

KITP

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The ground state wave function of Matrix Theory


Revisited

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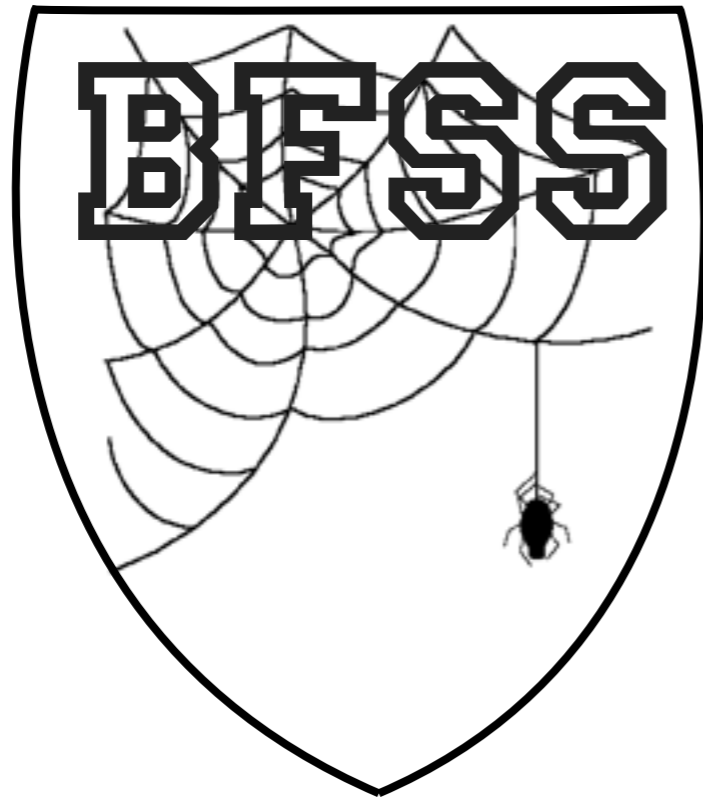
Harvard University

work in progress with Ying-Hsuan Lin

It has become increasingly evident that to understand semi-classical regimes of gravity using holography requires one to understand **genuinely** strongly coupled large N gauge theories.

...would like a clean setup where we have a better chance of tackling strong coupling (and...  Lorentzian) questions.

Turn back the clock 15 years



The holographic duality between

11-dimensional M-theory on the spacetime

$$ds^2 = f(r)dy^2 - 2dydt + \sum_{i=1}^9 dx_i dx_i,$$

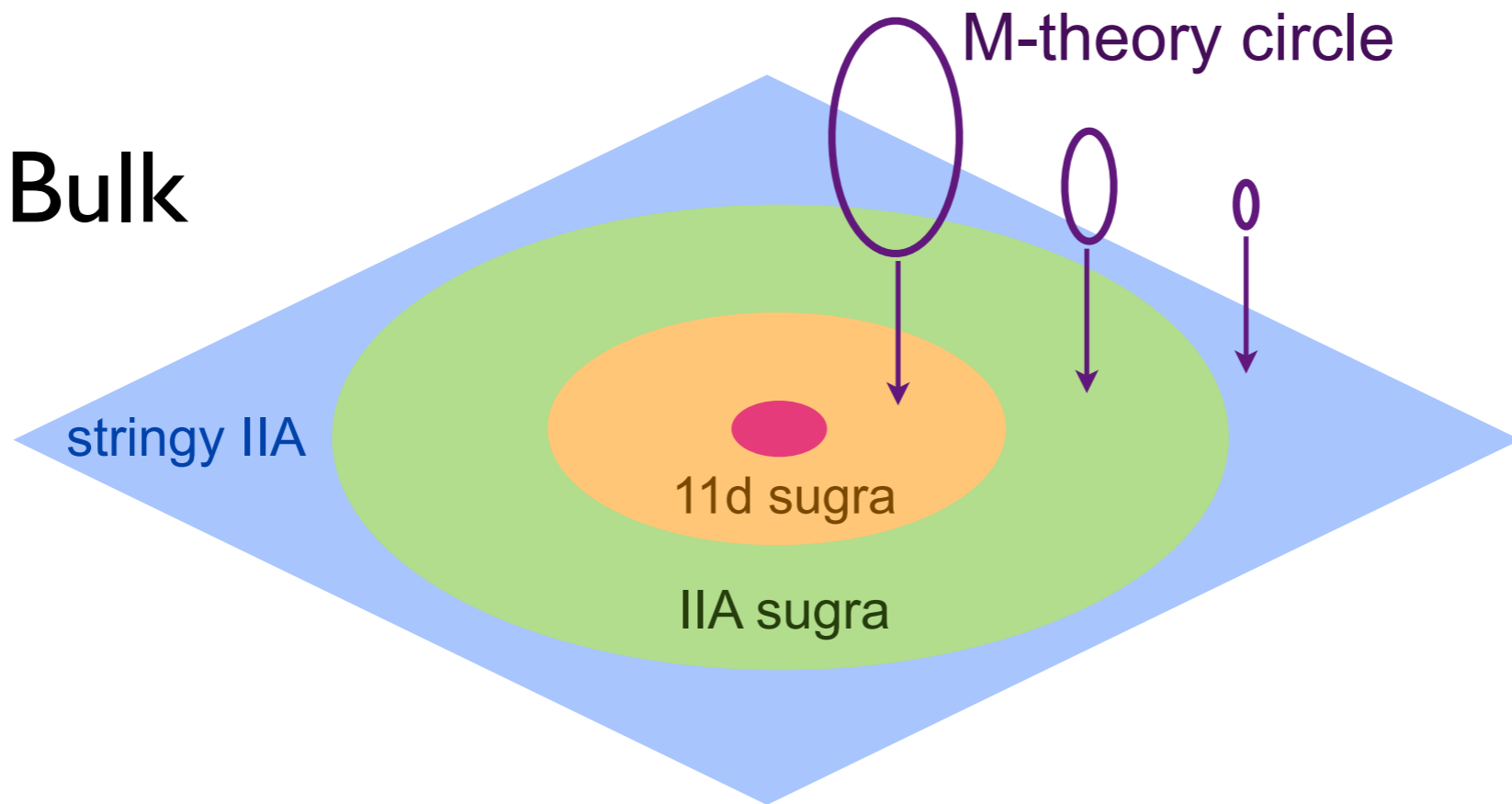
where $r^2 = \sum x_i^2$, $f(r) = N/r^7$. y is periodically identified:
 $y \sim y + 2\pi$.

16-supercharge SU(N) gauged matrix quantum mechanics

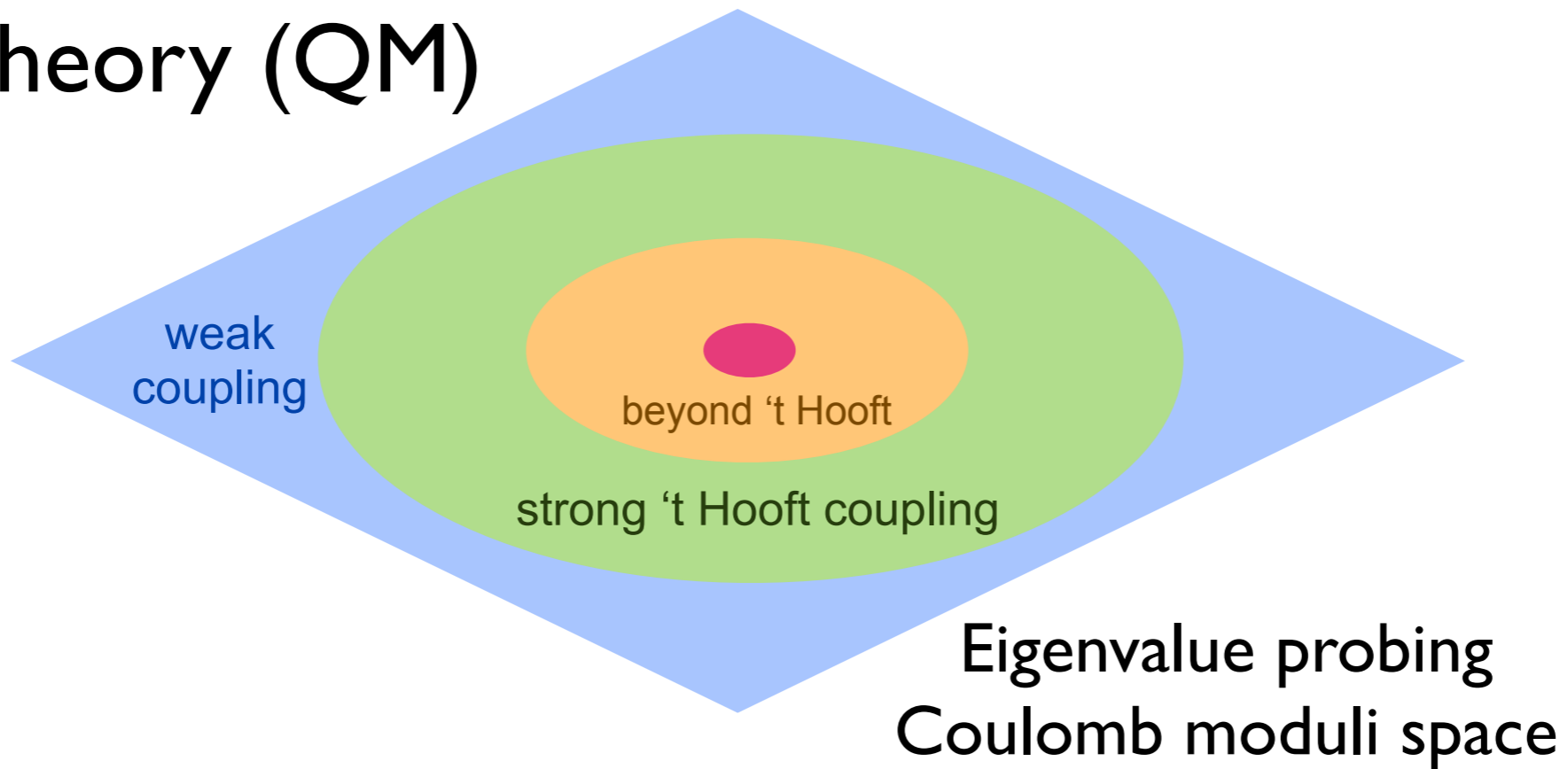
$$H = \text{Tr} \left(\frac{1}{2} P_i P_i - \frac{1}{4} [X_i, X_j]^2 - \frac{1}{2} \Theta^T \Gamma^i [X_i, \Theta] \right)$$

