The local RG
&
the Structure of 4D RG Flows

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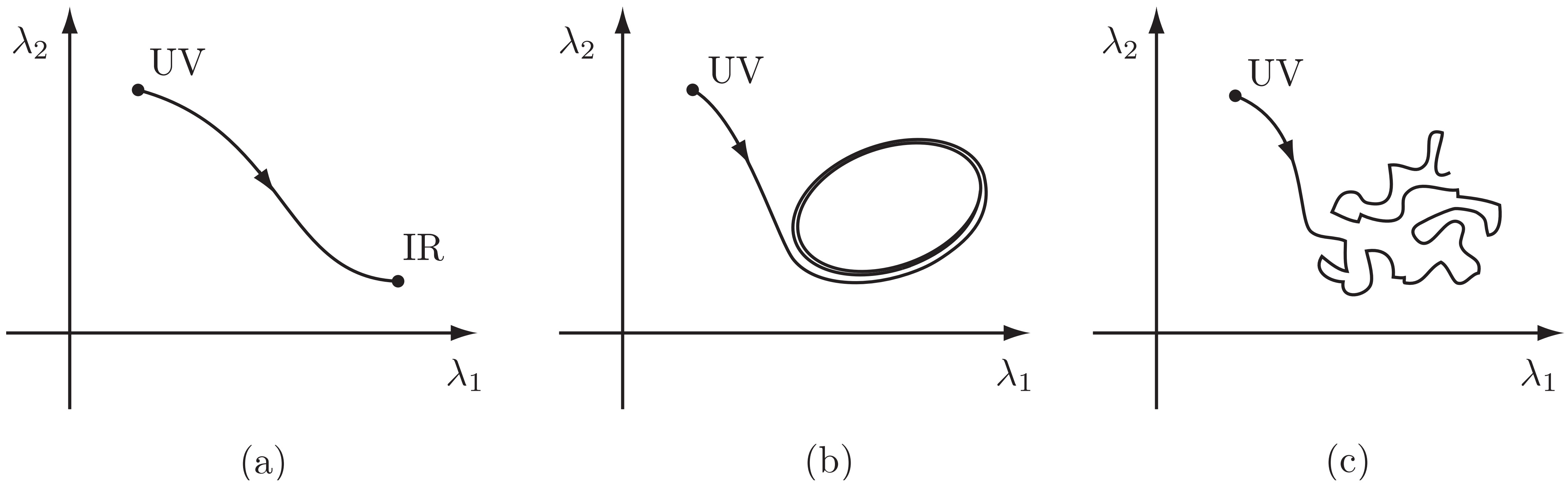
F. Baume, B. Keren-Zur, RR, L. Vitale arXiv:1401.5983

conceivable RG flows



but all known examples asymptote a CFT

How do we understand that?

