The Superconformal Bootstrap Program

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Based on work with
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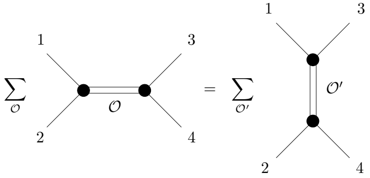
New Methods in Non-Perturbative QFT

In recent years, explosion of results for SuperConformal Field Theories in $d > 2$.

- A huge list of new models, mostly with no Lagrangian description.
- A hodgepodge of techniques (localization, large N integrability, AdS/CFT). Powerful but with limitations.

Time is ripe for a more systematic approach.
Bootstrap philosophy: abstract operator algebra, obeying general consistency requirements from symmetries, unitarity and crossing.



Two sorts of questions

What is the space of consistent SCFTs in various dimensions?

- 32 Qs: plausibly, complete catalogues in $d = 3$, $d = 4$ and $d = 6$.
- 16 Qs: proposed catalogue in in $d = 6$, beginning of a classification scheme in $d = 4$ (class S, $\dots$)
- 8 Qs: wide open.
  E.g. Conjectural landscape of AdS_4 string vacua $\leftrightarrow$ $d = 3$ SCFTs.

Can we bootstrap concrete models of special interest?

The bootstrap should be particularly powerful for models that are uniquely cornered by a few discrete data.
It is the only method presently available for finite N, non-Lagrangian theories, such as the $6d$ (2,0) theory.

Do the conformal bootstrap equations in dimension $d > 2$ admit a solvable truncation in the case of superconformal field theories?

A priori, there are two primary scenarios in which the constraints of crossing symmetry are nontrivial, yet solvable:

(I) *Meromorphic (and rational) conformal field theories in $d = 2$*
(II) *Topological quantum field theories.*

(I) is realized in $\mathcal{N} \geqslant 2$ theories in $d = 4$ and in $(2, 0)$ theories in $d = 6$.
This will be our focus.
(II) is realized in $\mathcal{N} \geqslant 4$ theories in $d = 3$.

The Superconformal Bootstrap Program

The bootstrap of $d = 4$, $\mathcal{N} \geqslant 2$ and of $d = 6$, $\mathcal{N} = (2, 0)$ SCFTs can be organized into two steps:

1. The bootstrap for a protected subsector of BPS operators ("minibootstrap")
2. The full-fledged bootstrap for generic operators.

Indeed, crossing-symmetry constraints for a BPS 4pt function neatly split into

1. Equations that describe intermediate BPS operators.
   They can be solved analytically.
2. Equations that describe intermediate non-BPS operators.
   They can be analyzed numerically.

Step (1) serves as essential input for Step (2).
Step (1) is captured by carving out a $2d$ chiral algebra inside the $d = 4$ or $d = 6$ SCFT.

In this talk, I'll focus on (1), and flash some results for (2).

Warm-up: $\mathcal{N} = 1$ chiral ring

By definition, chiral operators in an $\mathcal{N} = 1$, $d = 4$ QFT are annihilated by both components of the right-handed supercharge,

$$\{\tilde{Q}_{\dot{\alpha}}, \mathcal{O}(x)] = 0\,, \quad \dot{\alpha} = \dot{+}, \dot{-}\,.$$

An operator is chiral if and only if $\Delta = \frac{3}{2} r$.
One further defines the cohomology class $[\mathcal{O}(x)]_{\tilde{Q}}$ identifying

$$\mathcal{O}(x) \sim \mathcal{O}(x) + \{\tilde{Q}_{\alpha}, \ldots]\,.$$

From the susy algebra $P = \{Q, \tilde{Q}\}$:

$$\frac{\partial}{\partial x}\mathcal{O}(x) = [P, \mathcal{O}(x)] = \{\tilde{Q}, \mathcal{O}'(x)]\,, \quad \mathcal{O}'(x) = \{Q, \mathcal{O}(x)]\,,$$

so cohomology classes are position independent.
Concretely, correlators where *all* operators are chiral are position independent,

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \ldots \mathcal{O}_n(x_n) \rangle = \text{constant} \qquad x_i \in \mathbb{R}^4\,.$$

In fact, in an $\mathcal{N} = 1$ superconformal theory, they vanish identically since $r_i > 0$.