Banks-Fischler-Susskind-Shenker, Seiberg, Sen,
Balasubramanian-Gopakumar-Larsen, Susskind, Polchinski....

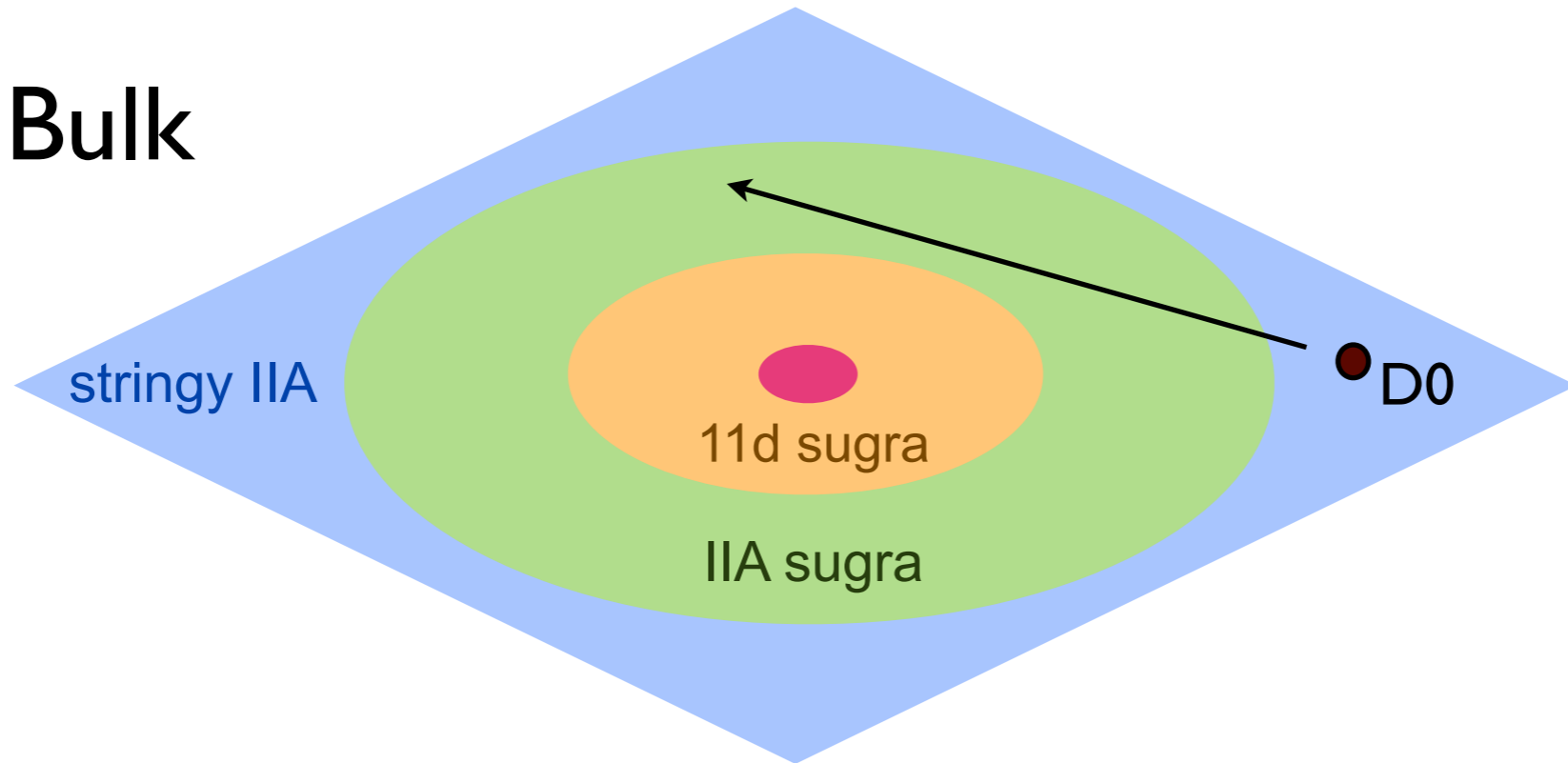
Bulk



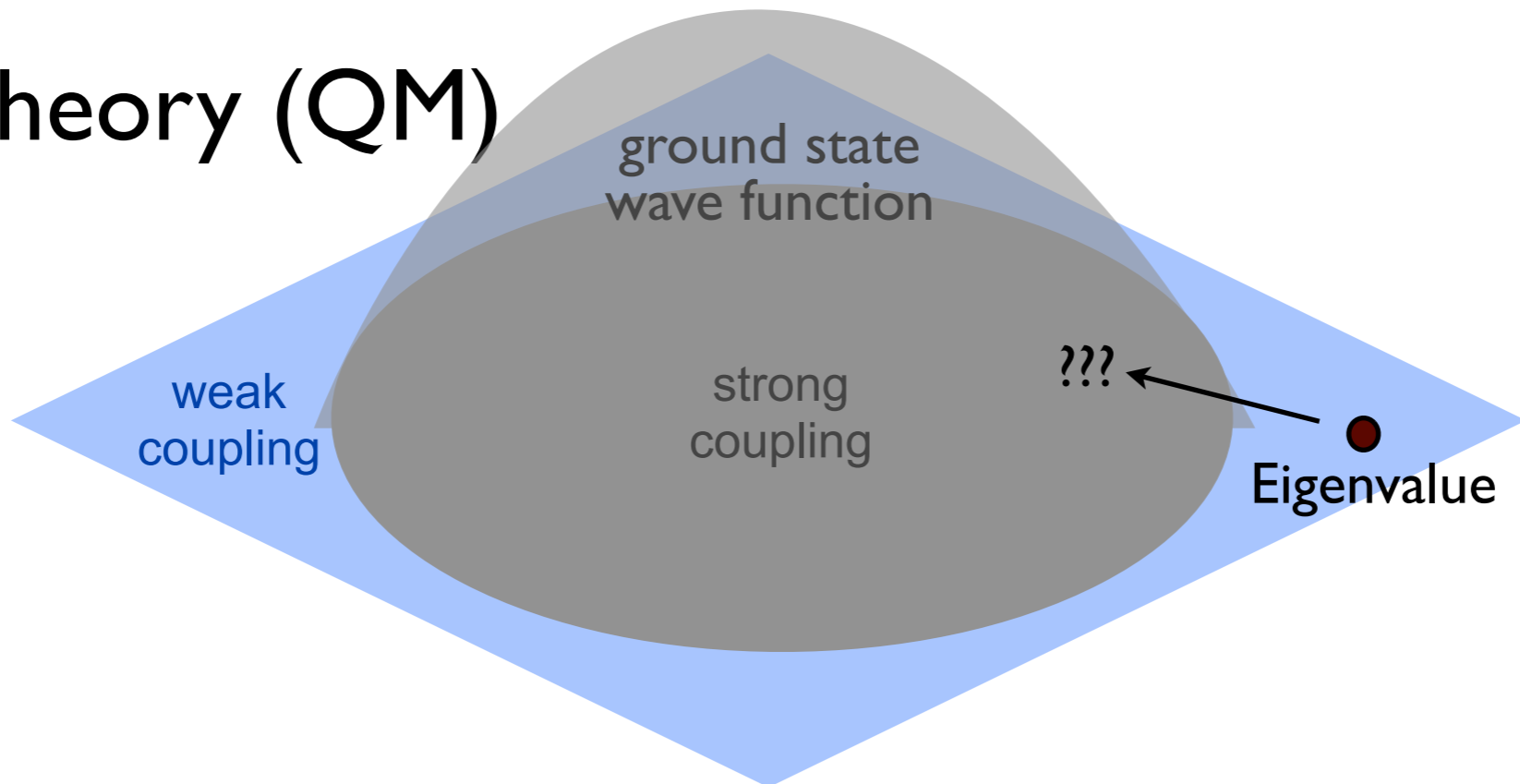
Gauge Theory (QM)



