Entanglement hydrodynamics
and comments on tensor networks

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Geometry from the Quantum, KITP, 01/13/2020
Phenomena associated with chaotic dynamics:

- Transport
- Thermalization
- Butterfly effect
Quantum chaotic dynamics

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- Thermalization
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**Ultimate goal:** understand these phenomena and their relation in quantum systems, relate them to gravity through AdS/CFT

**Goal of talk:**
- Develop an effective theory of entanglement dynamics in the hydrodynamic limit
- Study its interplay with other chaotic phenomena
- Comment on relations to tensor network approaches to AdS/CFT
Quantum chaotic dynamics

Setup:
- Degrees of freedom interacting strongly through local chaotic Hamiltonian.
- In highly excited state, out of equilibrium at $t = 0$, in equilibrium for $t \to \infty$.
- Foundational question in statistical physics. Subject of intense current activity in HEP, CMT, QI, and AMO experiments.
Quantum chaotic dynamics

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Study the setup using holographic duality:
• A QFT settling to thermal equilibrium is dual to a collapsing black hole.
• No small parameters, holography is indispensable in understanding real time quantum dynamics.
• Entanglement plays a crucial role in thermalization.
We have an effective theory for describing conserved densities.

- Hydrodynamics applies universally for all chaotic systems. Generalized hydrodynamics for integrable systems.

- Navier-Stokes equations: \( \partial_t v + (v \cdot \nabla) v - \nu \nabla^2 v = -\nabla p \)
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- Navier-Stokes equations: \( \partial_t v + (v \cdot \nabla)v - \nu \nabla^2 v = -\nabla p \)

- Relativistic hydro from hep-th POV is an EFT based on systematic long distance, late time expansion. Fluid variables:
  \[ T_{ab} = (\rho + p)u_a u_b + p \eta_{ab} + \Pi_{ab} \]

- Hydrodynamics follows from the conservation of \( T_{ab} \). Solution determines \( \langle T_{ab} \rangle \) out of equilibrium.
Hydrodynamics

We have an effective theory for describing conserved densities.

- Fluid/gravity constructs black holes with bumpy horizons from fluid flows. [Bhattacharyya et al.]

\[
\frac{ds^2}{z^2} = \frac{1}{z^2} \left[ 2u_a(x)dx^a\,dz + \eta_{ab} + \left( 1 - a \left( \frac{dz}{4\pi T(x)} \right) \right) u_a(x)u_b(x) \right] dx^a\,dx^b
\]

+ (gradients)

- Alternative history: String theorists discover hydrodynamics by studying AdS black holes.
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- Alternative history: String theorists discover hydrodynamics by studying AdS black holes.

- Interested in more data than $\langle T_{ab} \rangle$: entanglement entropy, butterfly effect, etc.

- I want to follow the “alternative history” path to discover a hydrodynamic effective theory of entanglement dynamics (and operator growth).

- Hydrodynamics is universal, there is evidence for the universality of the effective theory of entanglement hydrodynamics.
Outline

**Transport**
- Hydro as an EFT
- Holography for real time dynamics

**Thermalization**
- Entanglement entropy as a probe
- Membrane theory is the EFT
- Interplay with hydro

**Comments on tensor networks**
- Membrane theory from random circuits
- Interplay with operator growth

**Conclusions and open questions**
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Conclusions and open questions
Quantum thermalization and subsystems

Quantum thermalization

- Pure state with nonzero energy density: $|\psi(0)\rangle$
  Unitary time evolution: $|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$

- $\rho(t) \equiv |\psi(t)\rangle \langle \psi(t) | \not\Rightarrow \frac{e^{-\beta H}}{Z}$ cannot mean thermalization.