Two approaches

- Local RG: Wess-Zumino consistency conditions for Weyl anomaly off-criticality

Jack, Osborn 1990 Osborn 1991

- Dispersion relations for $\langle T \ldots T \rangle$

  Optical theorem for scattering amplitudes of background dilaton

Komargodski and Schwimmer 2011

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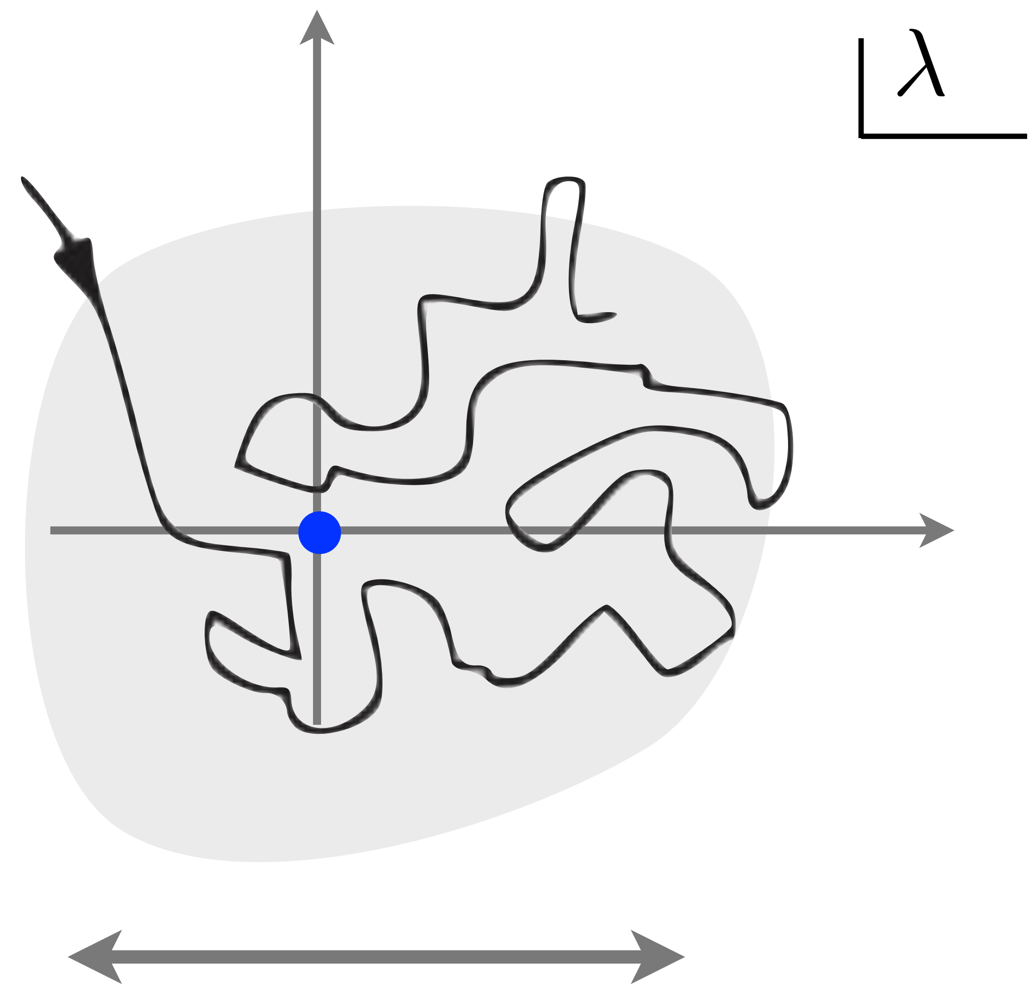
Goal: study RG flow in a domain around a fixed point

- $\mathcal{L} = \mathcal{L}_{CFT} + \sum_I \lambda^I \mathcal{O}_I$

- CFT, not necessarily free



$\boldsymbol{\beta}$ -function and
anomalous dimensions
are ‘small’

RG flow $\longleftrightarrow$ $T \equiv T^{\mu}_{\mu}$

Goal: systematically study $\langle T(x_1) \ldots T(x_n) \rangle$ including contact terms

Ex.:

$$T(x_1)T(x_n) = \beta^I \mathcal{O}_I(x_1)\beta^J \mathcal{O}_J(x_2) + \delta(x_1 - x_2) \; ?$$

$\Downarrow$

Effective action for the sources of composite operators

$$\left.\begin{array}{c} T_{\mu\nu} \leftrightarrow g_{\mu\nu}(x) \\ \mathcal{O}_I \leftrightarrow \lambda_I(x) \\ J_\mu^A \leftrightarrow A_\mu^A(x) \\ \mathcal{O}_a \leftrightarrow m_a(x) \end{array}\right\} \equiv \mathcal{J}$$

$$W \equiv W[g_{\mu\nu}, \lambda^I, A_\mu^A, m_a, \ldots]$$

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}} W \qquad \mathcal{O}_I(x) = \frac{1}{\sqrt{g}}\frac{\delta}{\delta \lambda_I(x)} W$$

etc ...

Ward identity for Weyl symmetry: local RG equation

Osborn 1991

$$\Delta_\sigma W \equiv$$

$$\int d^4x \Big\{ \sigma(x) \left[2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} + \tilde{m}^a \frac{\delta}{\delta m^a(x)} \right] +$$

$$+ \nabla_\mu \sigma(x) \left[\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} - S^A \frac{\delta}{\delta A_\mu^A(x)} \right] - \Box \sigma(x) t^a \frac{\delta}{\delta m^a(x)} \Big\} W =$$

$$= \int d^4x \, \mathcal{A}_\sigma(x)$$

- $$2\tilde{m}^a = 2m^b(\delta_b^a + \gamma_b^a) + \frac{1}{3}\eta^a R + d_I^a \Box \lambda^I + \frac{1}{2}\epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J$$
- $\mathcal{A}_\sigma(x) =$ all possible dim 4 covariant terms