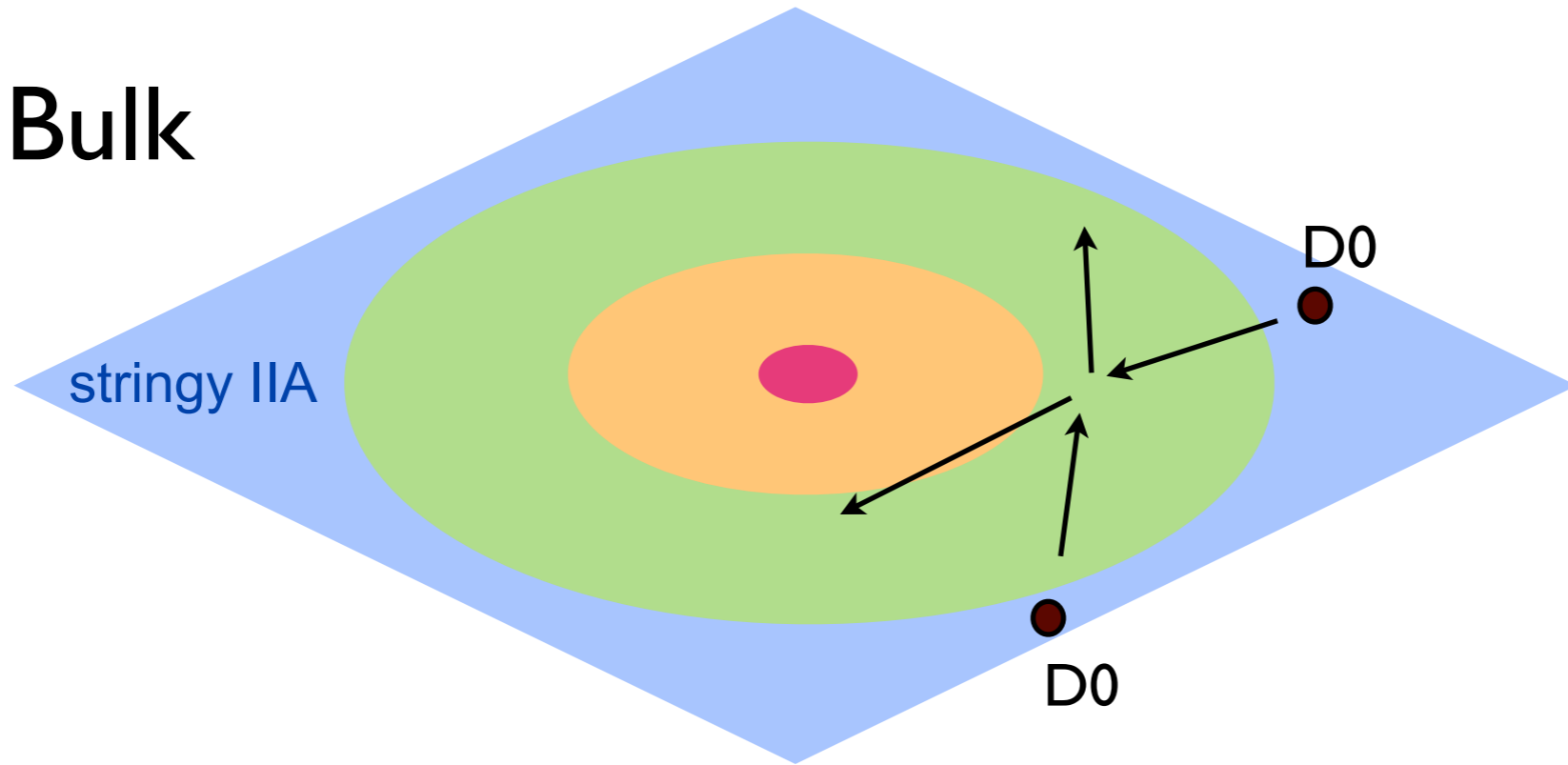
Bulk



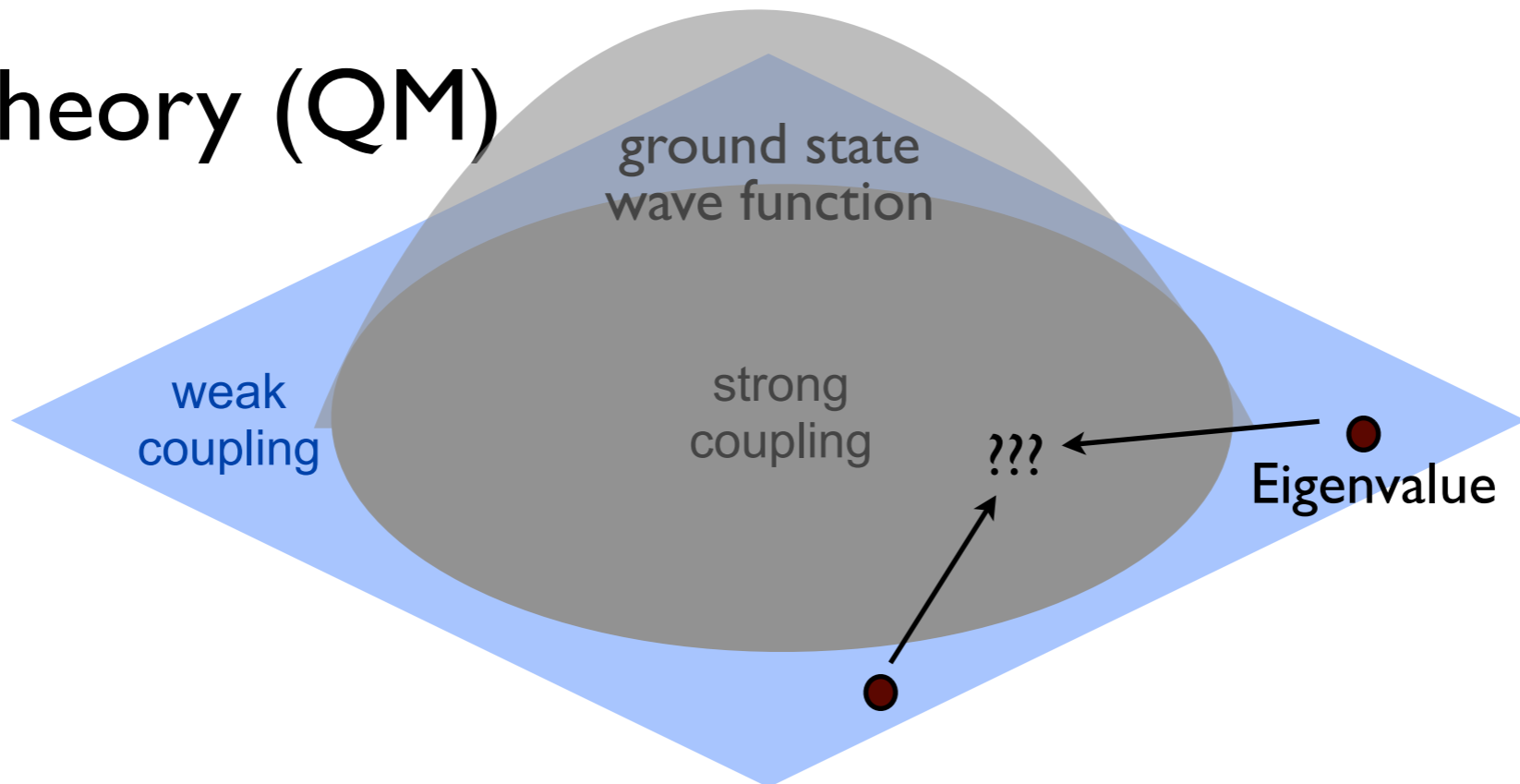
Gauge Theory (QM)



