Entropic constraints on causal structures

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joint with Mirjam Weilenmann

arXiv:1709.08988 (review) and arXiv:1605.02078

Why are we interested in causal structures?

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- Attempt to explain how correlations come about

Observe $P_{ABCD\ldots}$ → Why do we get these correlations?

What caused these things to be correlated?

What constitutes explanation?

What constitutes explanation?

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- Reichenbach's principle:

Observe two correlated things, i.e. $P_{AB} \neq P_A P_B$

$A \rightarrow B \qquad B \rightarrow A \qquad A \leftarrow \Lambda \rightarrow B$

What constitutes explanation?

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- Reichenbach's principle:

Observe two correlated things, i.e. $P_{AB} \neq P_A P_B$

$A \rightarrow B$ | $B \rightarrow A$ | $A \leftarrow \Lambda \rightarrow B$

$P_{AB} = P_A P_{B|A}$ | $P_{AB} = P_B P_{A|B}$ | $P_{AB\Lambda} = P_\Lambda P_{A|\Lambda} P_{B|\Lambda}$

What constitutes explanation?

- Reichenbach's principle:

Observe two correlated things, i.e. $P_{AB} \neq P_A P_B$

$A \to B$	$B \to A$	$A \leftarrow \Lambda \to B$
$P_{AB} = P_A P_{B\|A}$	$P_{AB} = P_B P_{A\|B}$	$P_{AB\Lambda} = P_\Lambda P_{A\|\Lambda} P_{B\|\Lambda \not{A}}$
trivial	trivial	non-trivial (unless Λ unseen)

What constitutes explanation?

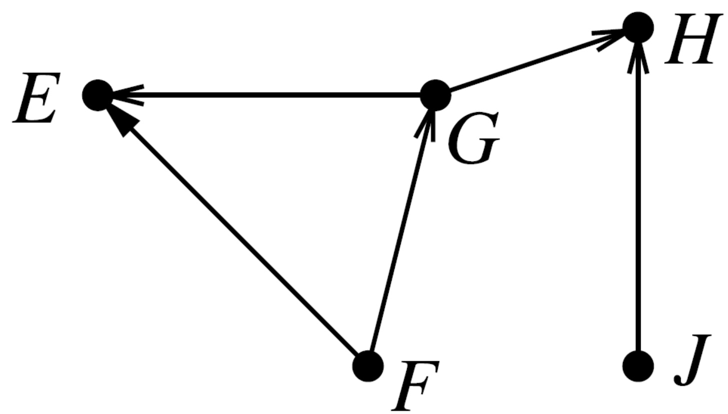
- Reichenbach's principle:

Observe two correlated things, i.e. $P_{AB} \neq P_A P_B$

		Λ unseen
$A \rightarrow B$	$B \rightarrow A$	$A \leftarrow \Lambda \rightarrow B$
$P_{AB} = P_A P_{B\|A}$	$P_{AB} = P_B P_{A\|B}$	$P_{AB} = \sum_{\Lambda} P_{\Lambda} P_{A\|\Lambda} P_{B\|\Lambda}$
trivial	trivial	trivial

Causal structure

- Directed Acyclic Graph



- Encodes: each variable is conditionally independent of its non-descendants given parents e.g. $P_{G|FJ} = P_{G|F}$.
- Here: $P_{EFGHJ} = P_F P_J P_{G|F} P_{E|GF} P_{H|GJ}$

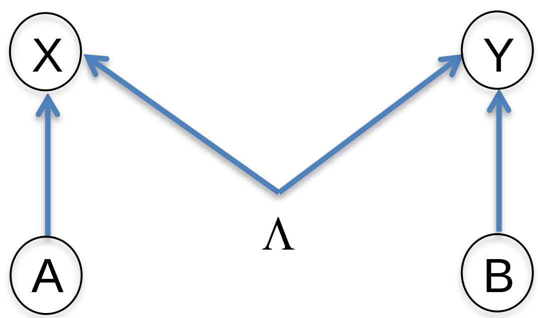
Example: Bipartite Bell scenario

- Two space-like separated measurements

- Observe X and Y correlated
- By Reichenbach's principle, something missing in the causal structure

Example: Bipartite Bell scenario

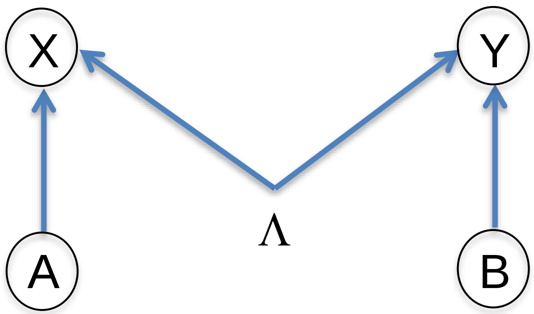
- Two space-like separated measurements



- Observe X and Y correlated
- Hypothesise the existence of additional common cause Λ.