Redundancies and scheme choices

- source reparametrization $\mathcal{J} \rightarrow f(\mathcal{J})$ Ex.: $m^a \rightarrow m^a + f_I^a \Box \lambda^I$
  $\mathcal{O}_I \rightarrow \mathcal{O}_I - f_I^a \Box \mathcal{O}_a$

- combine global symmetry $\Delta_\sigma \rightarrow \Delta_\sigma + \Delta^{Flavor}$

- add finite local functional $W[\mathcal{J}] \rightarrow W[\mathcal{J}] + F_{loc}[\mathcal{J}]$

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Consistency conditions

$$[\Delta_{\sigma_1}, \Delta_{\sigma_2}] = 0$$

I. On coefficients in Δ_σ

$$\Delta_\sigma = \sigma(x)\left[2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)} - B^I\frac{\delta}{\delta\lambda^I(x)} - P_I^A\nabla_\mu\lambda^I\frac{\delta}{\delta A_\mu^A(x)} + \tilde{M}^a\frac{\delta}{\delta m^a(x)}\right]$$

- $\dfrac{\delta}{\delta A_\mu^A}$

$$\boxed{B^I P_I^A = 0}$$

$$T(x)T(y) = \cdots + \delta^4(x-y)\cancel{B^I P_I^A}\partial^\mu J_{A\mu}$$

- $\dfrac{\delta}{\delta m^a}$ similar story

II. genuine WZ condition: $\Delta_{\sigma_2} \int \mathcal{A}_{\sigma_1} - \Delta_{\sigma_1} \int \mathcal{A}_{\sigma_2} = 0$

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$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\sigma\mathcal{A} = {} & \sigma\left(\beta_a W^2+\beta_b E_4+\frac{1}{9}\beta_c R^2\right)-\nabla^2\sigma\left(\frac{1}{3}dR\right)\\
&+\sigma\Big(\frac{1}{3}\chi^e_I\nabla_\mu\lambda^I\nabla^\mu R+\frac{1}{6}\chi^f_{IJ}\nabla_\mu\lambda^I\nabla^\mu\lambda^J R+\frac{1}{2}\chi^g_{IJ}G^{\mu\nu}\nabla_\mu\lambda^I\nabla_\nu\lambda^J\\
&\quad+\frac{1}{2}\chi^a_{IJ}\nabla^2\lambda^I\nabla^2\lambda^J+\frac{1}{2}\chi^b_{IJK}\nabla_\mu\lambda^I\nabla^\mu\lambda^J\nabla^2\lambda^K+\frac{1}{4}\chi^c_{IJKL}\nabla_\mu\lambda^I\nabla^\mu\lambda^J\nabla_\nu\lambda^K\nabla^\nu\lambda^L\\
&+\nabla^\mu\sigma\Big(G_{\mu\nu}w_I\nabla^\nu\lambda^I+\frac{1}{3}RY_I\nabla_\mu\lambda^I+S_{IJ}\nabla_\mu\lambda^I\nabla^2\lambda^J+\frac{1}{2}T_{IJK}\nabla_\nu\lambda^I\nabla^\nu\lambda^J\nabla_\mu\lambda^K\Big)\\
&-\nabla^2\sigma\left(U_I\nabla^2\lambda^I+\frac{1}{2}V_{IJ}\nabla_\nu\lambda^I\nabla^\nu\lambda^J\right)\\
&+\sigma\Big(\frac{1}{2}p_{ab}\hat{m}^a\hat{m}^b+\hat{m}^a\left(\frac{1}{3}q_aR+r_{aI}\nabla^2\lambda^I+\frac{1}{2}s_{aIJ}\nabla_\mu\lambda^I\nabla^\mu\lambda^J\right)\Big)\\
&+\nabla_\mu\sigma\left(\hat{m}^a j_{aI}\nabla^\mu\lambda^I\right)-\nabla^2\sigma\left(\hat{m}^a k_a\right)\\
&+\sigma\left(\frac{1}{4}\kappa_{AB}F^A_{\mu\nu}F^{B\mu\nu}+\frac{1}{2}\zeta_{AIJ}F^A_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J\right)+\nabla^\mu\sigma\left(\eta_{AI}F^A_{\mu\nu}\nabla^\nu\lambda^I\right) \qquad (2.49)
\end{aligned}
$$