Bulk



Gauge Theory (QM)



Previous last century perturbative computation in Matrix Theory

Becker, Becker, Polchinski, Tseytlin,
Helling, Plefka, Serone, Waldron,

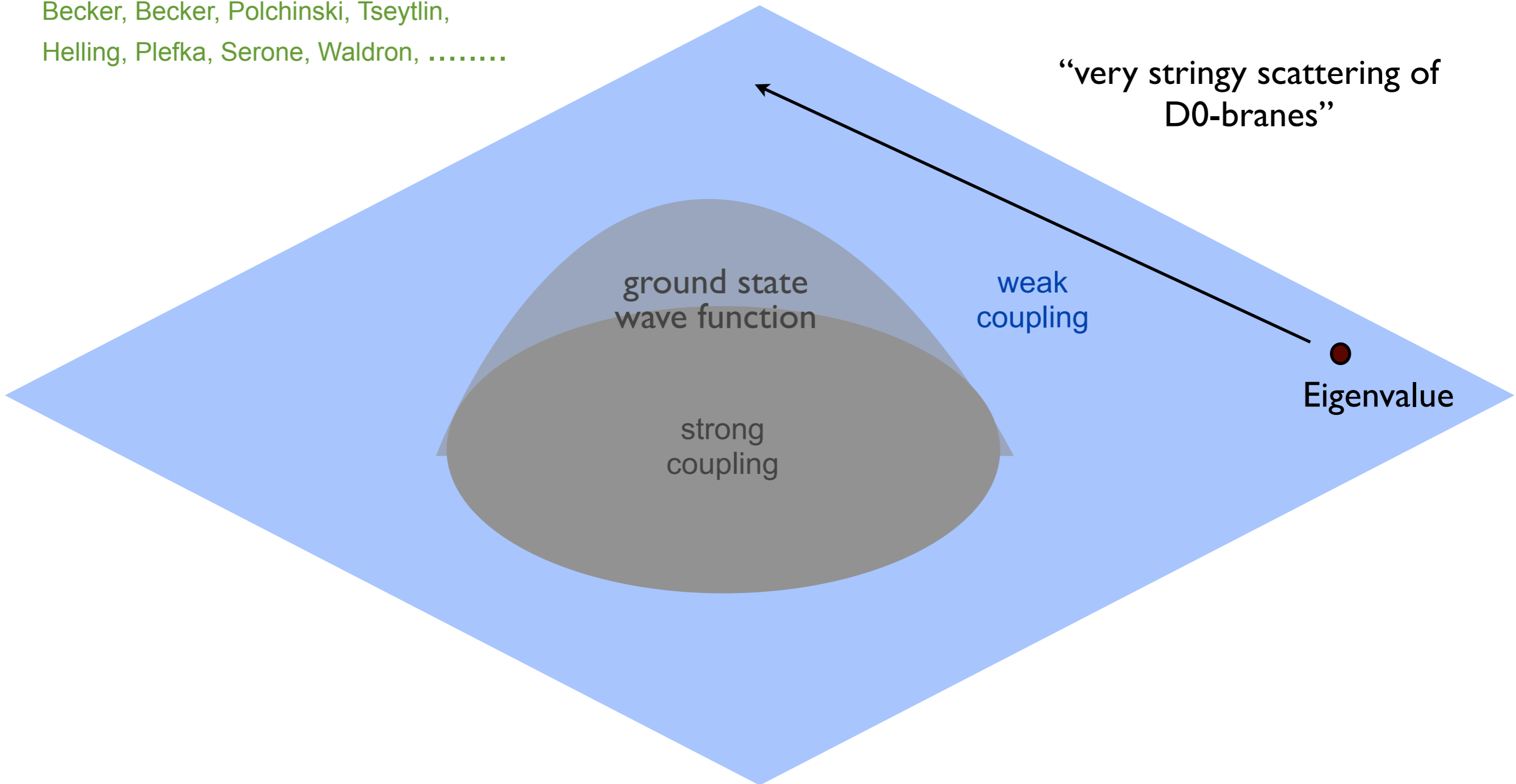
“very stringy scattering of
D0-branes”

ground state
wave function

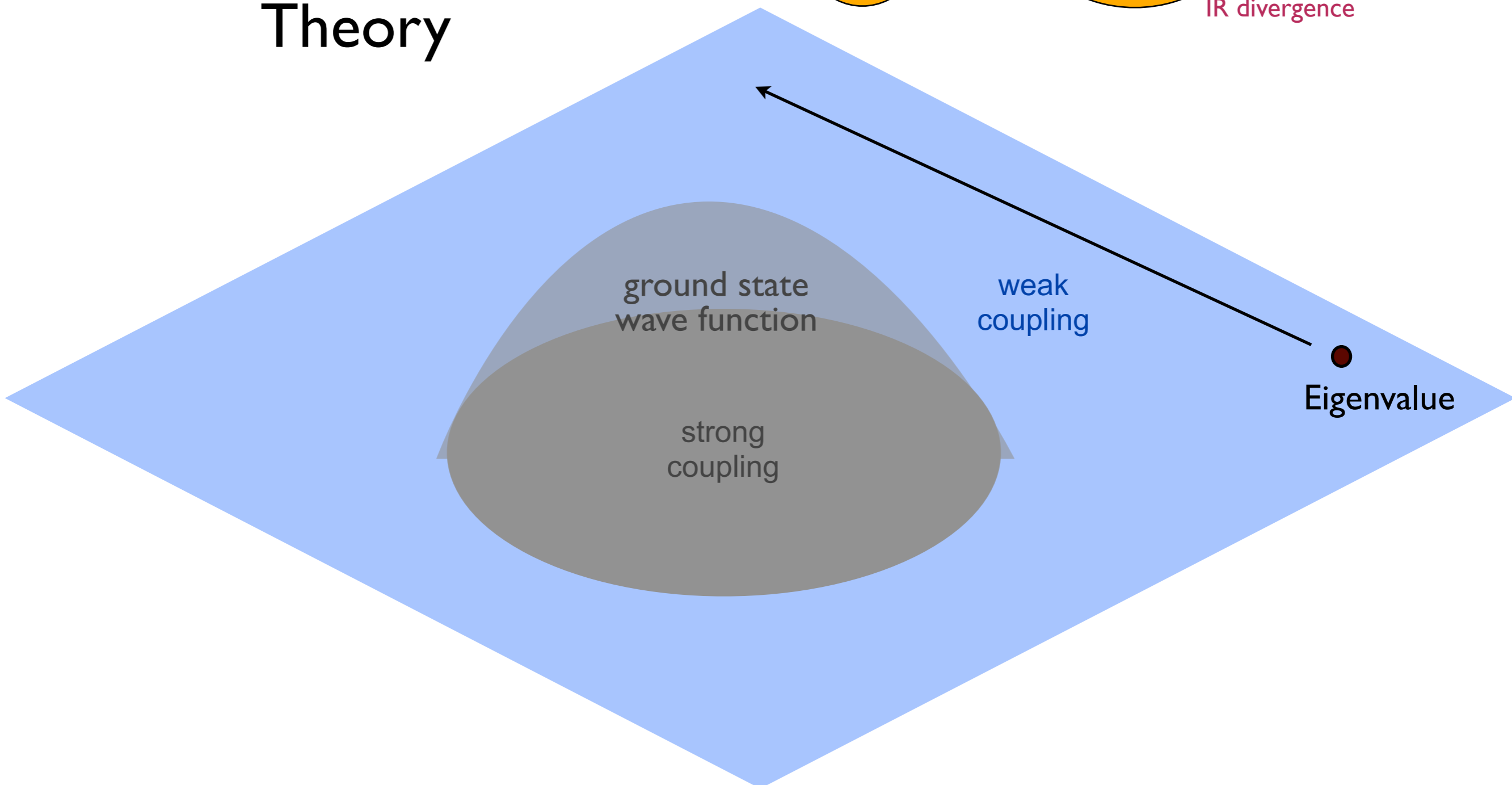
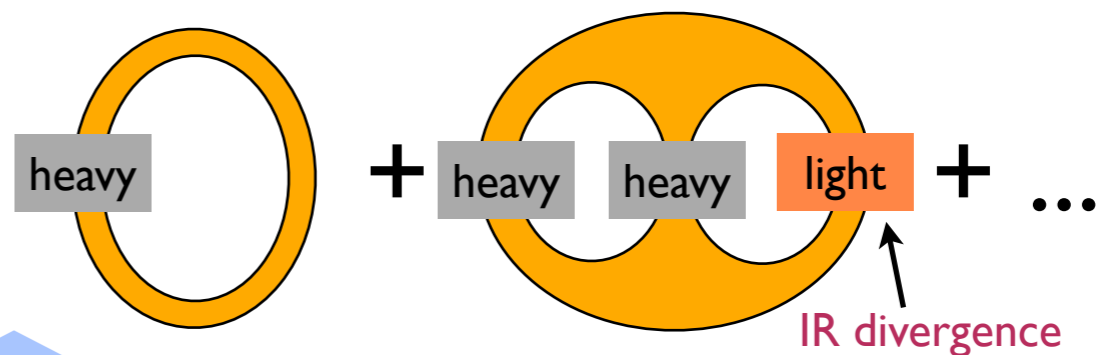
weak
coupling

