Analytic properties of correlation functions in CFTs

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KITP, January 2014

Work in progress with

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Introduction

One can ask the following basic question:

- What is the analytic structure of correlation functions of local operators in CFTs?

or even a simpler one first:

- What are the singularities of correlation functions of local operators in CFTs?

Introduction

It is well known in the case of scattering amplitudes that knowing the position and the nature of singularities is very useful.

- Unitarity/on-shell methods

- Dispersion relations

Introduction

In case of CFTs we can also hope that a better understanding of singularities and their properties can shed some light on general properties of strongly coupled systems as well as emergence of local spacetime in holography.

Introduction

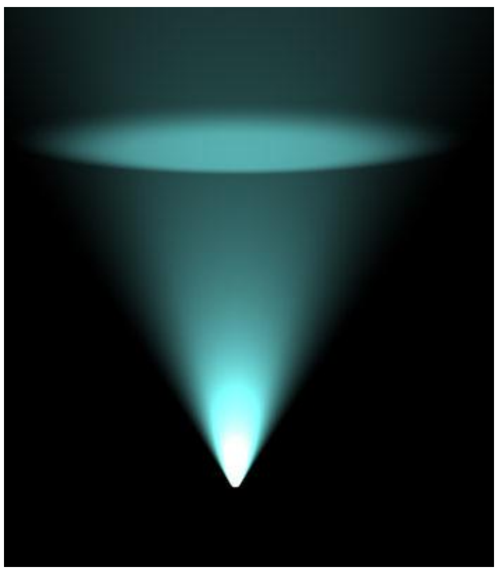
Consider a correlation function of local operators in an abstract CFT

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)\rangle$$

In Euclidean space the only singularities come from the coincident points. This is the usual OPE limit governed by the scaling dimensions Δ.

Introduction

In Minkowski space we can consider a situation when points approach each others light cone.

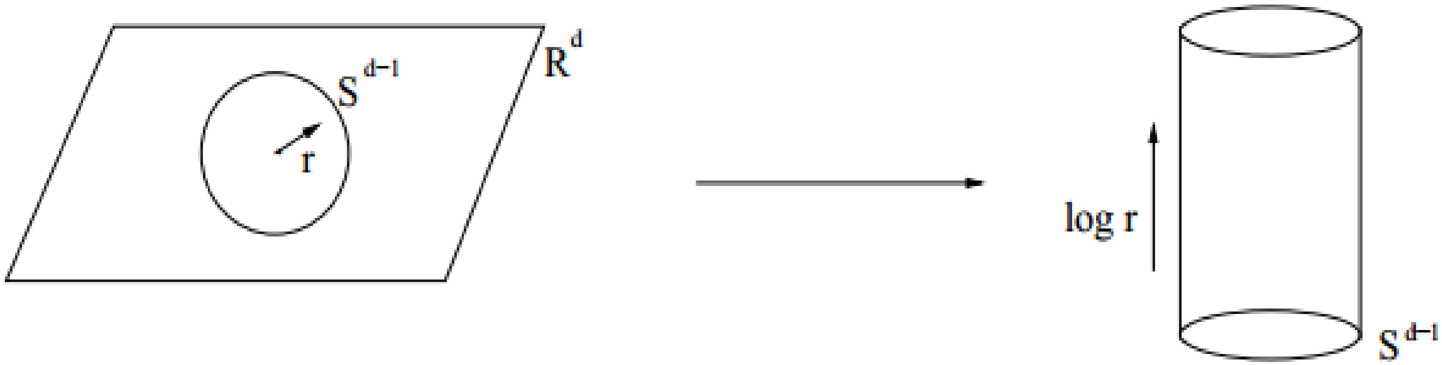


In this case the limit is governed by the so called twists of the operators

$$\tau = \Delta - s$$

Introduction

Recall that d-dimensional CFT is naturally defined on the cylinder $R \times S^{d-1}$



On the cylinder it is also natural to consider the limit when points are connected by the family of light cones that fill the whole sphere (Regge limit).