Meromorphic correlators in $d = 4$, $\mathcal{N} = 2$ SCFTs

Fix a plane $\mathbb{R}^2 \subset \mathbb{R}^4$, parametrized by complex coordinates $(z, \bar{z})$.

Claim : Any $\mathcal{N} = 2$ SCFT contains a subsector $\mathcal{A}_\chi = \{\mathcal{O}_i(z_i, \bar{z}_i)\}$ of protected local operators, with meromorphic correlation functions,

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \ldots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = R(z_i)\,.$$

Rationale: $\mathcal{A}_\chi$ is defined by the cohomology of a nilpotent $\mathbb{Q}$, of the form

$$\mathbb{Q} = \mathcal{Q} + \mathcal{S}\,.$$

where $\mathcal{Q}$ is a Poincaré and $\mathcal{S}$ a conformal supercharge.
The $\bar{z}$ dependence turns out to be $\mathbb{Q}$-exact.

Richer structure than the $\mathcal{N} = 1$ chiral ring because of z dependence.

Schur operators

$\mathcal{O}(0,0) \in \mathcal{A}_\chi$ if and only if its quantum numbers obey

$$\Delta = 2R + j_1 + j_2\,,$$

where R is the $SU(2)_R$ Cartan and (j_1, j_2) the Lorentz spins.
These are the operators that contribute to the Schur limit of the SC index.
They are killed by 2 real Poincaré supercharges (out of 8), one $\mathcal{Q}$ and one $\tilde{\mathcal{Q}}$, an intrinsically $\mathcal{N} = 2$ condition.

The Schur class includes:

- The $\frac{1}{2}$ BPS operators that parametrize the Higgs branch (but *not* the $\frac{1}{2}$ BPS operators of the Coulomb branch).
- The $SU(2)_R$ conserved current.
- A menagerie of operators obeying less familiar semi-shortening conditions.

To remain $\mathbb{Q}$-closed away from the origin, $\mathcal{O}(z,\bar{z})$ must acquire a certain position dependence, because

$$[\mathbb{Q}, L_n] = 0 \quad \text{but} \quad [\mathbb{Q}, \bar{L}_n] \neq 0, \quad n = -1, 0, 1\,,$$

where $L_n = -z^{n+1}\partial_z$, $\bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$.
We must twist the right-moving generators by $SU(2)_R$,

$$\widehat{L}_{-1} = \bar{L}_{-1} + \mathcal{R}^-\,, \qquad \widehat{L}_0 = \bar{L}_0 - \mathcal{R}\,, \quad \widehat{L}_1 = \bar{L}_1 - \mathcal{R}^+\,.$$

$$\widehat{L}_n = \{\mathbb{Q}, \dots\}\,, \quad n = -1, 0, 1\,.$$

$\mathbb{Q}$-closed operators have standard z dependence and twisted $\bar{z}$ dependence,

$$\mathcal{O}(z,\bar{z}) = e^{zL_{-1}+\bar{z}\widehat{L}_{-1}}\,\mathcal{O}(0)\,e^{-zL_{-1}-\bar{z}\widehat{L}_{-1}}\,.$$

The $SU(2)_R$ orientation is correlated with the position on the plane.

NB: the Schur condition $\Delta = 2R + j_1 + j_2$ is nothing but $\widehat{L}_0 = 0$.

By the usual formal argument, the $\bar{z}$ dependence is exact,

$$[\mathcal{O}(z,\bar{z})]_{\mathbb{Q}} \quad \Longrightarrow \quad \mathcal{O}(z) \ .$$

Cohomology classes define left-moving $2d$ operators, with conformal weight

$$L_0 = \frac{\Delta - (j_1 + j_2)}{2} = R + j_1 + j_2 \, ,$$

which are closed under OPE.