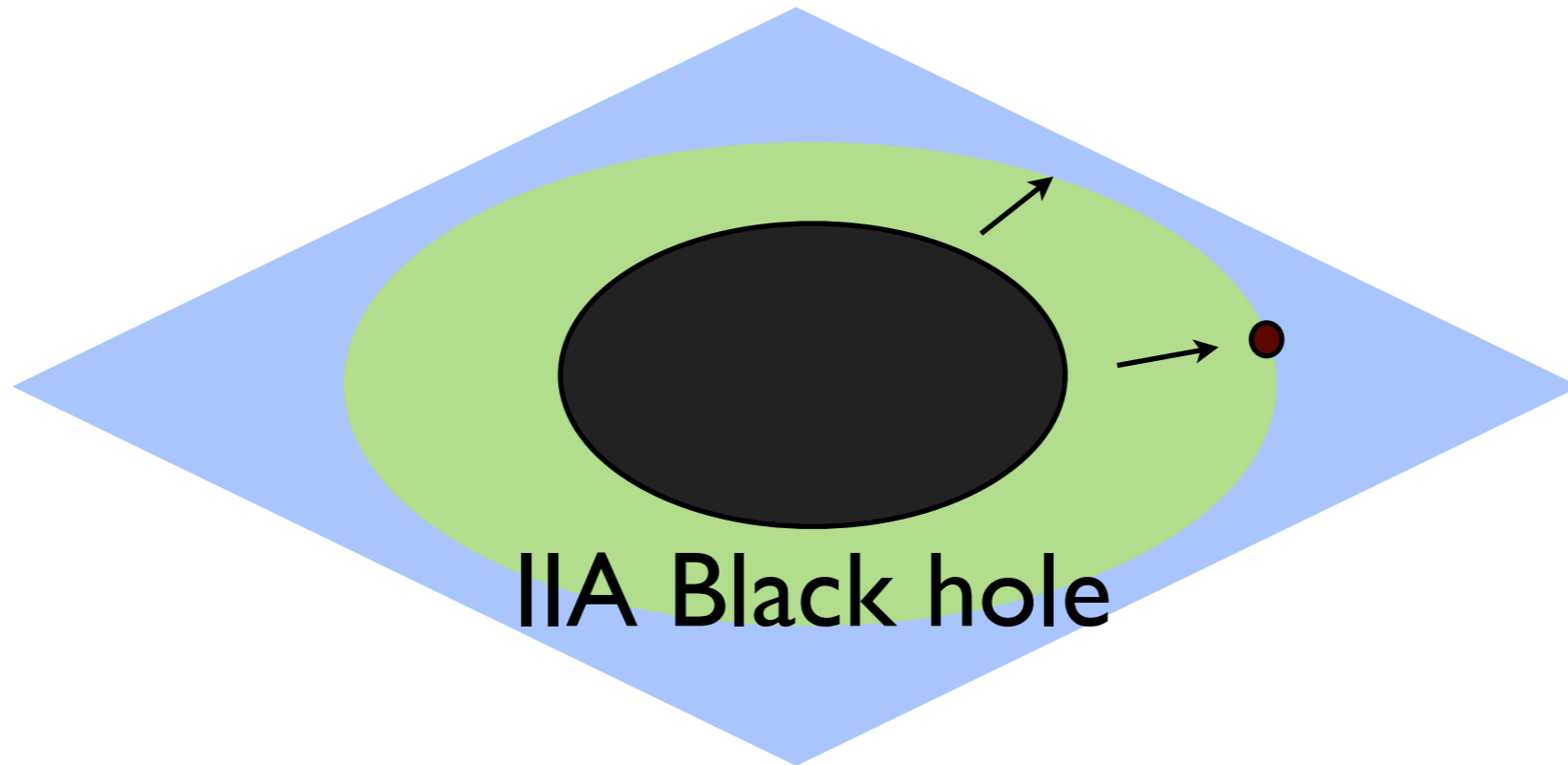
Eigenvalue

strong
coupling

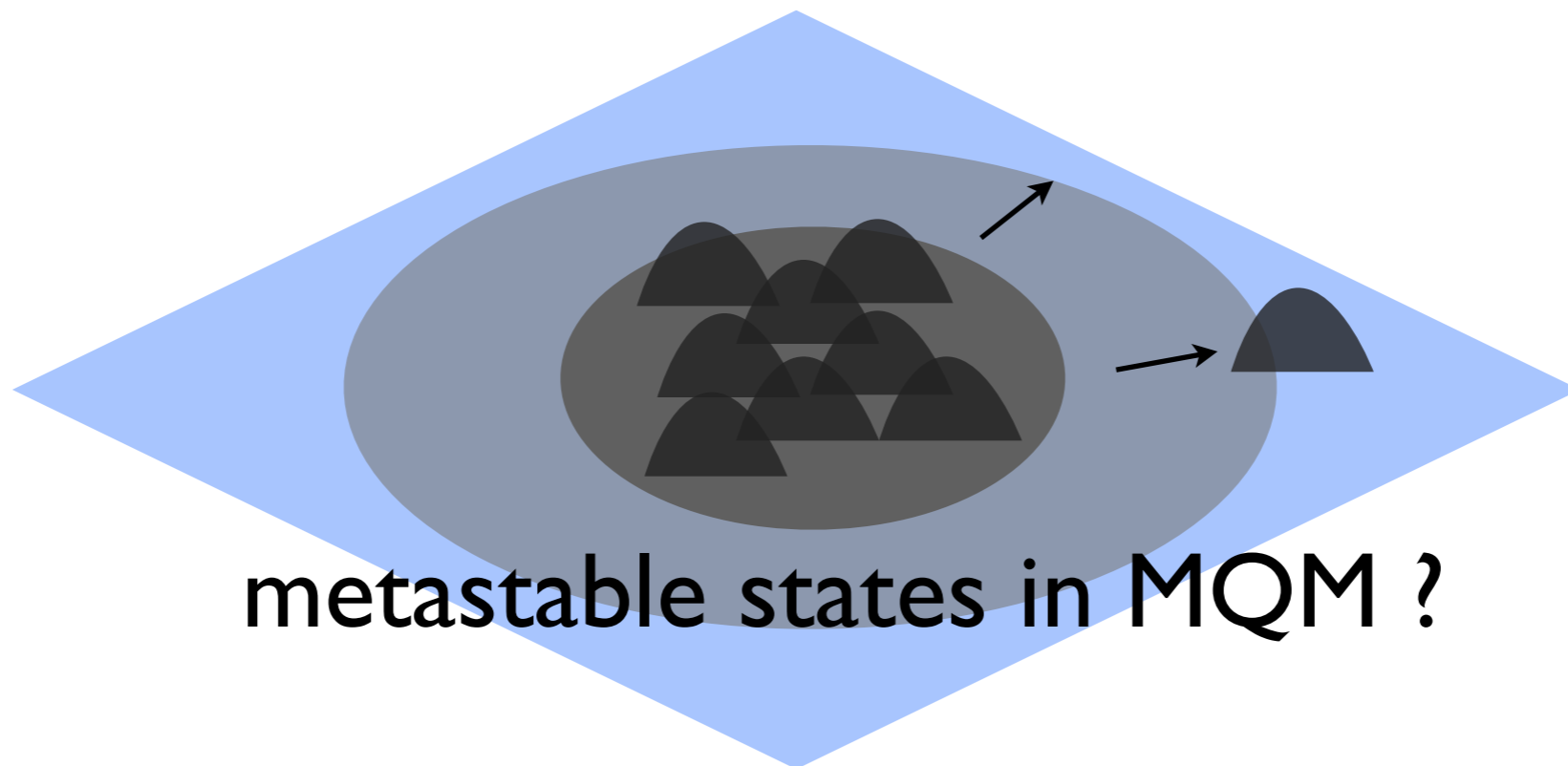


Last century perturbative computation in Matrix Theory





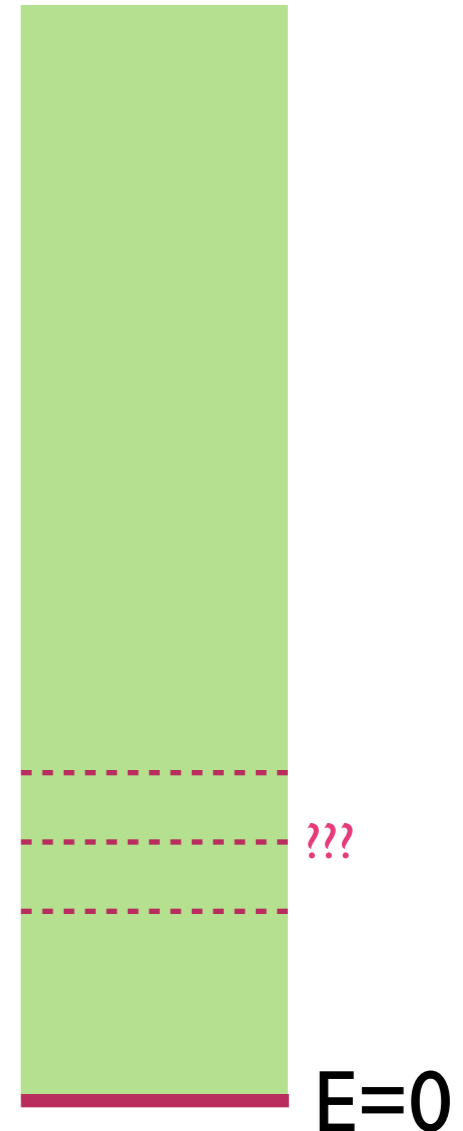
IIA Black hole



metastable states in MQM ?

The Hilbert space of Matrix Theory

1. There is a unique normalizable supersymmetric ground state.
2. There is a continuum of scattering states.
3. These are *probably* all the energy eigenstates. Unless the bulk black hole has remnants.
4. But, there must be **lots** of long lived metastable states to account for the microstates of the black hole.



What did we know about the ground state?

1. There is a unique normalizable ground state
(highly nontrivial index computation)

Yi, Sethi, Stern, Moore, Nekrasov, Shatashvili

2. Its wave function should spread to size $r \sim N^{1/3}$

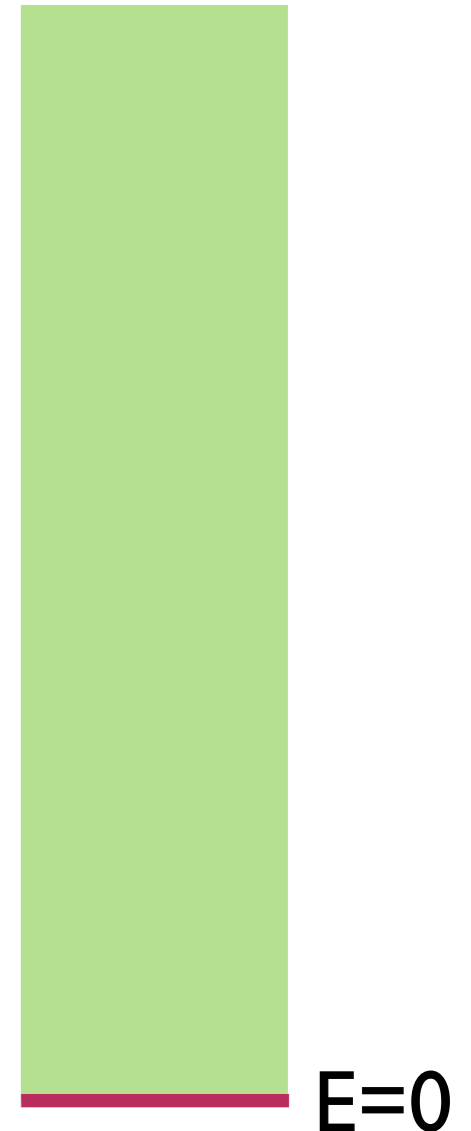
Polchinski

3. Asymptotic expansion for SU(2) case

Frolich, Graf, Hasler, Hoppe, Yau, Kac, Smilga

4. Some proposal for the asymptotic expansion for $N > 2$ (I'll present a new proposal.)

Hoppe, Plefka, Hasler



Why is this a hard problem?

1. MQM strongly coupled at low energies, and we are asking questions about its ground state.
2. The $SU(N)$ wave function has $2^{8(N^2-1)}$ components, each component a function of $9(N^2-1)$ coordinates.

Why is it an interesting problem?

1. The ground state wave function “creates” the semi-classical geometry in the bulk. It somehow turns the non-relativistic dispersion relation of an eigenvalue into a relativistic one (just like the fermi sea).
2. A starting point toward constructing metastable states that describe microstates of the black hole in the bulk.

The method of solving $Q\Psi=0$

Asymptotic expansion near the moduli space (Coulomb branch) with large separation of eigenvalues.

Frolich, Graf, Hasler, Hoppe, Yau

Neither existence nor uniqueness of solution is obvious / guaranteed.

SU(2) ✓

SU(N) need additional physical inputs

The asymptotic expansion

Step 1. Remove gauge redundancy, and separate heavy modes from light modes.