Cornalba, Costa, Penedones
Caron-Huot '13

Introduction

Given a generic CFT what are all the singularities and what is their physics?

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A quick look at some examples immediately reveals that the naive singularities that I just discussed do not exhaust all possibilities.

Introduction

A similar situation existed in the 60s in the context of the S-matrix theory. One of the general questions that people were interested in was:

What are the analytic properties of an abstract S-matrix?

Introduction

The approach taken back then is summarized in the paper by Landau from 1959 “On analytic properties of vertex parts in quantum field theory”

In recent years many papers have been concerned with dispersion relations. As is known, the latter express the analytic properties of various quantities of quantum field theory. The problem of localizing the singularities of these quantities is therefore highly important. As has become clear recently [1,2]), a direct study of graphs is the most effective method of investigating the location and nature of the singularities of vertex parts.

Introduction

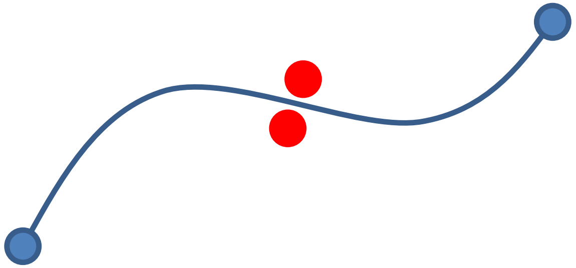
Let us try to apply the same strategy to CFTs.

Historically, people discuss Landau equations in momentum space. However, the absence of any scale in the theory makes perturbation theory in coordinate space very similar to the one in momentum space.

Landau equations

Problem: Given a Feynman graph find external configurations for which it is singular.

This happens when poles of the propagators pinch the contour of the loop integration.



Answer: Solution of Landau equations.

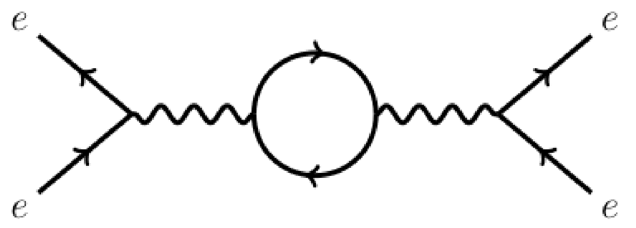
Landau equations

A very clear interpretation of Landau equations was given by Coleman and Norton in 1965.

Solution of Landau equations = Existence of a physical process that involves energy-momentum preserving scattering of on-shell particles moving forward in time.

From this point of view singularities are something that are especially “real”!

Example



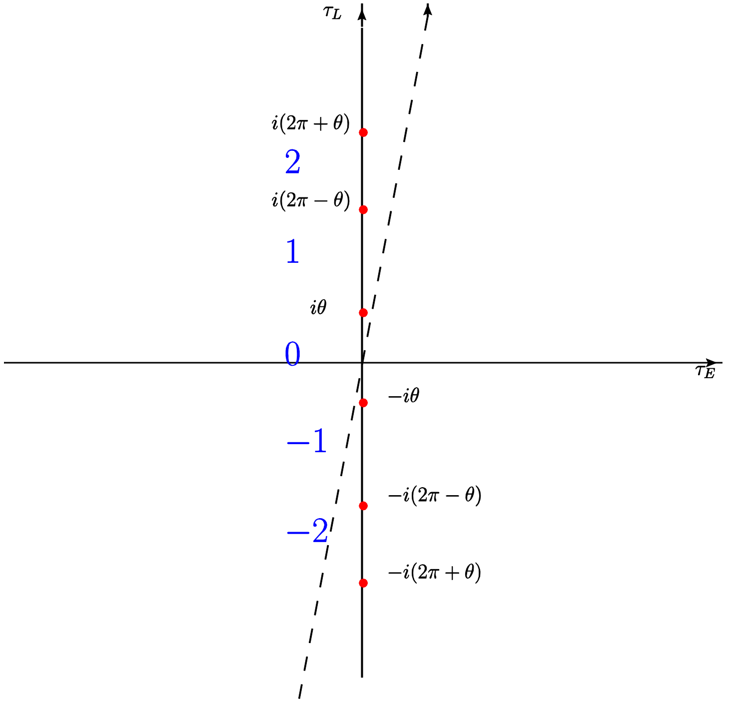
The on-shell process for the pinch at s= $4m^2$ is jus the creation of two particles at rest and their subsequent decay.