10 differential constraints involving 25 tensorial coefficients

all but a few constraints can be “solved”

$$\mathcal{A} = \underbrace{\mathcal{A}_{R^2} + \mathcal{A}_{W^2}}_{\text{manifestly consistent}} + \overbrace{\mathcal{A}_{E_4} + \mathcal{A}_{F^2}}^{\text{non-trivial}} + \underbrace{\delta_{Weyl} F_{local}}_{\substack{\text{trivial} \\ \text{(scheme dep)}}}$$

$$\frac{1}{\sqrt{g}}\sigma\mathcal{A}_{R^2} = \sigma\left(\frac{1}{2}b_{ab}\Pi^a\Pi^b + \frac{1}{2}b_{aIJ}\Pi^a\Pi^{IJ} + \frac{1}{4}b_{IJKL}\Pi^{IJ}\Pi^{KL}\right)$$

$$\Pi^{IJ} = \nabla_\mu\lambda^I\nabla^\mu\lambda^J - B^{(I}\Lambda^{J)}$$

$$\Lambda^J \propto \left(\Box\lambda^J + \frac{1}{6}B^J R(g)\right)$$

$$\Pi^a = m^a - \frac{1}{6}t^a R(g) - \theta^a_I\Lambda^I$$

$$\delta_\sigma\Pi^{IJ} = \sigma\,(\,\ldots\,) + \nabla_\mu\sigma\,(\,\ldots\,) + \nabla^2\sigma\,(\,\ldots\,)$$

absence of derivative terms: consistency is manifest

$$\frac{1}{\sqrt{g}}\sigma\mathcal{A}_{R^2} = \sigma\left(\frac{1}{2}b_{ab}\Pi^a\Pi^b + \frac{1}{2}b_{aIJ}\Pi^a\Pi^{IJ} + \frac{1}{4}b_{IJKL}\Pi^{IJ}\Pi^{KL}\right)$$

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$$\begin{aligned}\Delta_\sigma\Pi^{IJ} &= \sigma\left[2\Pi^{IJ} - \gamma^I_K\Pi^{KJ} - \gamma^J_K\Pi^{IK} + \gamma^{IJ}_{KL}\Pi^{KL}\right] \\ \Delta_\sigma\Pi^a &= \sigma\left[2\Pi^a - \gamma^a_b\Pi^b + \gamma^a_{IJ}\Pi^{IJ}\right]\end{aligned}$$

absence of derivative terms: consistency is manifest

Non-trivial anomalies

$$\frac{1}{\sqrt{g}}\mathcal{A}_{E_4} = \sigma\, a E_4 + \sigma\,\frac{1}{2}\chi_{IJ} G_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J + \nabla^\mu\sigma\, w_I G_{\mu\nu}\nabla^\nu\lambda^I + \ldots \cdots$$

$$\frac{1}{\sqrt{g}}\mathcal{A}_{F^2} = \sigma\,\frac{1}{4}\kappa_{AB} F^A_{\mu\nu}F^{B\mu\nu} + \sigma\,\frac{1}{2}\zeta_{AIJ} F^A_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J + \nabla^\mu\sigma\,\eta_{AI} F^A_{\mu\nu}\nabla^\nu\lambda^I +$$

$$\boxed{\begin{aligned} \mathcal{L}[w_I] &= -8\partial_I a + \chi_{IJ}B^J \\ \mathcal{L}[\eta_{AI}] &= \kappa_{AB}P^B_I + \zeta_{AIJ}B^J - \chi_{IJ}(T_A\lambda)^J \\ 0 &= \eta_{AI}B^I + w_I(T_A\lambda)^I \end{aligned}}$$

Gradient flow equation

$$\tilde{a} \equiv a + \frac{1}{8} w_I B^I$$

$$\boxed{8\partial_I \tilde{a} = \left(\chi_{IJ} + \partial_I w_J - \partial_J w_I + P_I^A \eta_{AJ}\right) B^J}$$

Jack,Osborn 2013

- non-trivial constraint on perturbative expansion of B^I
- at fixed points $\tilde{a}(\lambda)$ is stationary
- along line of fixed points $\tilde{a} = a = \text{const}$

$$\boxed{8\mu \frac{d\tilde{a}}{d\mu} \equiv 8B^I \partial_I \tilde{a} = \chi_{IJ} B^I B^J}$$

