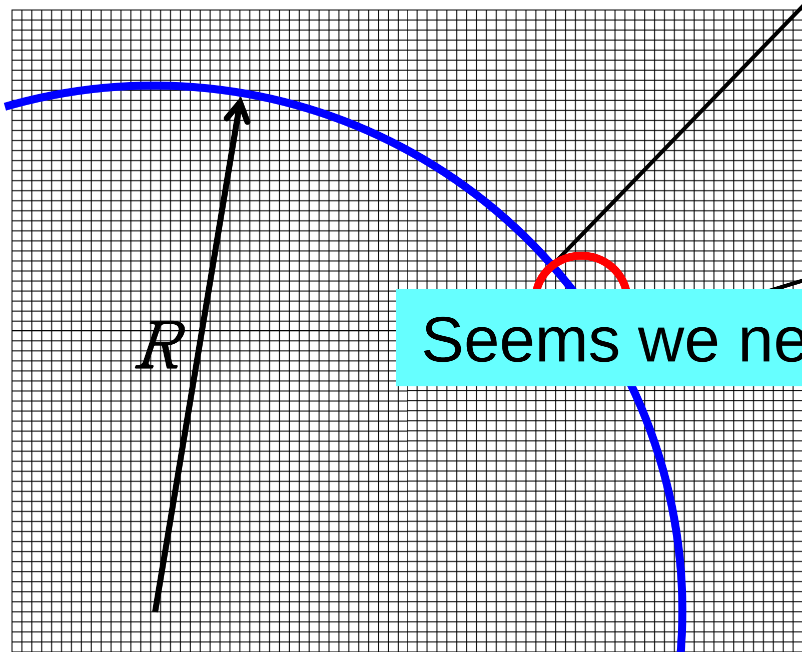
- if a_3 is physical, we should be able to use any regularization which defines the continuum QFT

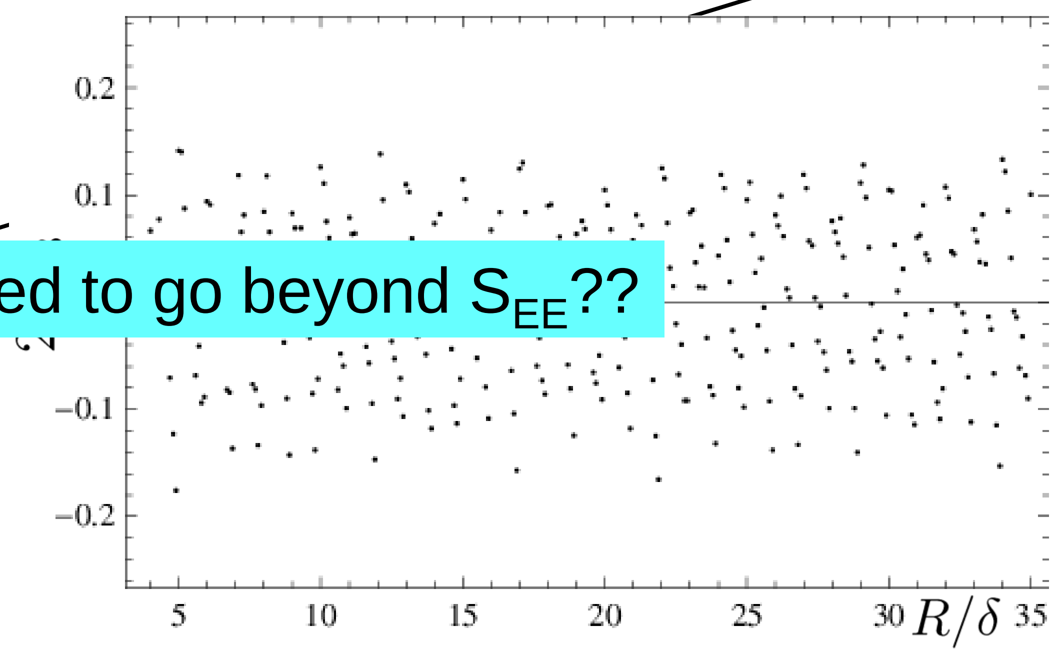
$$d = 3: \; S(R) = \frac{c_0}{\delta} R - 2\pi a_3$$