Propagator on the cylinder

Let us consider now a particle propagation on the cylinder parameterized by (τ, n), where τ stands for the global time and n is the unit vector on the sphere.

$$D_E = \frac{1}{(\cosh \tau_E - \cos\theta)^{\Delta}}$$

$$\tau_L = e^{\frac{i\pi}{2}} \tau_E$$



Landau equation for weakly coupled CFTs

On shell: $\cos(\tau - \tau_*) = n.n_*$

Equivalently:

$$E^2 = \frac{1}{2} J_{ij} J^{ij}$$

Energy and angular momentum are constant along geodesic.

$$J^{ij} = E \frac{n^{[i} n_*^{j]}}{\sin(\tau - \tau_*)}$$

The pinch condition implies that at every interaction, vertex energy and angular momentum are conserved

$$\sum_{ingoing} E_k = \sum_{outgoing} E_k$$

$$\sum_{ingoing} J_k^{ij} = \sum_{outgoing} J_k^{ij}$$

Landau equation for weakly coupled CFTs

Solution of Landau equations =

Physical process that involves energy-(angular momentum) preserving scattering of on-shell particles moving forward in time on the cylinder.

Thus, if we have weak coupling expansion and we are able to draw some nontrivial Landau graph for a given configuration of external points we should expect a singularity of the correlator there.

Landau equations for CFTs

The free particle propagator describes propagation of a basic excitation on top of a trivial vacuum. In a generic strongly coupled CFT both the vacuum and the excitations on top of it can behave very differently producing a priori very different structure of singularities in correlation functions.

It is easy to see that it is the case for CFTs with AdS dual. In this case the vacuum takes the form of free AdS space and elementary excitations on top of it are described by particles in AdS.

The strong coupling perturbative expansion is described by Witten diagrams and the condition for the pinch again corresponds to some physical processes this time, however, involving particles in AdS.

AdS/CFT and locality

Basic AdS/CFT fact: Landau equations at weak and at strong coupling are different.

Consequence: Singularities of local correlation functions are different.

Their appearance is related to locality in the emergent radial direction and they serve as some sort of order parameter for this emergence.

AdS locality is encoded in analytic properties of correlation functions.

Contact interaction. Kinematics

To get some insight into this phenomenon let us consider a correction to the n-point correlation function from the contact interaction φ^n in d-dimensional CFT.

Let us find the configuration of external points which are light-like separated from the interaction point.

$$
\begin{aligned}
X_i.Y &= 0, \quad i = 1, .., n \\
Y^2 &= -R^2_{AdS} \qquad \text{- bulk} \\
Y^2 &= 0 \qquad \text{- boundary}
\end{aligned}
$$

Contact interaction. Kinematics

Result: for (d+2)-point function in d-dimensional CFT the interaction point exists in the bulk but not on the boundary.