$\rho(t)$ encodes all the information in $|\psi(0)\rangle$, but at late times in a very nonlocal way.
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- Consider subsystems, reduced density matrix:
  $\rho_A = \text{Tr}_{\bar{A}} |\psi\rangle \langle \psi|$
- Thermalization: $\rho_A(t) \to \rho_A^{(eq)}(\beta) = \text{Tr}_{\bar{A}} \frac{e^{-\beta H}}{Z}$

For $t \to \infty$, in the thermodynamic limit $\bar{A} \to \infty$, with $\beta$ determined by the energy density. **Entanglement is crucial in making this possible.**
Entanglement entropy is a good diagnostic of thermalization, we focus on this quantity.

- In ground states of local Hamiltonians the entropy scales with the area:
  \[ S_A = \# \frac{\text{area}(\Sigma)}{\delta^{d-2}} + \ldots \]

- A generic state in the Hilbert space shows volume scaling:
  \[ S_A = s_{\text{th}} \text{vol}(A) + \ldots \]
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• Purest setup is a quench: start with ground state of a local Hamiltonian, change the Hamiltonian suddenly, and let the system evolve. (No transport.)
Qualitative picture of entanglement entropy at time $t$ of a region of characteristic size $R$,
$R, t \gg t_{\text{loc}}$ [Cardy, Calabrese; Hartman, Maldacena; Liu, Suh]

$S_A(t)$ reaches a thermal value at $t_{\text{loc}} \ll R$

One-point functions reach thermal value at $t_{\text{loc}} \sim \beta$
Qualitative picture of entanglement entropy at time $t$ of a region of characteristic size $R$, $R, t \gg t_{\text{loc}}$  
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Entrophy in the hydrodynamic limit

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Saturation takes $t_{s} \sim R$ similarly to $\langle \phi(R) \phi(0) \rangle$
Membrane theory of entanglement dynamics

Can reformulate holographic surface extremization in d+1 dimensions as membrane minimization in d dimensions in the limit $R, t \gg t_{\text{loc}}$. [MM$_2$]

- Detailed understanding of HRT surfaces. The surface has three parts: [MM$_1$]
  1. Outside the horizon part gives (divergent) area law.
  2. Behind the horizon region.
  3. Behind the shell part gives entropy in the vacuum.
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  1. Outside the horizon part gives (divergent) area law.
  2. **Behind the horizon region.**
  3. Behind the shell part gives entropy in the vacuum.
- Only the **2. part** contributes to the extensive part of the entropy.

$$S(t) = s_{\text{th}}R^{d-1}S_{\text{ext}}\left(\frac{t}{R}\right) + \ldots$$
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- Push HRT surface to the boundary along constant infalling time.
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Horizon $\sim$ boundary
Shell
Projection
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- Scaling limit: $x^\mu \rightarrow R x^\mu, \quad z \rightarrow z$

Area functional independent of the derivatives of $z$. Solve algebraic EoM, plug back into action to derive membrane theory.

\[ S[A] = s_{th} \int d^{d-1} \xi \sqrt{\gamma} \frac{\mathcal{E}(v)}{\sqrt{1 - v^2}} \]

**Horizon \sim \text{boundary}**

**Projection**

**EoM:** $z = f(v^2)$
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- Membrane theory:

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Membrane is projection of HRT to boundary. $\mathcal{E}(v)$ is repackaging of geometry, independent of quench details.

[Horizon ~ boundary]

Shell

$$v = \frac{(n \cdot \hat{t})}{\sqrt{1 + (n \cdot \hat{t})^2}}$$
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- Using the NEC, can prove the following properties of $\mathcal{E}(v)$.
  $\mathcal{E}(v)$ can be thought of as a transport coefficient.

\[ v = \frac{(n \cdot \hat{t})}{\sqrt{1 + (n \cdot \hat{t})^2}} \]
EE for strip, sphere, cylinder regions in the hydro limit is analytically solvable. \([\text{MM}_1; \text{MM}_2]\)

- Strip: $S_A(t)$

\[ t_S = \frac{R}{v_E} \]
EE for strip, sphere, cylinder regions in the hydro limit is analytically solvable. [MM₁; MM₂]

- Strip:

- Sphere:
EE for strip, sphere, cylinder regions in the hydro limit is analytically solvable. [MM₁; MM₂]

- Strip:

- Sphere:

- Stadium shape: [MM, van der Schee]
Applications

EE for strip, sphere, cylinder regions in the hydro limit is analytically solvable. \([\text{MM}_1; \text{MM}_2]\)

- Strip:

- Sphere:

- Stadium shape: \([\text{MM, van der Schee}]\)

- Simple bound on saturation time from operator growth: \([\text{MM, Stanford}]\) \(t_S \geq \frac{R}{v_B}\)

For elongated shapes in 4D we find: \(t_S = \frac{R}{v_B}\)

Black holes often saturate entanglement entropy the fastest.
The membrane theory is robust, can be generalized away from global quenches. [MM, Virrueta]

• Fluid/gravity black brane dual to an inhomogenous state in local thermal equilibrium. To subleading order, we get the membrane coupled to hydrodynamics:

\[
S = \int d^{d-1} \xi \sqrt{\gamma} s_{th}(x) \frac{\mathcal{E}(v)}{\sqrt{1 - v^2}} + \ldots, \quad v(x) \equiv \frac{(n \cdot u(x))}{\sqrt{1 + (n \cdot u(x))^2}}
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- Adaptable to other inhomogenous setups, can incorporate $\beta/R$ and $1/\lambda$ corrections without change in the structure of the membrane theory. $1/N$ corrections would be most interesting.

- Membrane theory is versatile, has connections to operator growth and hydrodynamics, and has all the features to be a universal theory.
Features of the thermalization:
• Conserved densities described by hydro.
• State of the entire system cannot become thermal. Small subsystem thermalize by becoming entangled with the rest of the system.

\[ S_A(t) \to S_A^{(eq)}(\beta) = s_{th}(\beta) \text{vol}(A) \]
Captures the essence of thermalization.

**Goal:** Find effective theory (akin to hydro) of entanglement dynamics.
• Alternative history method: Discovered membrane theory by studying AdS black holes, has structure applicable to all chaotic theories.
• In the following conduct further tests. Elucidate connections to other manifestations of chaotic dynamics.
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Conclusions and open questions
The same description of entanglement dynamics arises in CMT.

- Random quantum circuit model for the evolving wave function.

\[ |\psi(\Delta t)\rangle \]

\[ |\psi(0)\rangle \]
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- Minimal cut computes the entropy. [Nahum, Ruhman, Vijay, Haah]

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Minimal membrane phenomenology of entropy dynamics. [Jonay, Huse, Nahum]
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- Analytic arguments in Floquet systems. [Nahum, Zhou] Evidence in chaotic spin chains. [Jonay, Huse, Nahum]

- Remarkable unification of CMT and HEP approaches: Membrane description of EE growth in quenches.
The analogy between minimal cuts and the RT surface computing entanglement entropy has inspired toy models of holography.


- Suggestive results for maximal volume slice. [Hartman, Maldacena; Roberts, Stanford, Susskind] But HRT surfaces for different shapes do not lie on same Cauchy slice.
Tensor networks and holography

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- Entanglement of local operator with growing footprint is computed by membrane in time fold geometry. [Roberts, Stanford, Susskind; Jonay, Huse, Nahum; MM, Virrueta]

• Quantitative connection to TNs through EoM, bulk geometry encoded in $E(v)$. 
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I propose that there are two universality classes of entropy dynamics at long distances and late times (in translationally invariant systems).