- lattice regulator: circumference **always** uncertain to $O(\delta)$

→ a_3 **always** polluted by UV





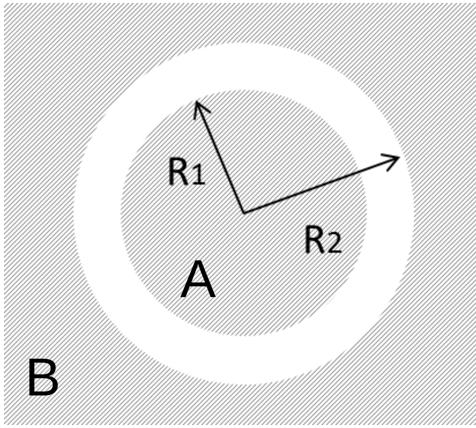


Seems we need to go beyond S_{EE}??

(Casini, Huerta, RM & Yale)

"Renormalized" Entanglement Entropy 2:

- S_{EE} is UV divergent, so must take care in defining universal term
- mutual information is intrinsically finite and so offers "universal" regulator for S_{EE} or alternative definition of a_3

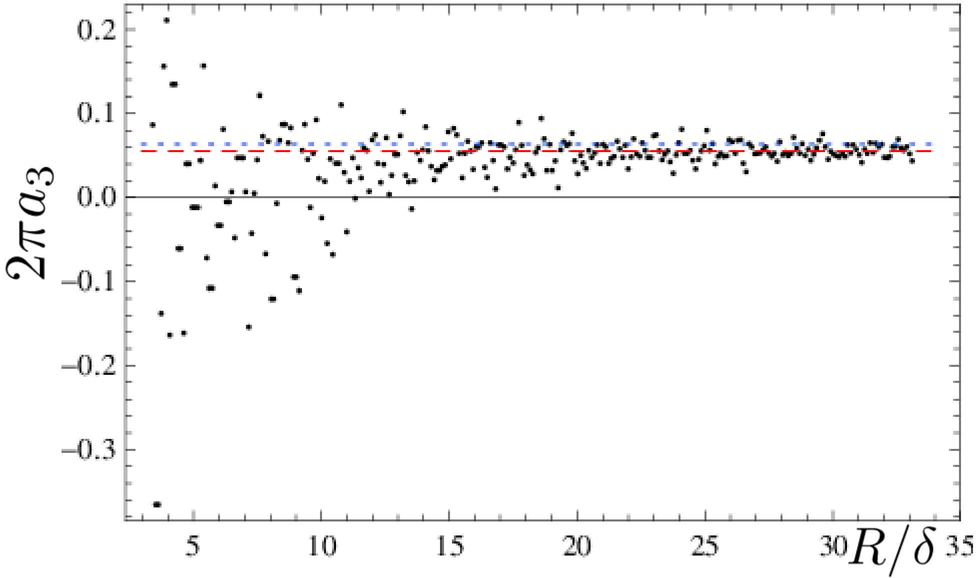


$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- with $R_{1,2} = R \pm \frac{\varepsilon}{2}$ and $R \gg \varepsilon \gg \delta$,

$$I(A,B) = 2\left(\frac{\tilde{a}}{\varepsilon} + b\right) R - 4\pi a_3 + O(\varepsilon)$$

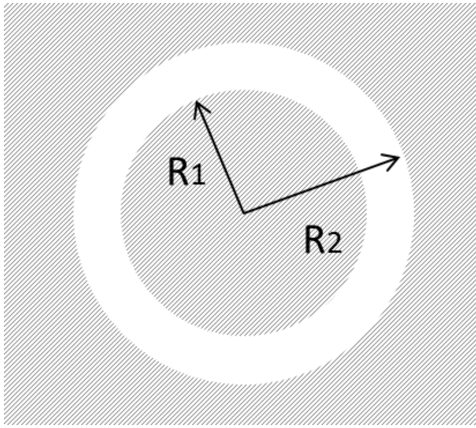
- choice ensures that a_3 is not polluted by UV fixed point



(Casini, Huerta, RM & Yale)

"Renormalized" Entanglement Entropy 2:

- S_{EE} is UV divergent, so must take care in defining universal term
- mutual information is intrinsically finite and so offers alternative regulator for S_{EE} or alternative definition of a_3



$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- with $R_{1,2} = R \pm \frac{\varepsilon}{2}$ and $R \gg \varepsilon \gg \delta$,

$$I(A,B) = 2\left(\frac{\tilde{a}}{\varepsilon} + b\right) R - 4\pi a_3 + O(\varepsilon)$$

