Properties of monopole operators in 3d gauge theories

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Based on:

- arXiv:1303.6125
- arXiv:1309.1160 (with Ethan Dyer and Mark Mezei)
- work in progress with Ethan Dyer, Mark Mezei, and Subir Sachdev

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Abelian gauge theories in 3d

- Consider (compact) $U(1)$ gauge theory in 3 dimensions:

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}^2 + \text{matter w/ integer } U(1) \text{ charges.}$$

- IR dynamics:
  - No matter $\Longrightarrow$ confinement [Polyakov] .
  - Lots of matter $\Longrightarrow$ interacting CFT [Appelquist, ...] . Maxwell term is irrelevant.
    - Can be studied in $1/N_f$ expansion.
  - Few matter fields: the theory *may* confine.
    - Analog of χSB if we have few fermions [Pisarski; Vafa, Witten; ...]

- *When* should we expect an interacting CFT? What is the operator spectrum of this CFT?

Monopole operators

- A $U(1)$ gauge theory in 3d has a global $U(1)_{\text{top}}$ symmetry with conserved current $j_\mu^{\text{top}} = \frac{1}{4\pi}\epsilon_{\mu\nu\rho}F^{\nu\rho}$. Monopole operators are local operators with non-zero $U(1)_{\text{top}}$ charge.

- Monopole operators are "disorder operators" that insert a monopole singularity in the gauge field
  - A monopole of charge $q \in \mathbb{Z}/2$ centered at the origin:

$$\int_{S^2} F = 4\pi q \,.$$

  - Monopole operator $\mathcal{M}_q(0)$ satisfies OPE

$$\mathcal{M}_q(0) \int_{S^2} F \sim \mathcal{M}_q(0) 4\pi q + \cdots \,.$$

  - Lots of operators with given q.

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Monopole operators in CFT

- In CFT: use state-operator correspondence and study the theory on $S^2 \times \mathbb{R}$.
- Let $\mathcal{M}_q(0)$ correspond to the ground state in the sector of monopole flux $\int_{S^2} F = 4\pi q$ [Borokhov, Kapustin, Wu '02] .

Roles played by monopole operators

- *Mechanism for confinement*: monopole proliferation in theories where $U(1)_{\text{top}}$ is explicitly broken microscopically [Polyakov] .
  - Key insight: failure of the Bianchi identity implies Wilson loop area law.
  - If monopole operators are irrelevant $\Longrightarrow$ ~~confinement~~ (e.g. [Hermele, Senthil, Fisher, Lee, Nagaosa, Wen]).
- *Order parameters* for continuous phase transitions between two ordered phases (i.e. which evade the Landau-Ginsburg-Wilson paradigm) [Sachdev, Read '89] .
  - A discrete $\mathbb{Z}_k$ subgroup of $U(1)_{\text{top}} \times SO(2)_{\text{rot}}$ is a symmetry of the lattice Hamiltonian.
  - Relevant monopole ops that transform under $\mathbb{Z}_k$: order parameters.
  - Relevant monopole ops invariant under $\mathbb{Z}_k$: can cause confinement.

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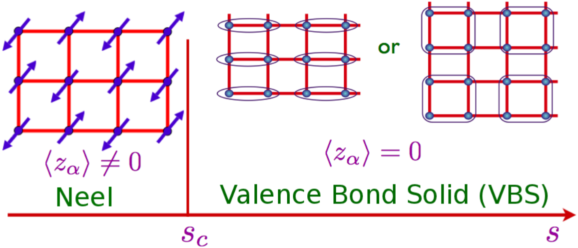
Monopole operators as order parameters

Square lattice antiferromagnet with $SU(N)$ spins at each site.

J-Q model:

$$H = J \sum_{\langle ij \rangle} S^{\alpha}{}_{\beta}(i) S^{\beta}{}_{\alpha}(j)$$

$$+ Q \times \text{(four spins)}$$



$$S = \int d^2r\, d\tau \left[|(\partial_\mu - iA_\mu) z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{4e^2} F_{\mu\nu}^2 \right]$$

- At the critical point $e = \infty$ and s is tuned to zero $\Longrightarrow$ same universality class as the $\mathbb{CP}^{N-1}$ model [Motrunich, Vishwanath '04; Senthil, Balents, Fisher, Sachdev, Vishwanath '04] .
- Néel order: $\langle z_\alpha \rangle \neq 0$; VBS order: $\langle \mathcal{M}_{1/2} \rangle \neq 0$. [Sachdev, Read]

Spin liquids

- In other models, obtain compact QED with some number of fermions $\Longrightarrow$ "spin liquids" [Wen, ...] .

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}^2 + \sum_{a=1}^{N_f} \psi_\alpha^\dagger (i\not{D} + \not{A}) \psi_\alpha \,.$$

- Monopole operators that transform under lattice symmetries $\Longrightarrow$ VBS order parameters.
- Monopoles invariant under lattice symmetries can cause confinement. Non-trivial CFT exists only if they are irrelevant [Heremele, Senthil, Fisher, Lee, Nagaosa, Wen '04] .