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(0)\rangle = \frac{\chi_{IJ}}{x^8} + O(\partial B, B)$$ by unitarity $\chi_{IJ} > 0$

$$\boxed{8\mu\frac{d\tilde{a}}{d\mu} = \chi_{IJ}B^I B^J \geq 0}$$

$$\tilde{a}(\lambda(\mu_1)) - \tilde{a}(\lambda(\mu_2)) = \frac{1}{8}\int_{\mu_1}^{\mu_2} \chi_{IJ}B^I B^J \, d\ln\mu$$

since $\tilde{a}$ is finite the only possible asymptotics must satisfy $B^I = 0$

CFT, free or interacting, is the only possible asymptotics

scheme dependence

$$W \;\to\; W \;+\; \frac{c_{IJ}}{2}\sqrt{g}G^{\mu\nu}\nabla_\mu\lambda^I\nabla_\nu\lambda^J$$

$$\tilde{a} \to \tilde{a} + B^I B^J c_{IJ} \qquad\qquad \chi^g_{IJ} \to \chi^g_{IJ} + \mathcal{L}(c_{IJ})$$

It would be desirable to have a statement based on physical quantities

The other perspective:

the dilaton effective action

$$W[\Omega]$$

- dilaton background $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$ $\nabla_\mu \lambda^I = m^a = A_\nu^A = 0$

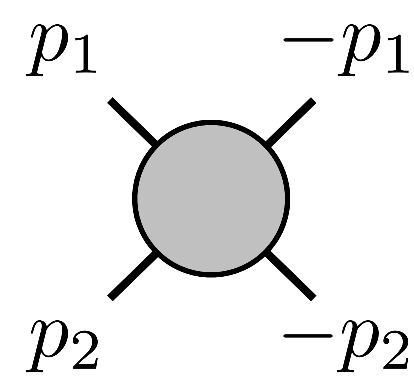
- extra UV divergences on curved background : $\mu \frac{d}{d\mu} W[\Omega] \neq 0$

  Ex.: $R(g)^2$ $R(g)\phi^2$

- 'on-shell' dilaton : $R(\Omega^2 \eta_{\mu\nu}) = 0 \quad \longleftrightarrow \quad \Box\Omega = 0$

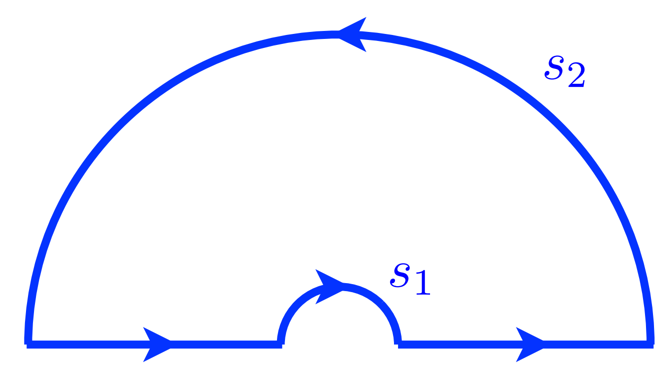
$$\boxed{\mu \frac{d}{d\mu} W[\Omega]\Big|_{\Box\Omega=0} = 0 \qquad \text{'on-shell' amplitudes are finite !}}$$

forward amplitude



$$\mathcal{M}(s) \;=\; \frac{\delta}{\delta\Omega(p_1)}\frac{\delta}{\delta\Omega(p_2)}\frac{\delta}{\delta\Omega(-p_1)}\frac{\delta}{\delta\Omega(-p_2)}W\Big|_{\Omega=1} \;=\; -8\alpha(\lambda(\sqrt{s}))\,s^2$$

$$\bar{\alpha}(s) \;\equiv\; \frac{1}{\pi}\int_0^{\pi} d\theta\,\alpha(se^{i\theta})$$



$$\boxed{s\frac{d\bar{\alpha}}{ds} \;=\; -\frac{2}{\pi}\,\mathrm{Im}\alpha(s) \;\geq\; 0}$$

$$\boxed{\bar{\alpha}(s) \quad \text{finite} \quad \Rightarrow \quad \lim_{s\to\pm\infty}\mathrm{Im}\,\alpha(s) \;=\; 0}$$

$$\mathcal{J}_0(\Omega) \equiv \begin{cases} g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}\,, & \lambda^I = \lambda^I(\mu) \\ A^A_\nu = m^a = 0 & \end{cases}$$

$$\mathcal{J}_1(\Omega) \equiv e^{-\Delta_{\ln\Omega}}\mathcal{J}_0(\Omega) = \begin{cases} g_{\mu\nu} = \eta_{\mu\nu} & \lambda^I = \lambda^I(\Omega\mu) \\ A^A_\mu = 0 & m^a = m^a[\Omega] \end{cases}$$

$$W[\Omega] = \underbrace{W[\mathcal{J}_0] - W[\mathcal{J}_1]}_{\text{local}} + W[\mathcal{J}_1]$$