Example: Bipartite Bell scenario

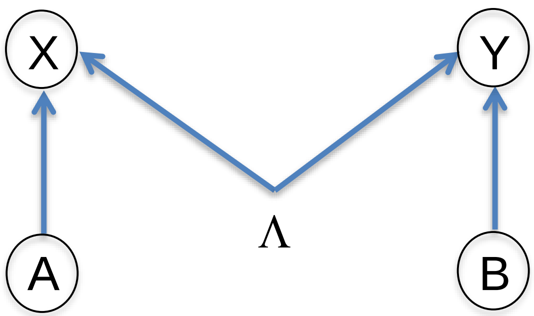
- Two space-like separated measurements



- Observe X and Y correlated
- Hypothesise the existence of Λ.
- This diagram encodes local causality and free choice.

Example: Bipartite Bell scenario

- Two space-like separated measurements



- This diagram encodes local causality and free choice.
- $P_{ABXY} = \sum_{\Lambda} P_{\Lambda} P_A P_B P_{X|A\Lambda} P_{Y|B\Lambda}$

Bell's theorem

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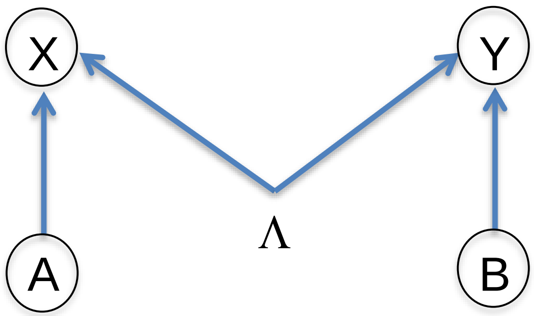
- There exist quantum correlations that are incompatible with this causal structure



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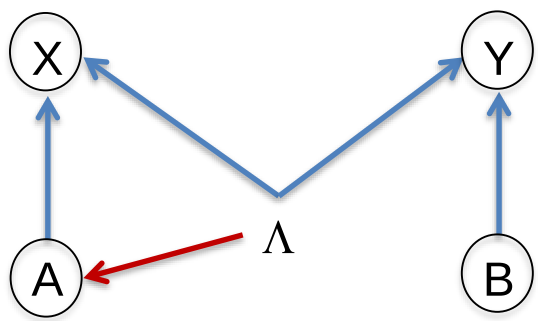
Bell's theorem

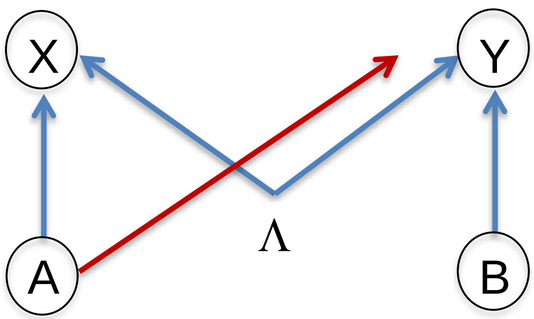
- There exist quantum correlations that are incompatible with this causal structure:



- Options
  - Reject free choice

Bell's theorem

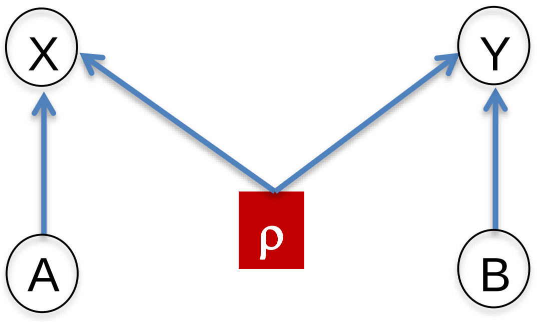
- There exist quantum correlations that are incompatible with this causal structure:



- Options
  - Reject free choice
  - Reject locality

Bell's theorem

- There exist quantum correlations that are incompatible with this causal structure:



- Options
  - Reject free choice
  - Reject locality
  - Extend the notion of cause

Bell's theorem

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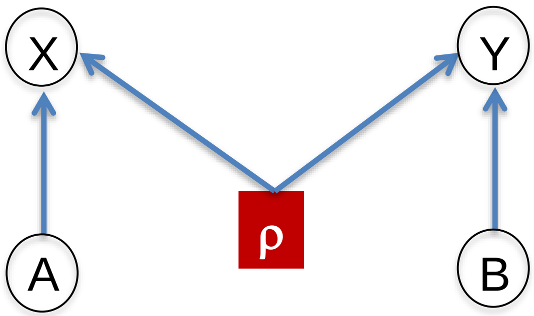

- Options
  - Reject free choice
  - Reject locality
  - Extend the notion of cause

Fine-tuned explanation [Wood Spekkens]

Quantum cause

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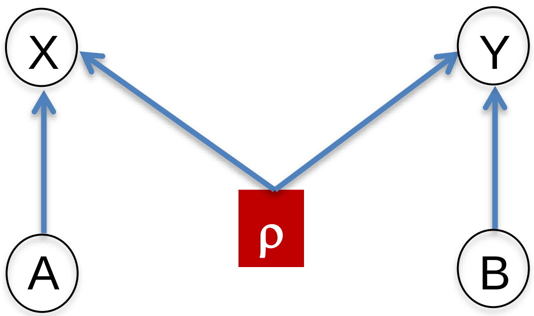

- Think of the “usual” quantum explanation of the correlations as a quantum causal explanation.
- i.e., correlations arise because an entangled state is shared by the source.

POVMs

$$P_{ABXY} = P_A P_B \mathrm{tr}(\rho(E^{a,x} \otimes F^{b,y}))$$

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Post-Quantum cause

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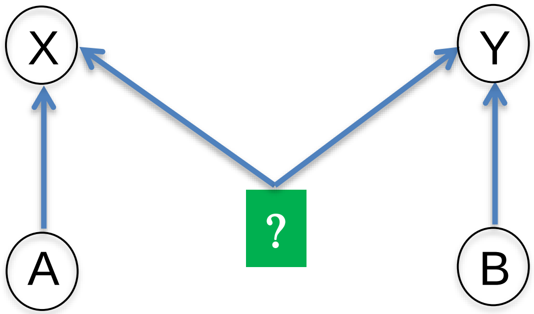

- Correlations arise because a resource is shared by the source (e.g. a no-signalling distribution).

$$P_{ABXY} = P_A P_B R_{XY|AB}$$

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Quantum-classical separation

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- Natural questions:
  - Given some correlations, which causal structures are compatible?
  - Which casual structures have a separation between different theories?
  - What are good ways to detect the separation?
  - In a given theory, how can different causal structures be separated?

Quantum-classical separation

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Application: cryptography

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- Cryptographic protocols involve exchanges of information and hence always take place within a causal structure.

- Finding good ways to detect quantum-classical separations is crucial for device-independent cryptography.

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Detecting the separation

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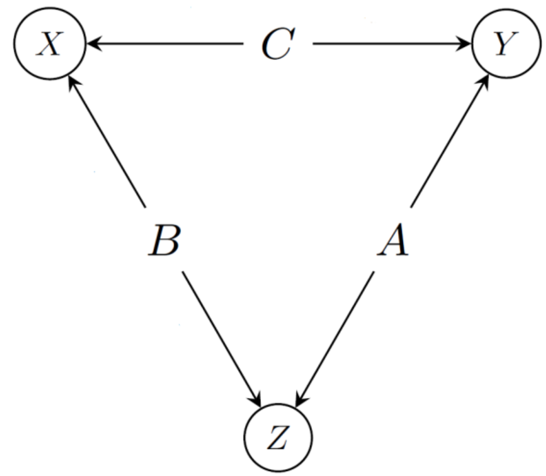
- In the bipartite Bell scenario this is relatively well-understood, at least for small alphabet sizes (note that the number of Bell inequalities grows very rapidly)

$P_{XY|AB} = \sum_{\Lambda} P_{\Lambda} P_{X|A\Lambda} P_{Y|B\Lambda}$ or $\mathrm{tr}(\rho(E^{a,x} \otimes F^{b,y}))$