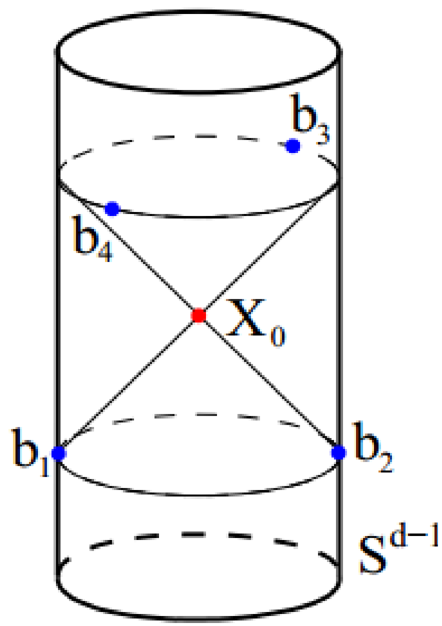
Points 1 and 2 are at time $-\frac{\pi}{2}$.

Points 3 and 4 are at time $\frac{\pi}{2}$.

A point that is null-separated from all of them exists only in the bulk.

Similar configuration can be drawn in higher dimensions.

For fewer number of points the solution of Landau equations exists both on the boundary and in the bulk.



AdS_3/CFT_2