$\Downarrow$

$$\boxed{\begin{array}{c}\text{effectively generated by} \\ \mathcal{L}_{\text{eff}} = \lambda^I(\Omega\mu)\mathcal{O}_I + m^a[\Omega]\mathcal{O}_a\end{array}}$$

$$W[\Omega]\Big|_{\mathrm{on-shell}} \;=\; W_{\mathrm{loc}} \;+\; W_{\mathrm{non-loc}}$$

$$= \left[\tilde{a}(\lambda(\Omega\mu)) + O(B^2)\right] \, (\partial \ln \Omega)^4$$

$$W[\Omega]\Big|_{\text{on-shell}} = W_{\text{loc}} + W_{\text{non-loc}}$$

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$$\mathcal{L}_{eff} = \lambda^I(\Omega\mu)\,\mathcal{O}_I + \frac{1}{2}B^I(\Omega\mu)\theta_I^a(\Omega\mu)(\Box \ln \Omega)\,\mathcal{O}_a$$

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scheme choice $\theta_I^a = 0$

$$= \left[\tilde{a}(\lambda(\Omega\mu)) + O(B^2)\right]\ (\partial \ln \Omega)^4$$

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scheme choice $\theta^a_I = 0$

(Jujitsu)

$$\bar{\alpha}(s) \;=\; \tilde{a}(s) + O(B^2)$$

ensures a scheme choice exists where $\bar{\alpha}(s) \;=\; \tilde{a}(s)$

$$-\mathrm{Im}\,\alpha(s) \;=\; \frac{1}{s^2}\sum_{\Psi}\left|\langle\Psi|B^I(\delta_I^J + \partial_I B^J)\mathcal{O}_J(p_1+p_2) + B^I B^J \mathcal{O}_I(p_1)\mathcal{O}_J(p_2)|0\rangle\right|^2$$

$$= B^I B^J G_{IJ}$$

$$G_{IJ} = \frac{1}{s^2}\sum_{\Psi}\langle 0|\mathcal{O}_I + \partial_I B^L \mathcal{O}_L + B^L \mathcal{O}_I\mathcal{O}_L|\Psi\rangle\langle\Psi|\mathcal{O}_J + \partial_J B^K\mathcal{O}_K + B^K\mathcal{O}_J\mathcal{O}_K|0\rangle \;\geq\; 0$$

$$\boxed{s\frac{d\bar{\alpha}(s)}{ds} \;=\; \frac{2}{\pi}G_{IJ}B^I B^J}$$

strictly > 0

at

$B,\ \partial B \ll 1$

G_{IJ} is the 4D analogue of Zamolodchikov metric in 2D

but 2D case simpler
(just 2-point funtions)

$$G_{IJ} = \frac{1}{p^2} \sum_{\Psi} \langle 0|\mathcal{O}_I(p)|\Psi\rangle \langle \Psi|\mathcal{O}_J(p)|0\rangle$$