$\mathcal{A}_\chi$ has the structure of a $2d$ chiral algebra.

Example: free hypermultiplet

Look for states with $\widehat{L}_0 = \frac{\Delta - j_1 - j_2}{2} - R = 0$.
The complex scalars Q and $\tilde{Q}$ fit the bill, since $\Delta = 1$, $R = \frac{1}{2}$, $j_1 = j_2 = 0$. They are top components of $SU(2)_R$ doublets,

$$Q^{\mathcal{I}} = \left(\begin{array}{c} Q \\ \tilde{Q}^* \end{array} \right) , \qquad \tilde{Q}^{\mathcal{I}} = \left(\begin{array}{c} \tilde{Q} \\ -Q^* \end{array} \right) .$$

Away from the origin we must consider twisted-translated operators

$$q(z,\bar{z}) := Q(z,\bar{z}) + \bar{z}\tilde{Q}^*(z,\bar{z}), \qquad \tilde{q}(z,\bar{z}) := \tilde{Q}(z,\bar{z}) - \bar{z}Q^*(z,\bar{z}) .$$

Elementary exercise:

$$q(z,\bar{z})\,\tilde{q}(0) \sim \bar{z}\,\tilde{Q}^*(z,\bar{z})\,\tilde{Q}(0) \sim \frac{\bar{z}}{z\bar{z}} = \frac{1}{z} .$$

The cohomology classes $[q(z,\bar{z})]_{\mathbb{Q}}$, $[\tilde{q}(z,\bar{z})]_{\mathbb{Q}}$ define a pair of symplectic bosons of weight $L_0 = \frac{1}{2}$, for which $c_{2d} = -1$.
Normal ordered products of $\partial^n q$ and $\partial^n \tilde{q}$ reproduce the entire spectrum of the chiral algebra associated to the free hypermultiplet.

Example: free vector multiplet

The gauginos λ^1_+ and $\tilde{\lambda}^1_{\dot{+}}$ satisfy $\widehat{L}_0 = 0$.

The twisted-translated operators

$$\lambda(z,\bar{z}) := \lambda^1_+(z,\bar{z}) + \bar{z}\lambda^2_+(z,\bar{z}) \,, \quad \tilde{\lambda}(z,\bar{z}) := \tilde{\lambda}^1_{\dot{+}}(z,\bar{z}) + \bar{z}\tilde{\lambda}^2_{\dot{+}}(z,\bar{z})$$

give rise in cohomology to chiral fields $\lambda(z)$, $\tilde{\lambda}(z)$, with

$$\tilde{\lambda}(z)\lambda(0) \sim \frac{1}{z^2} \,, \quad \lambda(z)\tilde{\lambda}(0) \sim -\frac{1}{z^2} \,.$$

Setting

$$\tilde{\lambda}(z) := b(z) \,, \quad \lambda(z) := \partial c(z) \,.$$

we recognize a bc ghost system of weights (1, 0), for which $c_{2d} = -2$.

χ : 4d $\mathcal{N} = 2$ SCFT $\longrightarrow$ 2d Chiral Algebra.

Some universal properties:

- Virasoro enhancement of $\mathfrak{sl}(2)$, with $T(z)$ arising from a component of the $SU(2)_R$ conserved current, $T(z) := [\mathcal{J}_R(z, \bar{z})]_{\mathbb{Q}}$, with

$$c_{2d} = -12\, c_{4d}\,,$$

  where c_{4d} is one of the conformal anomaly coefficient.

- Affine symmetry enhancement of global flavor symmetry, with $J(z)$ arising from the moment map operator, $J(z) := [M(z, \bar{z})]_{\mathbb{Q}}$, with

$$k_{2d} = -\frac{k_{4d}}{2}\,.$$

- Generators of the $4d$ Higgs branch $\Rightarrow$ generators of the chiral algebra.