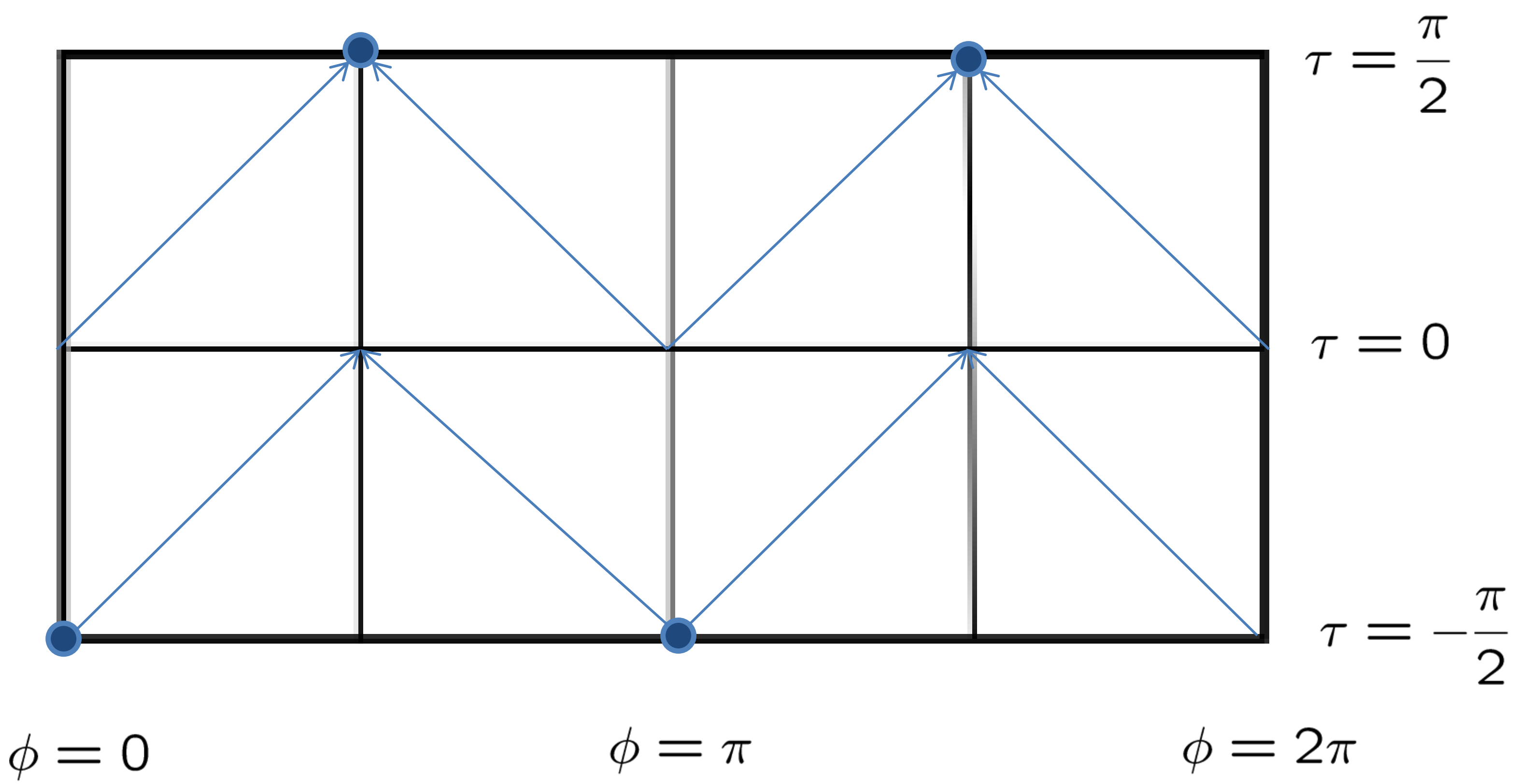
Polchinski, Susskind, Toumbas '99

Gary, Giddings, Penedones '09

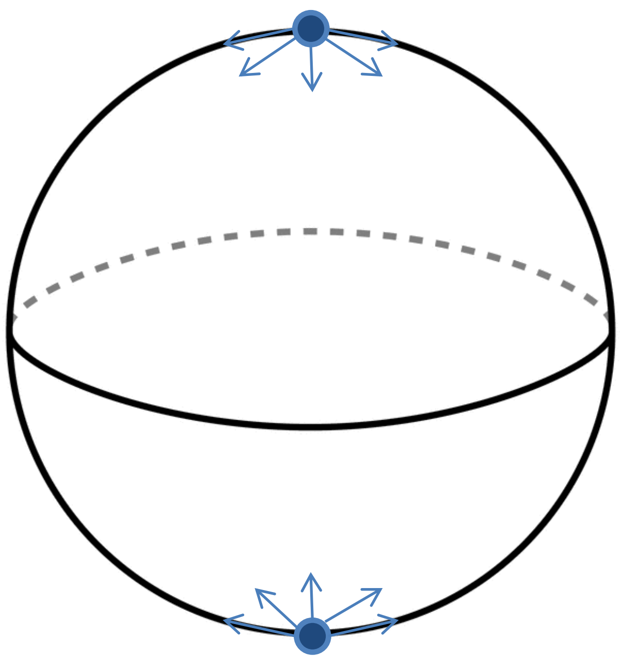
Heemskerk, Penedones, Polchinski, Sully '09

No Landau graph in 2d

We consider configuration when two points are at $\tau = -\frac{\pi}{2}$ and $\varphi = 0, \pi$ and two points are at $\tau = \frac{\pi}{2}$ and $\varphi = \alpha, \alpha + \pi$.