without dilaton as guideline harder to figure things out in 4D

Summary

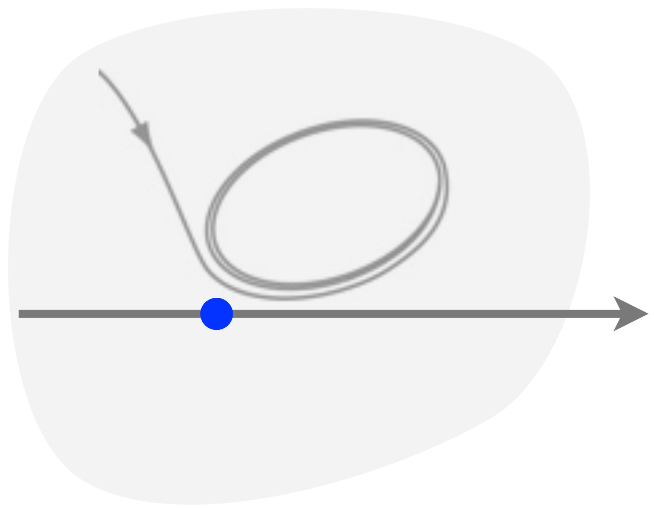
- Weyl anomaly off-criticality

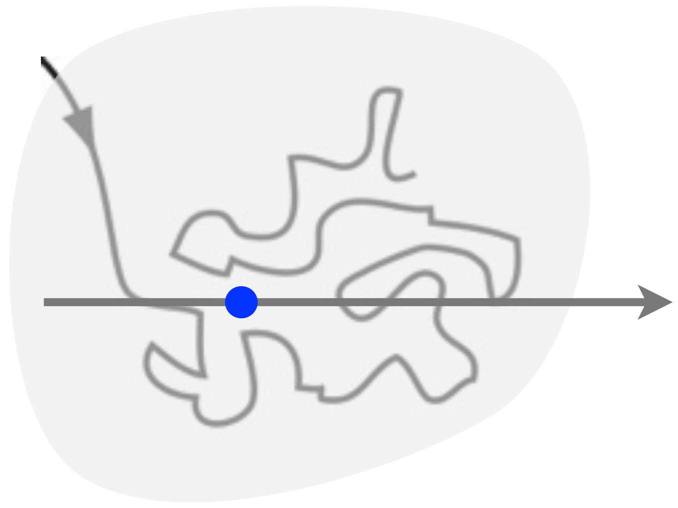
$$\mathcal{A} = \underbrace{\mathcal{A}_{R^2} + \mathcal{A}_{W^2}}_{\text{manifestly consistent}} + \overbrace{\mathcal{A}_{E_4} + \mathcal{A}_{F^2}}^{\text{non-trivial}} + \underbrace{\delta_{Weyl} F_{local}}_{\text{trivial}}$$

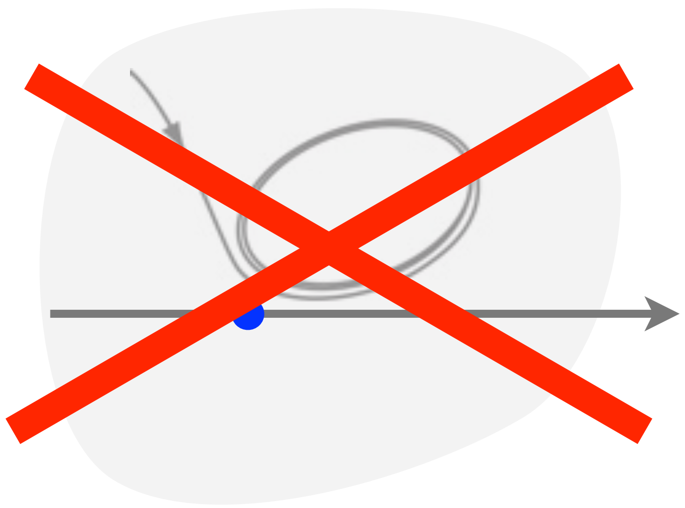
- 4D analogue of Zamolodchikov c-theorem

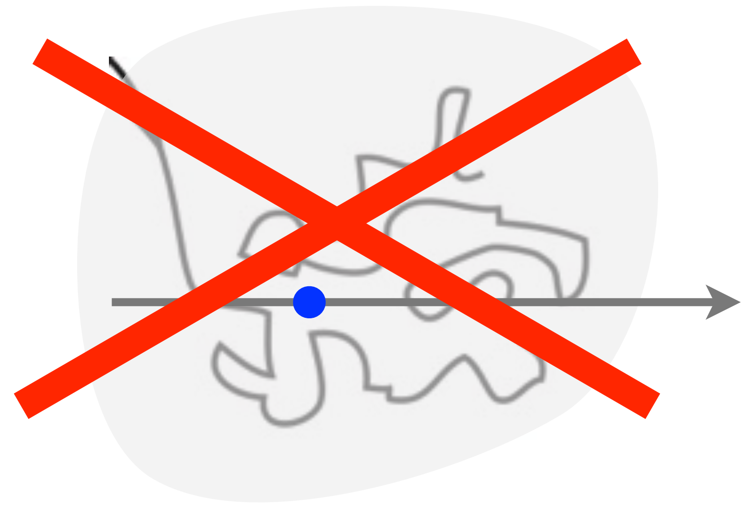
$$s\frac{d\bar{\alpha}(s)}{ds} = \frac{2}{\pi} G_{IJ} B^I B^J$$