Properties of monopole operators in CFT

- Task: determine quantum numbers under conformal group (scaling dimension, Lorentz spin) and under the flavor symmetry.
- Method: $1/N_f$ expansion. Good because gauge field fluctuations are suppressed.
- In fermionic theory, for instance:

$$S = \int d^3x \sqrt{g} \sum_{\alpha=1}^{N_f} \psi_\alpha^\dagger (i\not{D} + \not{A}) \psi_\alpha \,.$$

- Integrate out ψ:

$$Z = \int DA \exp\left[N_f \operatorname{tr} \log(i\not{D} + \not{A})\right]$$

- Saddle point approximation $A = \mathcal{A} + a$ for some saddle $\mathcal{A}$.
- At large N_f, gauge field fluctuations are suppressed by $1/\sqrt{N_f}$.

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A brief history of the $1/N_f$ expansion

The leading order epoch (treat gauge field as background):

- [Murthy, Sachdev '90] : bosonic theory, scaling dim's to order $O(N_f)$.
- [Borokhov, Kapustin, Wu '02] : fermionic theory, scaling dim's to order $O(N_f)$, flavor charges of $\mathcal{M}_{1/2}$.
- [Metlitski, Hermele, Senthil, Fisher '08] : bosonic theory, scaling dim's to order $O(N_f)$.

Assumption: rotationally-invariant saddle.

The subleading order epoch (fluctuations of the gauge field):

- [SSP '13] : fermionic theory, scaling dim of $\mathcal{M}_{1/2}$ to order $O(N_f^0)$.
- [Dyer, Mezei, SSP '13] : fermionic theory, scaling dim's to order $O(N_f^0)$ for all $\mathcal{M}_q$; **flavor symmetry charges;** generalization to QCD.
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Scaling dimensions in **fermionic** theory

- The IR scaling dimension of $\mathcal{M}_q$ are [Borokhov, Kapustin, Wu '02; SSP '13; Dyer, Mezei, SSP '13] :

q	scaling dimension $[\mathcal{M}_q]$
0	0
$\mathbf{1/2}$	$\mathbf{0.265N_f - 0.0383 + O(1/N_f)}$
1	$0.673N_f - 0.194 + O(1/N_f)$
$3/2$	$1.186N_f - 0.422 + O(1/N_f)$
2	$1.786N_f - 0.706 + O(1/N_f)$
$5/2$	$2.462N_f - 1.04 + O(1/N_f)$

- $\mathcal{M}_{1/2}$ is irrelevant, provided:

$$[\mathcal{M}_{1/2}] > 3 \quad \Longrightarrow N_f \geq 12\,.$$

- So we expect the theory does not confine whenever $N_f \geq 12$.
- F-theorem $\Longrightarrow$ confinement impossible for $N_f \geq 12$. Also [Grover] .

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- $\mathcal{M}_{3/2}$ is irrelevant, provided:

$$[\mathcal{M}_{3/2}] > 3 \quad \Longrightarrow N_f \geq 4 \,.$$

- So we expect the theory does not confine whenever $N_f \geq 4$ **if** $\mathcal{M}_{1/2}$ and $\mathcal{M}_1$ transform non-trivially under lattice symmetries.

Flavor quantum numbers in **fermionic** theory

- N_f fermions in $U(1)$ gauge theory have $SU(N_f)$ flavor symmetry.
- How do monopole operators transform in representations of $SU(N_f)$?
- Step 1: Ground state of N_f fermions on $S^2 \times \mathbb{R}$ in the presence of uniform magnetic flux $F = q \sin\theta d\theta \wedge d\phi$.
- Step 2: Take into account the effect of having a dynamical gauge field.

Step 1: Fermions on S^2 in uniform magnetic flux

- The Dirac equation

$$(i\not{D} + \not{A})\psi^\alpha = 0$$

  has solutions with energy $\pm\sqrt{(j+1/2)^2 - q^2}$ transforming in the spin-j irrep of $SU(2)_{\text{rot}}$.

- $2|q|N_f$ zero-energy modes with $j = |q| - 1/2$ (for each flavor there are $2j+1$ zero-energy modes).
  - Creation and annihilation operators $c_{jm}^{\alpha\dagger}$ and c_{jm}^{α}, respectively.

- Ground state: non-zero energy modes are not excited.

- Ground state Hilbert space $\mathcal{G}$ has dimension $2^{2|q|N_f}$. It is spanned by $|\Omega\rangle$, $c_{jm}^{\alpha\dagger}|\Omega\rangle$, $c_{jm_1}^{\alpha_1\dagger}c_{jm_2}^{\alpha_2\dagger}|\Omega\rangle$, etc.