$$UX^iU^{-1} = \begin{pmatrix} x_1^i & 0 & & \\ 0 & x_2^i & & \\ & & \ddots & \\ & & & x_N^i \end{pmatrix} + \begin{pmatrix} 0 & q_{12}^i & & \\ (q_{12}^i)^* & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad \vec{q}_{ab} \cdot (\vec{r}_a - \vec{r}_b) = 0$$

$$U\hat{\Theta}_\alpha U^{-1} = \begin{pmatrix} \theta_{\alpha 1} & 0 & & \\ 0 & \theta_{\alpha 2} & & \\ & & \ddots & \\ & & & \theta_{\alpha N} \end{pmatrix} + \begin{pmatrix} 0 & (\Theta_\alpha)_{12} & & \\ (\Theta_\alpha)_{12}^* & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Assume $|\mathbf{r}_a - \mathbf{r}_b| \sim r$ large, expand in powers of $1/r$.

q_{ab} scales like $r_{ab}^{-1/2}$. Write $q_{ab} = r_{ab}^{-1/2} y_{ab}$.

The asymptotic expansion

Step 2. Expand the equation $Q\Psi = 0$.

$$Q = Q_0 + Q_1 + Q_2 + \dots$$

order $r^{1/2}$

$$Q^0 = \sum_{i,j=1}^9 \sum_{1 \leq a \neq b \leq N} |r_{ab}|^{\frac{1}{2}} \left(\Pi_{ab}^{ij} \frac{\partial}{\partial y_{ba}^j} \Gamma^i \Theta_{ba} + \frac{1}{2} \frac{r_{ab}^i}{|r_{ab}|} y_{ab}^j \Gamma^{ij} \Theta_{ba} \right)$$

order r^{-1}

$$Q^1 = \sum_a \frac{\partial}{\partial r_a^i} \Gamma^i \theta_a + \sum_{a \neq b} \frac{r_{ab}^i}{2|r_{ab}|^2} y_{ab}^j \frac{\partial}{\partial y_{ab}^j} \Gamma^i (\theta_a - \theta_b) + \sum_{a \neq b} \sum_{c \neq a,b} \frac{y_{ac}^i y_{cb}^j}{|\vec{r}_{ac}|^{\frac{1}{2}} |\vec{r}_{bc}|^{\frac{1}{2}}} \Gamma^{ij} \Theta_{ba}$$

$$+ \sum_{a \neq b} \frac{y_{ab}^i y_{ba}^j}{2|\vec{r}_{ab}|} \Gamma^{ij} (\theta_a - \theta_b) - \sum_{a \neq b} \frac{r_{ab}^i}{|\vec{r}_{ab}|^2} \Gamma^i \Theta_{ba} M_{ab}$$

$$- \sum_{a \neq b} \sum_{c \neq a,b} \left(\frac{|r_{bc}|^{\frac{1}{2}}}{|r_{ac}|^{\frac{1}{2}}} y_{ca}^k \Pi_{cb}^{kj} \frac{\partial}{\partial y_{cb}^j} - \frac{|r_{ac}|^{\frac{1}{2}}}{|r_{bc}|^{\frac{1}{2}}} y_{bc}^k \Pi_{ac}^{kj} \frac{\partial}{\partial y_{ac}^j} \right) \frac{\hat{r}_{ab}^i}{|r_{ab}|} \Gamma^i \Theta_{ab}$$

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \dots$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ r^{-\kappa} & r^{-\kappa-3/2} & r^{-\kappa-3} \end{array}$$

The asymptotic expansion

Step 2. Expand the equation $Q\Psi = 0$.

$$Q_{\alpha}^0 \Psi_0 = 0$$

$$Q_{\alpha}^0 \Psi_1 + Q_{\alpha}^1 \Psi_0 = 0$$

$$Q_{\alpha}^0 \Psi_2 + Q_{\alpha}^1 \Psi_1 + Q_{\alpha}^2 \Psi_0 = 0$$

etc.