  Higgs branch relations encoded in null states of the chiral algebra!
  (Crucial that k_{2d} takes special negative levels).

Consequences for $4d$ physics: new unitarity bounds

Consider the full-fledged $4pt$ correlator of some protected operators.

$$\langle \mathcal{O}_1^{\mathcal{I}_1}(x_1)\mathcal{O}_2^{\mathcal{I}_2}(x_2)\mathcal{O}_3^{\mathcal{I}_3}(x_3)\mathcal{O}_4^{\mathcal{I}_4}(x_4)\rangle\,.$$

The mere existence of our twist implies the superconformal Ward identities: the correlator can be expressed in terms of some unprotected functions $\mathcal{G}_i(z,\bar{z})$, and of some protected meromorphic functions $f_i(z)$.
The chiral algebra precisely captures $f_i(z)$.

Example: $4pt$ correlator of moment maps M, in the adjoint of the flavor group G_F. The $f_i(z)$ are uniquely fixed in terms of the flavor central charge k_{4d}.

Inserting the exact expressions for $f_i(z)$ in the double OPE expansion $\Rightarrow$ general unitarity bounds for k_{4d} valid in any interacting SCFT.

A crucial assumption is that the theory has no higher spin conserved currents.
Maldacena Zhiboedov

G_F		Bound	Representation
$\mathrm{SU}(N)$	$N \geqslant 3$	$k_{4d} \geqslant N$	$\mathbf{N^2-1}_{\mathrm{symm}}$
$\mathrm{SO}(N)$	$N = 4, \ldots, 8$	$k_{4d} \geqslant 4$	$\frac{\mathbf{1}}{\mathbf{24}}\mathbf{N}(\mathbf{N}-\mathbf{1})(\mathbf{N}-\mathbf{2})(\mathbf{N}-\mathbf{3})$
$\mathrm{SO}(N)$	$N \geqslant 8$	$k_{4d} \geqslant N-4$	$\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{N}+\mathbf{2})(\mathbf{N}-\mathbf{1})$
$\mathrm{USp}(2N)$	$N \geqslant 3$	$k_{4d} \geqslant N+2$	$\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{2N}+\mathbf{1})(\mathbf{2N}-\mathbf{2})$
G_2		$k_{4d} \geqslant \frac{10}{3}$	$\mathbf{27}$
F_4		$k_{4d} \geqslant 5$	$\mathbf{324}$
E_6		$k_{4d} \geqslant 6$	$\mathbf{650}$
E_7		$k_{4d} \geqslant 8$	$\mathbf{1539}$
E_8		$k_{4d} \geqslant 12$	$\mathbf{3875}$

Table : Unitarity bounds for k_{4d} arising from positivity in non-singlet channels.

These bounds are saturated by the SCFTs on $D3$ branes probing the F-theory singularities of type $H_1, H_2, D_4, E_6, E_7, E_8$, whose Higgs branches are one-instanton moduli spaces.

When the bounds are saturated, certain states become null in the affine Lie algebra. These nulls are interpreted in $4d$ as the "Joseph relations" on the moment map

$$(M \otimes M)|_{\mathcal{I}_2} = 0 \ , \qquad \mathrm{Sym}^2(\mathbf{adj}) = (2\,\mathbf{adj}) \oplus \mathcal{I}_2 \ .$$

In the singlet channel, the stress-tensor also contributes, and positivity implies a bound involving the conformal and flavor anomalies,

$$\frac{\dim G_F}{c_{4d}} \geqslant \frac{24h^\vee}{k_{4d}} - 12 \ .$$

When the bound is saturated,

$$c_{2d} = c_{Sugawara} = \frac{k_{2d} \dim G_F}{k_{2d} + h^\vee} \ .$$

Gauging prescription

Start with $4d$ SCFT $\mathcal{T}$, with flavor symmetry G_F.
We can generate a new SCFT $\mathcal{T}_G$ by gauging $G \subset G_F$, provided $\beta_G = 0$.