- 2d integrable models, RCFTs, d>2 free theories are described by the **quasiparticle theory**.
- The holographic results can be reformulated in terms of a **membrane theory**, which then can be adopted to any chaotic system. Applies to holographic theories, random circuits, evidence for chaotic spin chains. [Jonay et al., MM$_2$]
- Is there something in between?
- Analogous to the dichotomy between generalized hydrodynamics applicable to integrable systems (giving ballistic transport) and hydrodynamics (describing diffusive transport).
Entropy in the hydrodynamic limit

- Qualitative picture of entanglement entropy at time $t$ of a region of characteristic size $R$, $R, t \gg t_{loc}$. [Cardy, Calabrese; Hartman, Maldacena; Liu, Suh]

- EE in free scalar theory for a disk, dots are data points, line is quasiparticle theory [Cotler, Hertzberg, MM, Mueller]

- EE in holographic theories for a disk, data collapse, solid line is membrane theory, deviation is controlled by $1/R$ [MM$_1$]
Summary

Phenomena associated with chaotic dynamics:
• Hydrodynamics is the EFT for transport, serves as target
• Universality classes of thermalization:
  **Quasiparticle theory vs Membrane theory**
• Derived the membrane theory of entanglement dynamics from holography. Evidence for universality from CMT
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• Uncovered interplay with hydro, chaos and TNs:
  ➢ Membrane couples to hydrodynamics
  ➢ Key role of $\nu_B$, bounds on entropy, operator EE picture
  ➢ Membrane is a cut through TN, TN is obtained after solving bulk EoMs
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- Rich applications
  - Entropy cone inequalities generalized to time dependent settings. [Hayden, Headrick, Maloney; Bao et al.; Bao, MM]
  - Bit threads reformulation. [Freedman, Headrick; Agon, MM]
- **Membrane theory has all the features to be a universal theory.**
Open questions and some hints

• What does the membrane theory imply for holographic RG?
  
  *Hint:* The metric inside the horizon does not seem to be organized by scale.

Equally important at the longest scales

Organized by RG scale
Open questions and some hints

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  *Hint: The metric inside the horizon does not seems to be organized by scale.*

• Are new quantum extremal surfaces, islands be captured by the membrane theory?
  *Hint: It looks plausible that 1/N corrections can be captured by the membrane theory. It may be that we get multiple minimal membranes for evaporating BH.*
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• Is the membrane theory a good starting point to getting gravitational dynamics out of entanglement?
  *Hint: Slogan: “Gravity is the hydrodynamics of entanglement.” May have to go to shorter times and distances in CFT to see dynamical geometry.*
Calabrese-Cardy model: energy injection from quench creates a finite density of EPR pairs, subsequently travel freely at the speed of light isotropically.

- Leads to linear growth with $v_E = 1$ in 2d.

- Higher dimensions: entanglement spreading depends on entanglement pattern on the light cone $\mu[L_\Sigma]$. Contribution from each light cone has to be added. [Casini, Liu, MM]

\[
\begin{align*}
\text{FIG. 7. Space-time picture illustrating how the entanglement between an interval } A & \text{ and the rest of the system, due to oppositely moving coherent quasiparticles, grows linearly and then saturates. The case where the particles move only along the light cones is shown here for clarity.} \\
\text{Now consider these quasiparticles as they reach either } A & \text{ or } B \text{ at time } t. \text{ The field at some point } x' \in A \text{ will be entangled with that at a point } x'' \in B \text{ if a pair of entangled particles emitted from a point } x \text{ arrive simultaneously at } x' \text{ and } x'' \text{ (see Fig. 7).} \\
\text{The entanglement entropy between } x' \text{ and } x'' \text{ is proportional to the length of the interval in } x \text{ for which this can be satisfied. Thus the total entanglement entropy is} \\
S_A(t) & \approx \int_{x' \in A} dx' \int_{x'' \in B} dx'' \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dp'' f(p',p'') \delta(x' - x - v(p')t) \delta(x'' - x - v(p'')t). \\
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Bound on the entanglement speed from SSA:

$$v_E \leq v_E^{(EPR)} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)} < v_E^{(SBH)}$$

Slower than holography.
Quasiparticle model

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Slower than holography.

- In strongly coupled systems, entanglement grows faster than what’s possible for free particles streaming at the speed of light!