The asymptotic expansion

Step 3. Project onto zeroth order ground states of heavy modes. Solve for leading asymptotic wave function.

$Q_{\alpha}^0 \Psi_0 = 0 \quad \Rightarrow$ The off-diagonal modes of Ψ_0 sits in their ground states. Does not constrain dependence on Cartan degrees of freedom (r, θ) .

$Q_{\alpha}^0 \Psi_1 + Q_{\alpha}^1 \Psi_0 = 0 \quad \Rightarrow$ Not knowing Ψ_1 , can project LHS onto zero-eigenspace of Q^0 :

$$P_0 Q_{\alpha}^1 \Psi_0 = 0$$

The asymptotic expansion

Step 3. Project onto zeroth order ground states of heavy modes. Solve for leading asymptotic wave function.

$$P_0 Q_\alpha^1 \Psi_0 = 0$$

After a simple rescaling of Ψ_0 by a measure factor, this becomes an equation for $N-1$ **free** superparticles on $\mathbf{R}^{9|16}$.

Further impose $S_N \times SO(9)$ invariance.

Solution not unique when $N > 2$. Need additional input before we could carry out the asymptotic expansion order by order.

We'll discuss the explicit solution in a moment.

The asymptotic expansion

Step 4. Project the next equation onto zeroth order ground states of heavy modes. Solve for next-to-leading order asymptotic wave function.

$$P_0 Q_\alpha^1 \Psi_1 + P_0 Q_\alpha^2 \Psi_0 = 0 \quad \Rightarrow \quad \text{Solve for } \Psi_1 .$$

This case turns out to be particularly simple, and is determined from Ψ_0 algebraically.

$$P_0 Q_\alpha^1 \Psi_2 + P_0 Q_\alpha^2 \Psi_1 + P_0 Q_\alpha^3 \Psi_0 = 0$$

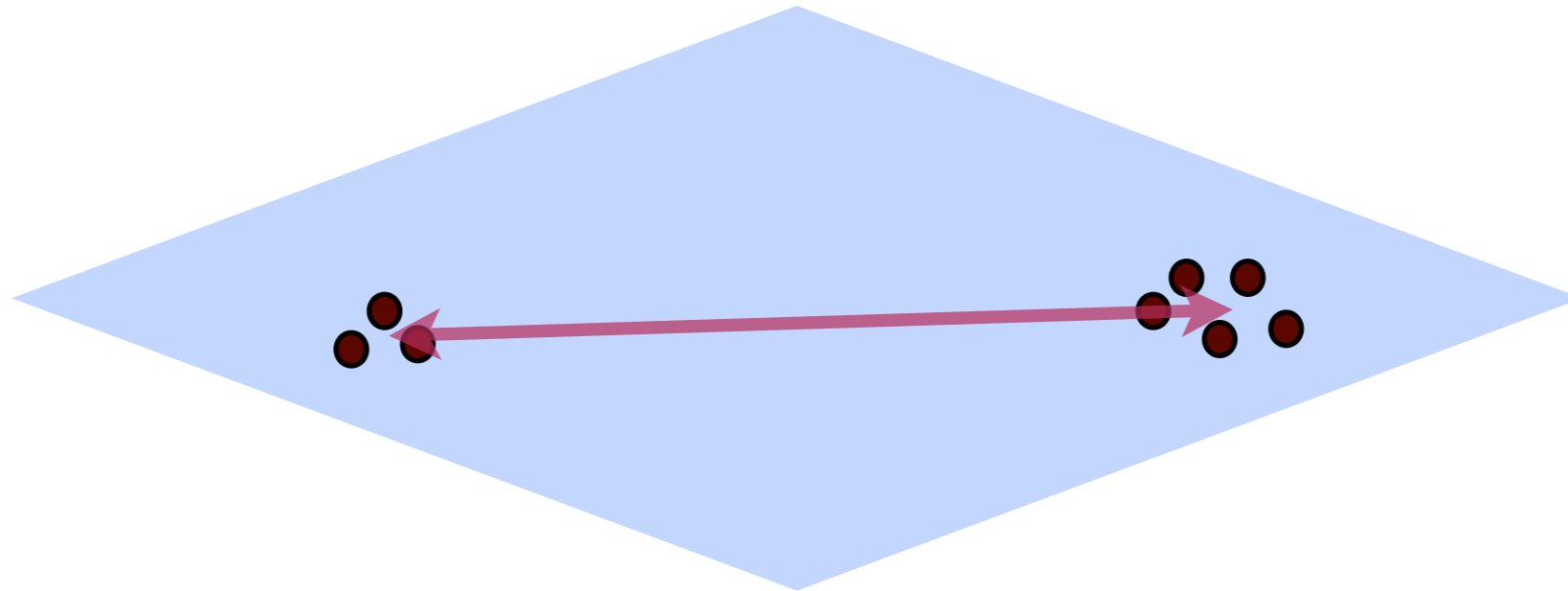
$$\Rightarrow \quad \text{Solve for } \Psi_2 .$$

Some more work required: need to solve Dirac-like equation for wave function of N-1 superparticles with sources.

With the right Ψ_0 , we could carry this on order by order.

Expected properties of Ψ_0

1. Factorization in various limits on the Coulomb branch.



2. Fall off as slow as possible at large r . Caution: a priori there could be more than one branches in the asymptotic expansion.

The SU(2) case Frolich-Graf-Hasler-Hoppe-Yau '99

16 Cartan U(1) fermions θ_α , quantization \rightarrow 256 states that branch into 44 \oplus 84 \oplus 128 under SO(9) rotation. Only the 44 can form an SO(9) singlet by combining with the bosonic part of the wave function.

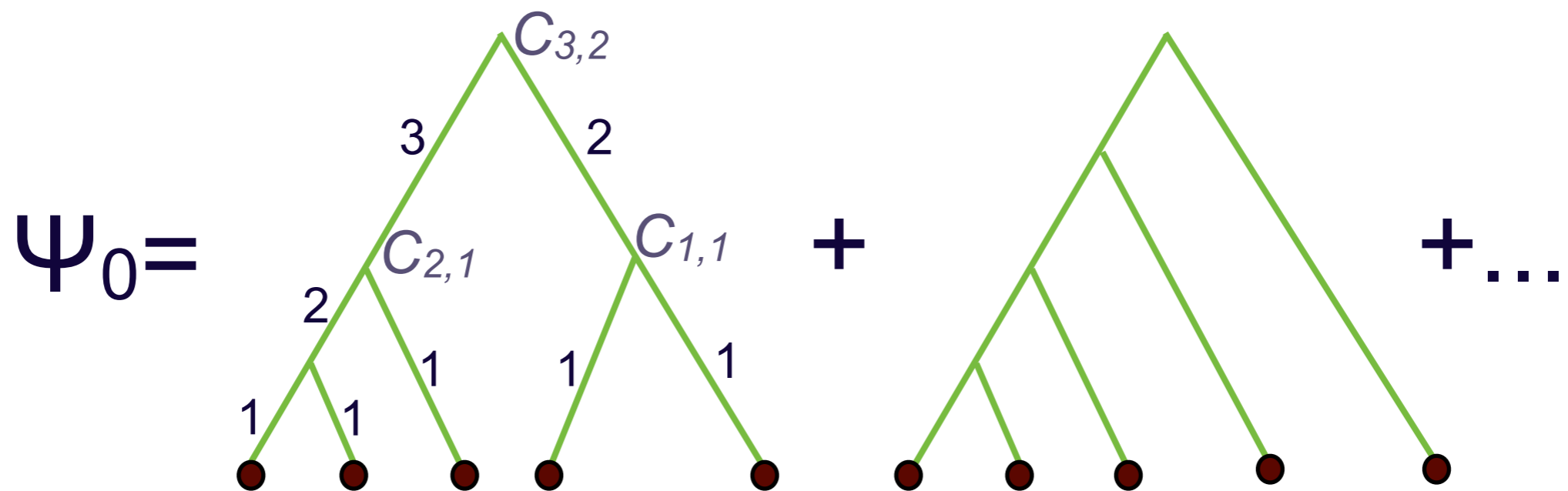
$$\Psi_{free} = \frac{\hat{r}^i \hat{r}^j}{|r|^9} |ij\rangle$$