If we already know the chiral algebra $\chi[\mathcal{T}]$, can we find $\chi[\mathcal{T}_G]$?

Extra $4d$ vector multiplet $\Rightarrow$ extra $(b^A c_A)$ ghost system, in the adjoint of G.
We must also restrict to gauge singlets.
This is the correct answer at zero gauge coupling. But at finite coupling, some states are lifted and the chiral algebra must be smaller.

Elegant prescription to find quantum chiral algebra. Pass to the cohomology of

$$Q_{\rm BRST} := \oint \frac{dz}{2\pi i}\, j_{\rm BRST}(z)\,, \quad j_{\rm BRST} := c_A \left[J^A - \frac{1}{2} f^{AB}{}_C\, c_B b^C \right],$$

where J^A is the G affine current of $\chi[\mathcal{T}]$.

$Q^2_{\rm BRST} = 0$ precisely when the $\beta_G = 0$, which amounts to $k_{2d} = -2h^\vee$.

By this prescription, we can in principle find $\chi[\mathcal{T}]$ for any Lagrangian SCFT $\mathcal{T}$.

Some non-trivial examples

By low level-calculations of BRST cohomology and guesswork, we find that in some interesting cases the chiral algebra is finitely generated:

- $SU(2)$ gauge theory with $N_f = 4 \Rightarrow \mathfrak{so}(8)_{-2}$ AKM algebra.
- E_6 SCFT $\Rightarrow (\mathfrak{e}_6)_{-3}$ AKM algebra.
- $\mathcal{N} = 4$ SYM with gauge group $G \Rightarrow \mathcal{N} = 4$ super $\mathcal{W}$-algebra, with generators given by chiral primaries of dimensions $\{h_i = \frac{r_i+1}{2}\}$, where $\{r_i\}$ are the exponents of G.
  (So for $G = SU(2)$, simply the $\mathcal{N} = 4$ algebra.)

In the first two examples the bounds for k_{4d} and for c_{4d} are saturated.
We don't know whether the chiral algebra is *always* finitely generated.

- SCFTs of class $\mathcal{S} \Rightarrow$ chiral algebras labelled by punctured Riemann surfaces, obeying remarkable gluing conditions.

Chiral algebras for $6d$ (2, 0) Beem, L.R., van Rees, to appear

• By very similar methods, (2, 0) SCFT of type ADE $\Longrightarrow$ 2d Chiral Algebra. The $\frac{1}{2}$ BPS operators can be argued to be generators of the chiral algebra.

Claim 1: the chiral algebra of the A_{N-1} theory is the $\mathcal{W}_N$ algebra, with central charge

$$c_{2d} = 4N^3 - 3N - 1\,.$$

(Similar proposals for D and E theories).
Check: the limit of the superconfornal index computed by Kim^3 is reproduced.

Connection with the AGT correspondence!

• Codimension-two defects of the $(2, 0)$ theory harbor $\mathcal{N} = 2$, $d = 4$ SCFTs. They are labelled by $\mathfrak{sl}(2)$ embeddings into ADE and have flavor group G_F equal to the commutant of this embedding.

Claim 2: the chiral algebra of these defect SCFTs is the ADE affine Lie algebra at the *critical level* $k_{2d} = -h^\vee$ for maximal flavor, and its quantum DS reduction for reduced flavor.

Connection to geometric Langlands?

Example of full-fledged bootstrap: $\mathcal{N} = 4$ Beem, L.R., van Rees

Natural to start from the universal 4pt function of the stress tensor multiplet,

$$\langle \mathcal{O}^{I_1}_{\mathbf{20'}}(x_1)\mathcal{O}^{I_2}_{\mathbf{20'}}(x_2)\mathcal{O}^{I_3}_{\mathbf{20'}}(x_3)\mathcal{O}^{I_4}_{\mathbf{20'}}(x_4)\rangle = \frac{A^{I_1 I_2 I_3 I_4}(u,v)}{x_{12}^4 x_{34}^4} \ .$$

$\mathbf{20'} \times \mathbf{20'} = \mathbf{1} + \mathbf{15} + \mathbf{20'} + \mathbf{84} + \mathbf{105} + \mathbf{175}$: a priori six functions of u and v, but susy Ward identities allow to reduce them to:

1. two meromorphic protected functions $f_1(z)$, $f_2(z)$,
2. one unprotected function $\mathcal{G}(u,v)$. Here $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$.