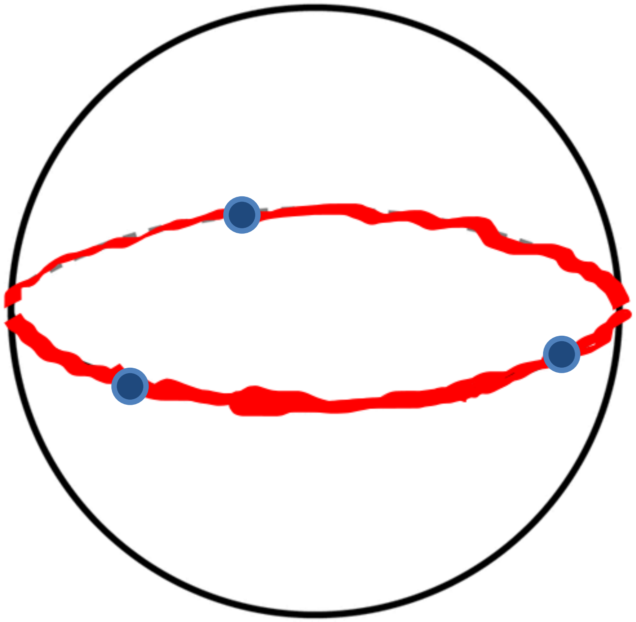


No Landau graph in 3d



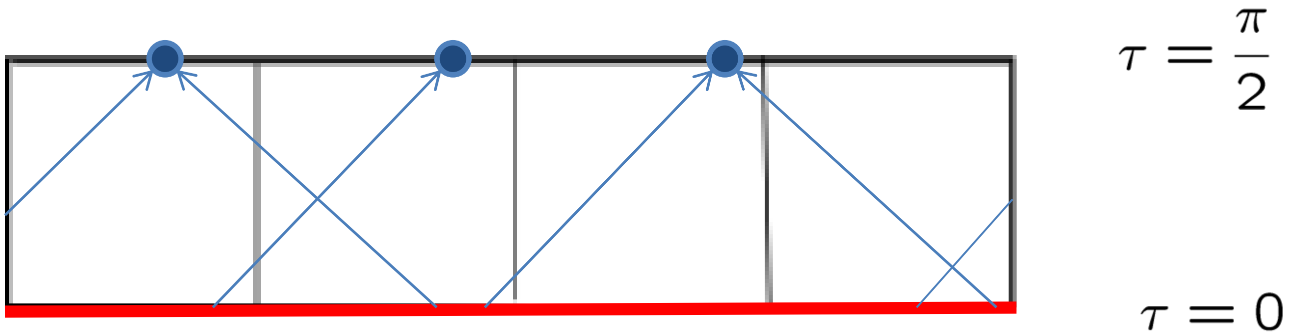
We consider configuration when two points are at $\tau = -\frac{\pi}{2}$ and three points are at $\tau = \frac{\pi}{2}$.

No Landau graph in 3d



We put three points in the future at the equator. Energy-momentum conservation then confines the dynamics to the equator.

No Landau graph in 3d

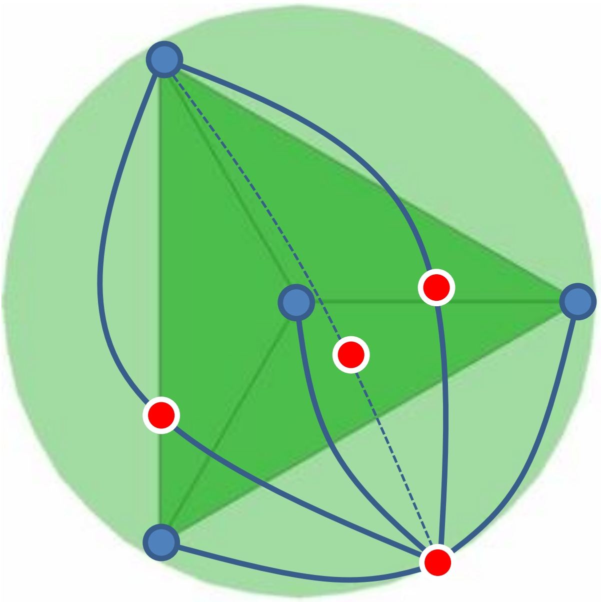


Comment on 4d

In this case the problem is reduced to the dynamics on the equatorial S^2 (in general S^{d-2}).

One can show easily that there are families of configurations for which Landau equations are satisfied.

The real question though is:
If there exists configuration of points for which the solution does not exist…



Bulk singularity

Thus, we see that there is a singularity that naturally appears in the bulk that <u>cannot appear to any order in perturbation theory</u>.

Moreover, assuming the convergence of the planar expansion we see that the singularity is absent at finite coupling and appears only in the limit of an infinite coupling.

A scenario for singularity emergence

The singularity is related to the UV structure of the bulk theory.

In string theory we expect the short distance structure to be non-singular.

HS symmetry gets restored and removes the singularity (it is also absent in free theories and Vasiliev theories).

At least, in the case of AdS this singularity is a way to define a point of “event” in the bulk.