- choice ensures that a_3 is not polluted by UV fixed point
- naturally extends to defining a_d in higher odd dimensions
- for d=3, entropic proof of F-theorem can be written in terms of mutual information

Counting degrees of freedom?:

- Susskind & Witten: density of degrees of freedom in N=4 SYM connected to area of holographic screen at large R in AdS_5

$$\frac{V_3}{\delta^3} \times N_c^2 \sim \frac{A(R)}{\ell_P^3}$$

cut-off scale defined by regulator radius: $\frac{1}{\delta} = \frac{R}{L^2}$

- given higher curvature bulk action, natural extension is to evaluate Wald entropy on holographic screen at large R

$$S = -2\pi \oint d^{d-1}x\sqrt{h}\ \hat{\varepsilon}^{ab}\,\hat{\varepsilon}_{cd}\ \frac{\partial \mathcal{L}_{bulk}}{\partial R^{ab}{}_{cd}}$$

- straightforward evaluate "entropy" density

$$\boxed{S = \frac{2}{\pi}\,a_d^*\,\frac{V_{d-1}}{\delta^{d-1}}}$$

for any covariant action: $\mathcal{L}_{bulk} = \mathcal{L}_{bulk}\left(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \cdots\right)$

(Komargodski & Schwimmer; see also: Luty, Polchinski & Rattazzi)

a-theorem and Dilaton Effective Action

- analyze RG flow as "broken conformal symmetry" (Schwimmer & Theisen)
- couple theory to "dilaton" (conformal compensator) and organize effective action in terms of $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$

  diffeo X Weyl invariant: $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \quad \tau \to \tau + \sigma$

- follow effective dilaton action to IR fixed point, eg,

$$S_{anomaly} = -\delta a \int d^4x \sqrt{-g} \Big(\tau E_4 + 4(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box \tau + 2(\partial\tau)^4 \Big)$$

$\delta a = a_{UV} - a_{IR}$: ensures UV & IR anomalies match

- with $g \to \eta$, only contribution to 4pt amplitude with null dilatons:

$$S_{anomaly} = 2\,\delta a \int d^4x\, (\partial\tau)^4$$

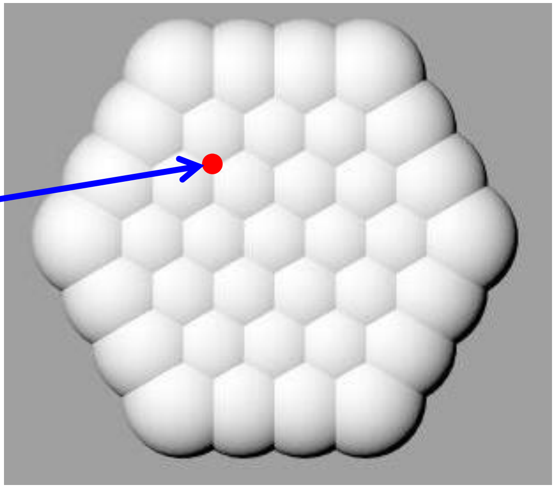
- dispersion relation plus optical theorem demand: $\delta a > 0$

Conclusions and Questions:

• is there entropic proof of c-theorem in higher dimensions?

→ need a new idea?

higher dim. intersections lead to subleading divergences which trivialize SSA inequality



• hybrid approach? (Solodukhin): needs work

• can c-theorems be proved for higher dimensions? eg, d=5 or 6

→ again, entropic approach needs a new idea

→ dilaton-effective-action approach requires refinement for d=6

(Elvang, Freedman, Hung, Kiermaier, RM & Theisen; Elvang & Olson)

Conclusions and Questions:

- how much of Zamolodchikov's structure for d=2 RG flows extends higher dimensions?

→ d=3 entropic C-function not always stationary at fixed points

(Klebanov, Nishioka, Pufu & Safdi)

→ same already observed for d=2; special case or generic? need a better C-function?

- does scale invariance imply conformal invariance beyond d=2?

→ "more or less" in d=4

(Luty, Polchinski & Rattazzi; Dymarsky, Komargodski, Schwimmer & Theisen)

- further lessons for RG flows and entanglement from holography?

→ translation of "null energy condition" to boundary theory?

- what can entanglement entropy/quantum information really say about RG flows, holography or nonperturbative QFT?