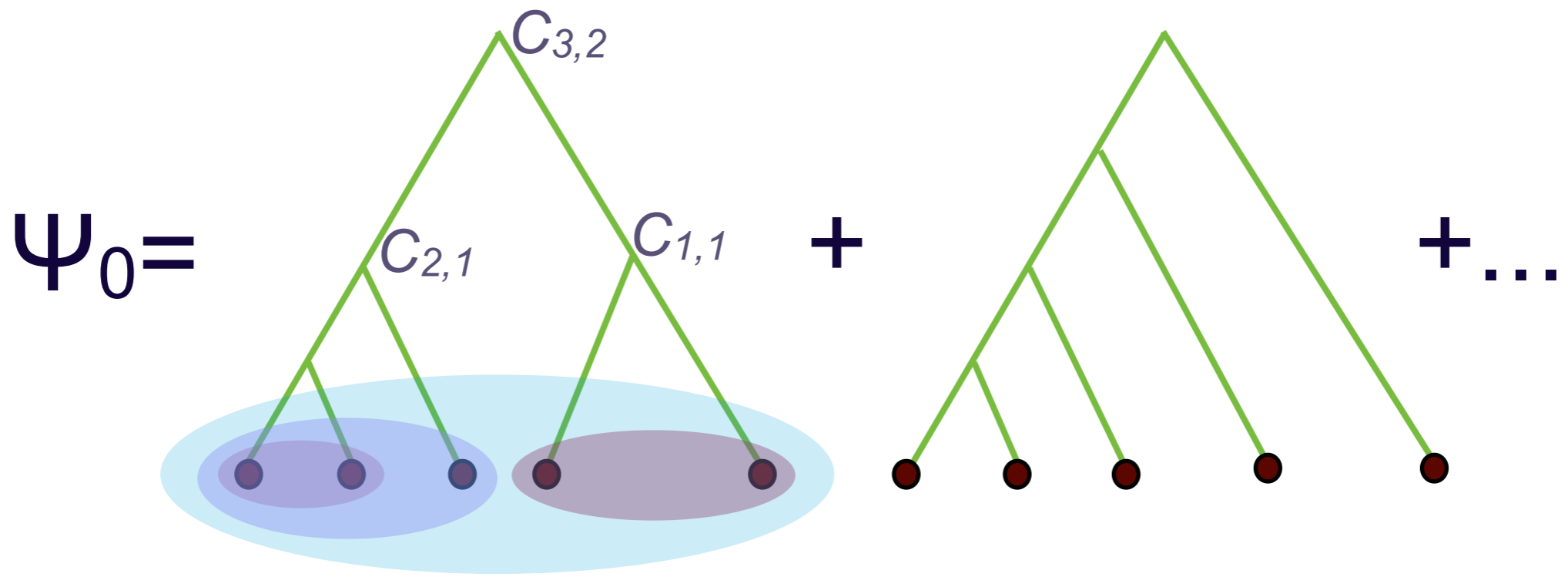
Note that the bosonic part are indeed a set of harmonic functions on R^9

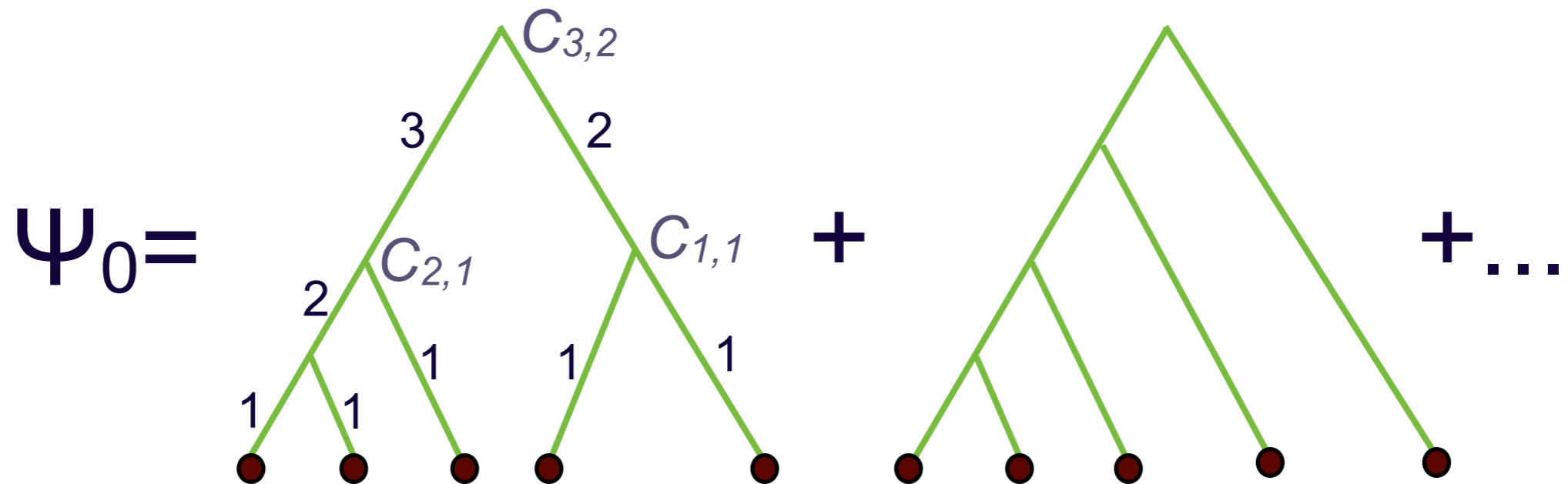
Ψ_0 is related to this by a measure factor and tensoring with the ground state of the off-diagonal modes (its relative phase fixed by SO(9) invariance).

Now for SU(N), I present a simple proposal that solves all the constraints we know of.

Summing over products of two-body wave functions, labeled by trees that group N particles in pairs:







Falls off like $r^{-9(N-1)}$ at large distances on Coulomb branch.
 Perhaps anticipated (Polchinski 99), but differs from previous proposal of Hoppe (of the form $Q^{16}|r|^{-9(N-1)+2} \sim r^{-9(N-1)-14}$).

I don't know any argument to fix the coefficients $C_{n,m}$.
 Unclear if constrained by asymptotic expansion to higher order... perhaps can be computed/tested via Monte Carlo.

Once we know Ψ_0 , no further constraint at next-to-leading order. Straightforward to compute Ψ_1 :

$$\Psi_1 = \sum_{\triangle} \text{[Diagram 1]} + \text{[Diagram 2]} + \dots$$

More explicitly,

$$\Psi_1 = -\frac{5}{8} \sum_{a,b} \frac{1}{|r_{ab}|^{\frac{3}{2}}} (\Theta_{ba}^+ \not{y}_{ab} (\theta_a - \theta_b)) \Psi_0 + \sum_{a,b,c} \frac{1}{|r_{ab}| + |r_{ac}| + |r_{bc}|} \left[\frac{15}{8} \frac{(\Theta_{bc}^+ \not{y}_{ca} \Theta_{ab}^+)}{|r_{ac}|^{\frac{1}{2}}} + \frac{1}{16} \left(\frac{1}{|r_{bc}|} - \frac{1}{|r_{ab}|} \right) \frac{(\vec{r}_{ab} \cdot \vec{y}_{ca})(\Theta_{ab}^+ \Theta_{bc}^+)}{|r_{ac}|^{\frac{1}{2}}} - 2 \frac{(\vec{r}_{cb} \cdot \vec{y}_{ac})(\vec{y}_{cb} \cdot \vec{y}_{ba})}{|r_{ab}|^{\frac{1}{2}} |r_{ac}|^{\frac{1}{2}} |r_{bc}|^{\frac{1}{2}}} \right] \Psi_0.$$

We may go on and solve for Ψ_2, Ψ_3 , etc. recursively. Not as easy as for Ψ_1 because now we will have to solve first order Dirac-like equations with source.

We have proposed an asymptotic form of the ground state wave function that is consistent with all physical constraints we know of, and carried out the asymptotic expansion to the next order.

What is it good for?

1. Revisit D0/graviton scattering calculations in matrix QM.

Even in the perturbative regime, ground state wave function important and spoils the structure naive analytic structure of perturbative expansion in v^2/r^4 and $1/r^3$, starting at order v^8 (when susy no longer fixes r dependence.)

Scattering off ground state wave function \rightarrow DBI for bulk probe?

2. Help address the question of what happens to an eigenvalue that falls into the “big” wave function and how it “sees” semi-classical relativistic space-time.

3. Speculation: reminiscent of tensor network?
Entanglement in fermion wave function...

4. Build metastable excited states (black hole microstates).

An old idea: the space-time is not emergent in BFSS matrix QM - it's already there on the Coulomb branch (so it seems). Perhaps the best way to probe the bulk geometry isn't through correlation functions, but still to use eigenvalues.

A big question: Black hole \iff long lived metastable states. Is the horizon (and the spacetime behind the horizon) visible in MQM?

“What is there to do in string theory besides inflation?”

“Matrices. You can play with matrices.”

- two prominent physicists