Eden Petkou Schubert Sokatchev, Dolan Osborn, . . .

Remarkably, crossing symmetry implies:

1. a set of equations involving f_1 and f_2 only – these are the bootstrap equations of the chiral algebra. There is unique family of solutions parametrized by the central charge a. Plugging back f_i, one derives
2. a single crossing symmetry equation for the unprotected part

$$\sum_{\Delta,\ell} a_{\Delta,\ell} F_{\Delta,\ell}(u,v) = F^{\rm short}(u,v;a) \ ,$$

where $F^{\rm short}(u,v;a)$ is a complicated but completely known function.
The sum is over the intermediate unprotected superconformal primaries, which are constrained by Ward identities to be $SU(4)_R$ singlets.
$\ell = 0, 2, 4, \ldots$ is the spin, $\Delta \geqslant \ell + 2$ the conformal dimension.

Formally very similar to the basic bootstrap sum rule for identical scalar operators, with $F^{\rm short}(u,v;a)$ replacing $I(u,v)$ (contribution of the identity).
Rattazzi Rychkov Tonni Vichi

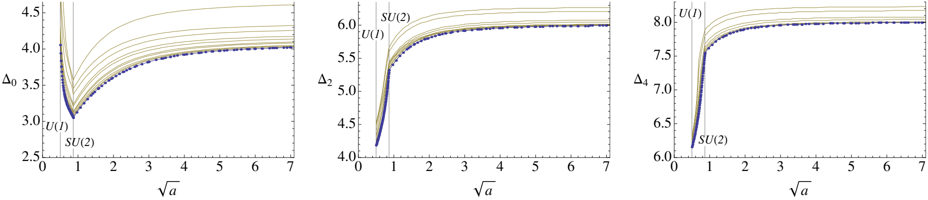


Figure : Bounds for the scaling dimension of the leading-twist unprotected operator of spin $\ell = 0, 2, 4$, as a function of the anomaly a.

- For $a = \frac{1}{4}$, saturated by the $U(1)$ (free) theory
- For $a \to \infty$, saturated by $AdS_5 \times S^5$ sugra, including $1/a$ corrections!

  In planar $\mathcal{N} = 4$ SYM for large 't Hooft coupling, leading-twist unprotected operators are double-traces of the form $\mathcal{O}_s = \mathcal{O}_{\mathbf{20'}} \partial^s \mathcal{O}_{\mathbf{20'}}$, with $\Delta_0 \approx 4 - \frac{4}{a}$, $\Delta_2 \approx 6 - \frac{1}{a}$, and $\Delta_4 \approx 8 - \frac{12}{25a}$ (from Witten diagrams).

Conjecture: the bounds are saturated also at finite a, by the theory at one of the cusp points $\tau = i$ or $\tau = e^{i\pi/3}$. Compatible with S-duality invariant resummation of four-loop perturbative results. Beem, L.R., van Rees, Sen

Prospects

Minibootstrap:

- For a given theory $\mathcal{T}$, develop systematic tools to characterize $\chi[\mathcal{T}]$ as $\mathcal{W}$ algebra.
- Classification of SCFTs related to classification of special chiral algebras.
- Add non-local operators.
  Particularly interesting in $d = 6$, where it should lead to AGT.

Maxibootstrap:

- (2, 0) bootstrap: in progress, stay tuned.
- Exploration of landscape of $\mathcal{N} = 2$ models, especially non-Lagrangian ones.
- More $\mathcal{N} = 4$.

Neat interplay of striking mathematical physics and numerical experiments.