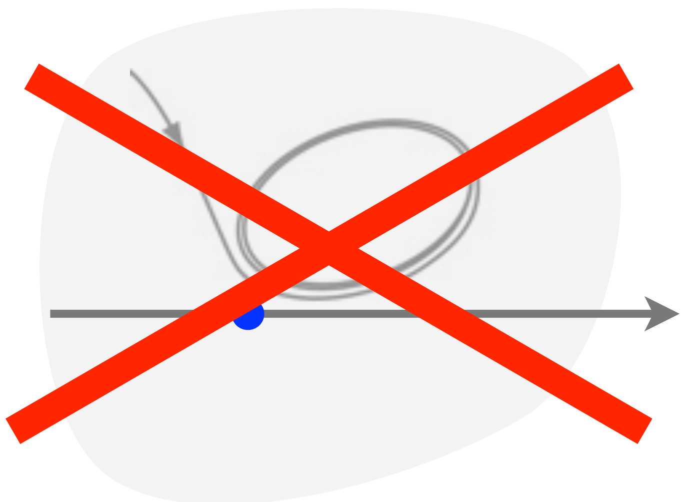
$$G_{IJ} = \frac{1}{s^2} \sum_{\Psi} \langle 0 | \mathcal{O}_I + \partial_I B^L \mathcal{O}_L + B^L \mathcal{O}_I \mathcal{O}_L | \Psi \rangle \langle \Psi | \mathcal{O}_J + \partial_J B^K \mathcal{O}_K + B^K \mathcal{O}_J \mathcal{O}_K | 0 \rangle \geq 0$$

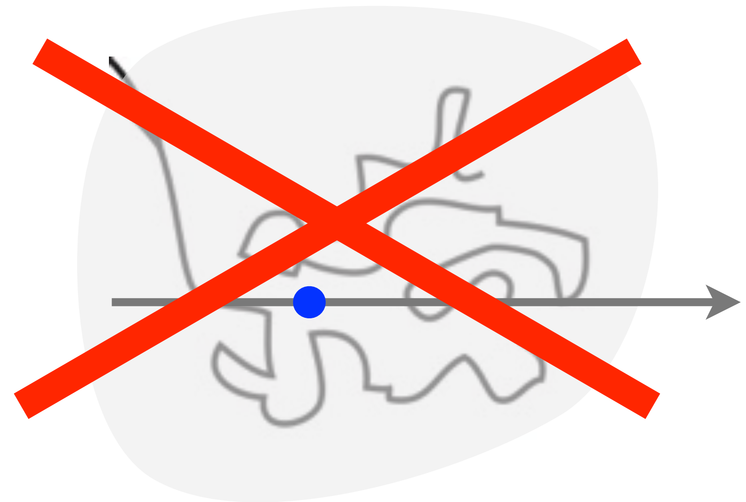




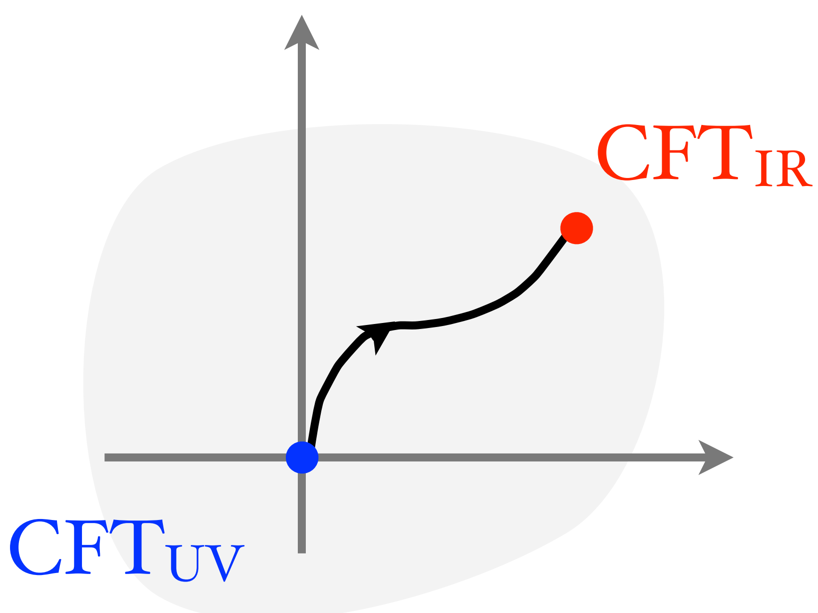








Only option



More on the local RG equation:

- Any lessons hidden in the remaining consistency condition?
- What about the special case of supersymmetry?
- What about flows around CFT that break parity?