A scenario for singularity emergence

The singularity is related to the point-like scattering of particles in the small portion of AdS. In this region the effects of curvature are negligible and the relevant process is the high-energy fixed angle scattering in flat space. This amplitude is very different in gravity and string theory at high energies.

$$x = z - \bar{z} \qquad I(x, \alpha') = \int_0^\infty dp \, e^{-ixp} e^{-\alpha' p^2}$$

If we sent first the string length to zero then we recover the singularity at

$$x = 0$$

Wave function that comes from insertion of local operators

Flat space scattering amplitude.

A scenario for singularity emergence

This picture emerges from many works on flat space limit of AdS/CFT.

The relevant data in terms of the OPE is the correction to the anomalous dimension of double trace operators.

Heemskerk, Penedones, Polchinski, Sully '09

Large energy anomalous dimensions of double trace operators are known to be related to the flat space scattering amplitudes.

Fitzpatrick, Katz, Poland, Simmons-Duffin '10

Singularity both in the bulk and the boundary

From the analysis of Landau equations it is also clear that the singularity should still be there for the four-point function in d>2.

It follows from the fact that the solution of Landau equations have support on non-compact H_{d-2} that touches the boundary and thus we expect the singularity to be present both at weak and strong coupling.

In some simple examples (like 4-point function of half-BPS operators in N=4) it is indeed the case.

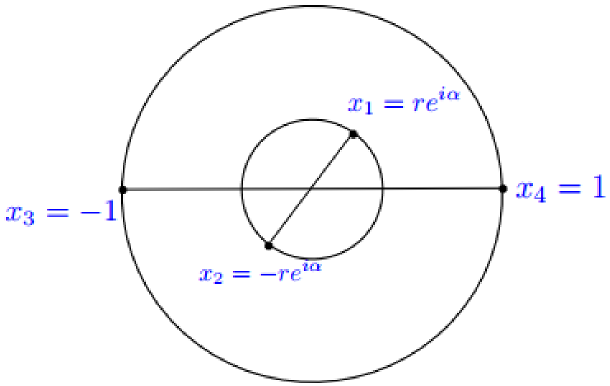
From the OPE point of view this is related to the universal singularity of conformal blocks.

An OPE-bound

One can also easily prove a nonperturbative bound on the maximal power of singularity of the four point function.

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle \leq \frac{1}{(z-\bar{z})^{4\Delta}} + \ldots$$

It follows from relating the singularity to the one due to the exchange of the unit operator.



Pappadopulo, Rychkov, Espina, Rattazzi '12

$$r \to e^{-i\pi}$$

Non-perturbative corrections

What about non-perturbative corrections to correlation functions?

In this case perturbations propagate on top of the instanton background. This fact potentially changes Landau equations and, thus, the singularities structure.

As a simple example consider the one-instanton correction to the four-point function of half-BPS operators in SU(2) N=4 SYM.

Bianchi, Brandhuber, Travaglini, Wen '13

Non-perturbative corrections

This is the weak coupling computation but the result is the same as the correction from the φ^4 in AdS!

The AdS coordinates are just the instanton moduli and the bulk-to-boundary propagators come out of the propators in the instanton background.

Dorey, Hollowood, Khoze, Mattis '02

At strong coupling the same type of corrections come from the D-instanton correction to the closed string amplitudes.

M. Green at al.

Non-perturbative corrections

Lesson: singularities that we showed cannot appear perturbatively seem to be potentially present non-perturbatively.

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An identical problem is known to be present for the instanton corrections to the scattering of W-bosons which was studied in the context of generating baryon asymmetry.

Rubakov, Shaposhnikov '96

Non-perturbative corrections

Curiously, the avatar of unitarity violation in the context of scattering amplitudes in the context of CFT correlation functions is that in the simplest example considered they can be shown to violate the OPE bound.

Of course, in both cases it is expected that corrections resolve the issue.

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  Perturbative vs non-perturbative singularities...