- Consider the effect of interactions: tensor network picture emerging from scattering particles is natural. [Hartman, Maldacena; Casini, Liu, MM]
Free field theory and the quasiparticle model

In a free theory for Gaussian states we can use the correlation matrix to compute EE.

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- Correlation matrix determines all correlation functions due to Wick’s theorem:

\[ \chi_I = \begin{pmatrix} \phi_i \\ \pi_i \end{pmatrix}, \quad [\chi_I, \chi_J] = i J_{IJ} \]

\[ \Gamma_{IJ} = \frac{1}{2} \langle \psi | \{ \chi_I, \chi_J \} | \psi \rangle \]
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  \[ \Gamma_{IJ} = \frac{1}{2} \langle \psi | \{ \chi_I, \chi_J \} | \psi \rangle \]
- The symplectic eigenvalues of the correlation matrix give the eigenvalues of the reduced density matrix:
  \[ \tilde{\chi} = S \chi, \quad SJS^T = J, \]
  \[ \tilde{\Gamma} = S \Gamma S^T = \begin{pmatrix} \text{diag}(\gamma_k) & 0 \\ 0 & \text{diag}(\gamma_k) \end{pmatrix} \]
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\[ \tilde{\Gamma} = STS^T = \begin{pmatrix} \text{diag} (\gamma_k) & 0 \\ 0 & \text{diag} (\gamma_k) \end{pmatrix} \]

- Numerical results for 3d boundary state quench for scalar field. [Cotler, Hertzberg, MM, Mueller]
Entanglement entropy in static holographic states obeys inequalities, that are not true in general in QM.

- The best known one is the monogamy of mutual information. [Hayden, Headrick, Maloney] It can be proven using the same steps as in the proof of SSA.

\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]
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\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]

• The inclusion-exclusion proof method can be used to derive many-party inequalities. [Bao et al.] Holography is not essential, only need that the entropy is proportional to a partionable geometric minimization problem.

• HRT is an extremization of codimension-2 surface, no proof (or counterexample) is known for many-party inequalities. Inclusion-exclusion applies to the membrane theory, hence proof for time dependent states (large regions, late times). [Bao, MM]
The Ryu-Takayanagi prescription can be reformulated in the language of bit threads. [Freedman, Headrick]

- Maximize $\int_A \sqrt{\hbar} n_\mu w^\mu$

  Constraints: $\nabla_\mu w^\mu = 0$, $1 - |w^\mu| \geq 0$

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\( \nabla_\mu w^\mu = 0 \), \( H(w_t) - |\bar{w}| \geq 0 \)

\( H(w_t) \) is the Legendre transform of \( \mathcal{E}(v) \):

\( H(w_t) \equiv \mathcal{E}(v) - v \mathcal{E}'(v), \quad w_t \equiv -\mathcal{E}'(v) \)
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- The map that reconstructs the HRT surface from the minimal membrane can be used to push the membrane theory bit thread into the bulk.

- Membrane theory is versatile, has connections to operator growth and hydrodynamics, and has all the features to be a universal theory.
Entanglement entropy obeys inequalities, natural to consider bounds in the quench setup.

• \( v_E \leq 1 \) can be proven using Lorentz invariance and the SSA inequality, [Casini, Liu, MM] or the monotonicity of relative entropy. [Afkhami-Jeddi, Hartman]
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- Monotonicity of (thermal) relative entropy for subsystems combined with emergent $v_B$ light cones at finite temperature in chaotic systems:
  \[ S[A(t)] \leq S[A'(t')] + s_{th} (V[A(t)] - V[A'(t')]) \]
  Gives bound for all times. Can be combined with another proposed inequality. [MM, Stanford]
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- Membrane theory proof: there exists a maximal membrane tension compatible with the general properties discussed before.

$$\mathcal{E}_{\text{max}}(v) = v_E + \left( 1 - \frac{v_E}{v_B} \right) |v| \quad (|v| \leq v_B)$$

The resulting minimal membrane is a combination of a cylinder and the cone saturating the combined inequalities. MM$_2$