- Violate Bell inequality → non-classical
- Semi-definite hierarchy → non-quantum

Other causal structures – examples

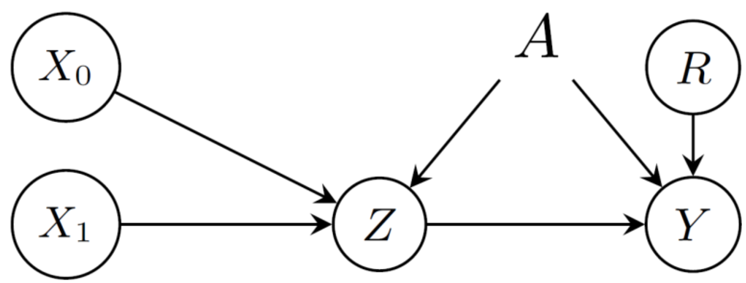
- Triangle



$$P_{XYZ} = \sum_{ABC} P_A P_B P_C P_{X|BC} P_{Y|AC} P_{Z|AB}$$

- Information causality

$$P_{X_0 X_1 ZYR} = \sum_{A} P_A P_{X_0} P_{X_1} P_R P_{Z|AX_0X_1} P_{Y|ARZ}$$



Entropy vectors

- Take the given correlations and construct a vector of all the joint entropies:

$$h(P_{ABC}) \coloneqq \big(H(A), H(B), H(C), H(AB), \dots, H(ABC)\big)$$

- Ask: which entropy vectors are compatible with a causal structure? [Fritz, Chaves, Majenz, Gross, …]

Entropy vectors

$$h(P_{ABC}) \coloneqq \big(H(A), H(B), H(C), H(AB), \dots, H(ABC)\big)$$

- Why might this help?
  - Useful way to distinguish different causal structures
  - Causal constraints, which are non linear for probabilities, become linear
    - E.g. $P_{X|AB\Lambda} = P_{X|A\Lambda}$ becomes $I(X\!:B|A\Lambda) = 0$
  - For many causal structures [in particular all classical ones], the set of achievable entropy vectors is convex.

    i.e. $\{v: \exists P$ valid for the causal structure with $h(P) = v\}$.

Classical entropy vectors

$$h(P_{ABC}) \coloneqq \big(H(A), H(B), H(C), H(AB), \ldots, H(ABC)\big)$$

- Shannon constraints:
  - Strong subadditivity ($H(A|B) \geq H(A|BC)$)
  - Positivity ($H(A) \geq 0$)
  - Monotonicity ($H(A|B) \geq 0$)
- Non-Shannon constraints:
  - Additional relations valid for all entropy vectors that don't follow from the above
  - Not well understood
- Causal constraints

Quantum entropy vectors

$$h(\rho_{ABC}) \coloneqq \big(H(A), H(B), H(C), H(AB), \ldots, H(ABC)\big)$$

- vN constraints:
  - Strong subadditivity ($H(A|B) \geq H(A|BC)$)
  - Positivity ($H(A) \geq 0$)
  - Weak monotonicity ($H(A|B) + H(A|C) \geq 0$)
- Non-vN constraints:
  - Additional relations valid for all quantum entropy vectors that don't follow from the above
  - Conjectured, but none are proven
- Causal constraints

Marginalizing

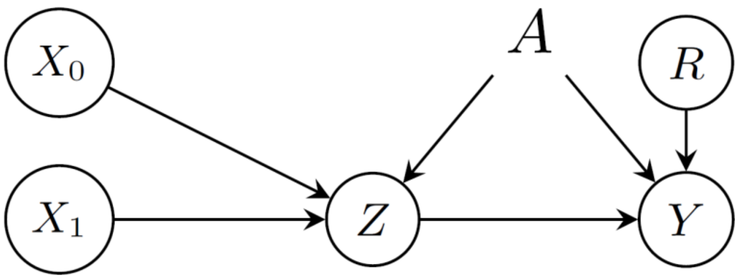
- We apply the constraints to the causal structure with all variables, but want constraints only for the observed (classical) variables.
- These can be derived using Fourier-Motzkin elimination [cf. Chaves et al.]

Overall algorithm

- Input: causal structure
- Output: set of linear entropic constraints that are necessary for this causal structure
- [We also have another technique for finding sufficient conditions.]