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Step 2: Dynamical gauge field

- At IR fixed point, ignore Maxwell field. The action is

$$S = \int d^3x \sum_{\alpha=1}^{N_f} \psi_\alpha^\dagger (i\not{D} + \not{A} + \not{a})\psi_\alpha \,.$$

- The path integral over a_μ imposes $j^\mu = \sum_{\alpha=1}^{N_f} \psi_\alpha^\dagger \gamma^\mu \psi_\alpha = 0$.
- The space of physical ground states $\mathcal{G}_{\text{phys}}$ consists of those states of $\mathcal{G}$ that satisfy

$$j^\mu(x)|\chi\rangle = 0 \,.$$

Step 2: Dynamical gauge field

- The current is:

$$j^\mu(x) = \sum_{m,m'} \left(c^{\alpha\dagger}_{jm} c^\alpha_{jm'} - \frac{N_f}{2}\delta_{mm'} \right) S^\dagger_{q,jm}(\theta,\phi)\gamma^\mu S_{q,jm'}(\theta,\phi) + \cdots .$$

- Integrate $j^\tau(x)$ against $Y_{00}(\theta,\phi) \Longrightarrow$ total charge constraint: $\sum_{m,\alpha} c^{\alpha\dagger}_{jm} c^\alpha_{jm} |\chi\rangle = |q| N_f |\chi\rangle$.
- Integrate $j^\tau(x)$ against $Y_{1m}(\theta,\phi) \Longrightarrow$ total spin vanishes.
- Integrate $j^\tau(x)$ against $Y_{\ell m}(\theta,\phi)$, $\ell > 1 \Longrightarrow$ more complicated constraints.

- All $\ell \geq 1$ constraints: invariance under the $SU(2|q|)$ that rotates the m index of $c^{\alpha\dagger}_{jm}$.

Fun with group theory

- Package all $c_{jm}^{\alpha\dagger}$ into a column vector of length $2|q|N_f$ and let $SU(2|q|N_f)$ act on it.
- Clearly, $SU(2|q|N_f) \supset SU(N_f) \times SU(2|q|)$.
- Group theory question: find $SU(2|q|)$ singlets under the decomposition of the rank-$|q|N_f$ anti-symmetric tensor irrep of $SU(2|q|N_f)$.
- Answer: only one $SU(2|q|)$ singlet transforming in the $SU(N_f)$ irrep with Young diagram

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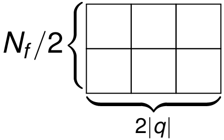
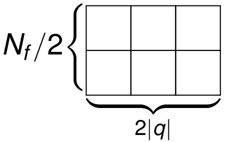


Quantum numbers of monopole operators in **fermionic** theory

- $\mathcal{M}_q$ transforms under $SU(N_f)$ as [Dyer, Mezei, SSP '13] :



- These operators are Lorentz scalars b/c $SU(2)_{\text{rot}} \subset SU(2|q|)$.
- Scaling dimensions

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Generalization to $U(N_c)$ QCD w/ N_f fundamentals

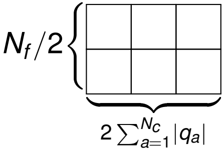
- Monopole operator with lowest scaling dimension:

$$\Delta = 0.265 N_f - 0.0383 - (N_c - 1) 0.516 + O(N_c^2/N_f)\,.$$

- Predictions:

  $U(1)$ deconfines when $N_f \geq 12$,
  $U(2)$ when $N_f \geq 14$,
  $U(3)$ when $N_f \geq 16$, etc.

- More complicated monopole operators are parameterized by $\{q_a \in \mathbb{Z}/2\}$, $a = 1, 2, \ldots, N_c$. They transform as



- Not all $\{q_a\}$ yield independent operators!! [Dyer, Mezei, SSP '13]

Open questions

- Can the properties of monopole operators I described be verified through lattice QED/QCD or through the conformal bootstrap?
- Do monopole operators play any role in non-supersymmetric (bosonization) dualities?
  - $U(N)_k +$ fundamental boson: $\Delta \propto k$ if $k \gg N$.
  - $U(N)_k +$ fundamental fermion: $\Delta \propto k^{3/2}$ if $k \gg N$.

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Tentative comments on the critical **bosonic** theory

- Assuming spherically symmetric flux through S^2 [Sachdev, Murthy '90; Metlitski, Hermele, Senthil, Fisher '08; Dyer, Mezei, SSP, Sachdev, **in progress**] :

q	scaling dimension $[\mathcal{M}_q]$
0	0
$\mathbf{1/2}$	$\mathbf{0.125\, N_b + 0.0603 + O(1/N_b)}$
1	$0.311 N_b - 0.233 + O(1/N_b)$
$3/2$	negative mode
$\vdots$	$\vdots$

- Lorentz scalars invariant under $SU(N)$ global symmetry.
- $\mathcal{M}_{1/2}$ is irrelevant, provided:

$$[\mathcal{M}_{1/2}] > 3 \quad \Longrightarrow N_b \geq 24 \,.$$

- So we expect the theory does not confine whenever $N_b \geq 24$.