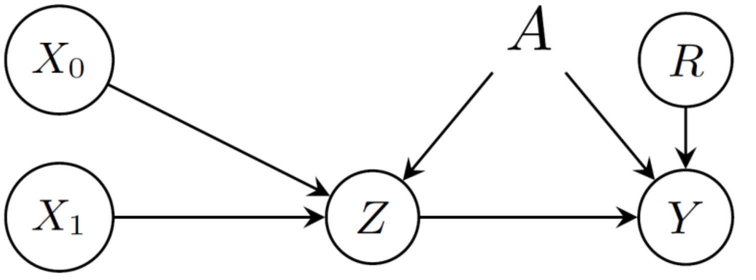
Post-selection

- Sometimes we can consider effective causal structures after post-selection. [BraunsteinCaves]
- Post-select on observed classical nodes.
- Example:

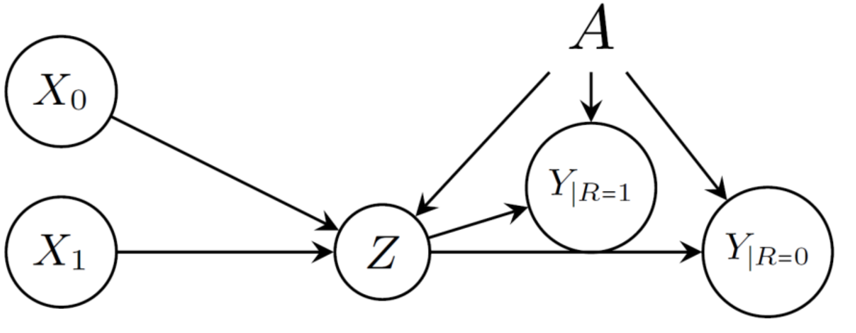


Post-selection – example

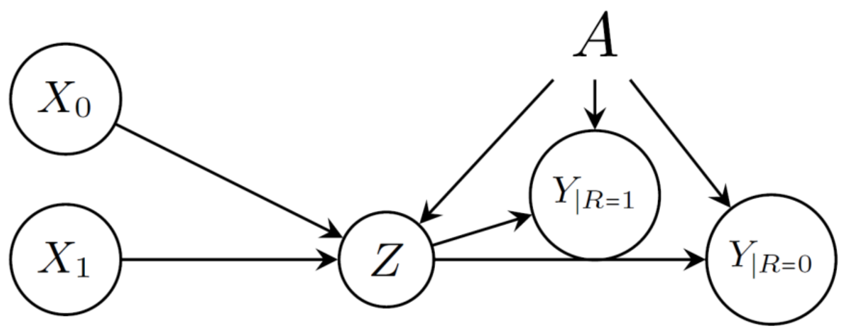
- Information causality



- Post-select on binary R



Post-selection – example



- Theory obeys information causality if

$$I(X_0 : Y_{|R=0}) + I(X_1 : Y_{|R=1}) \leq H(Z)$$

for all pre-shared resources allowed by the theory.

- The fact that this follows for classical and quantum theory follows immediately from the techniques I have discussed (as do lots of other inequalities for this causal structure).

Entropy vectors

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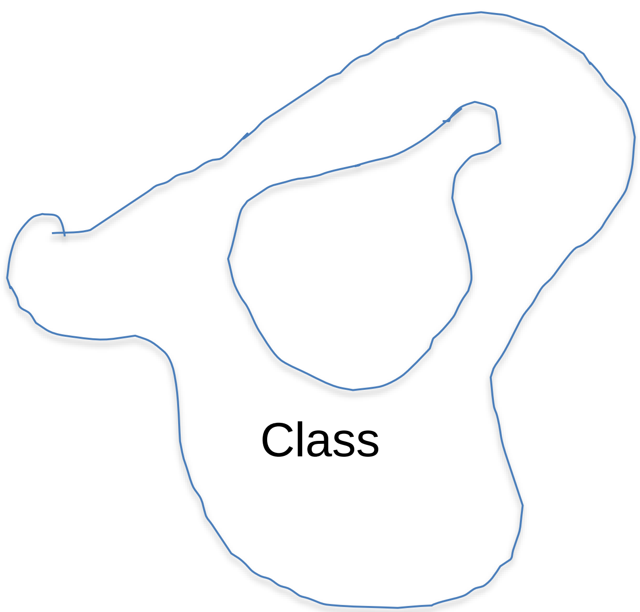
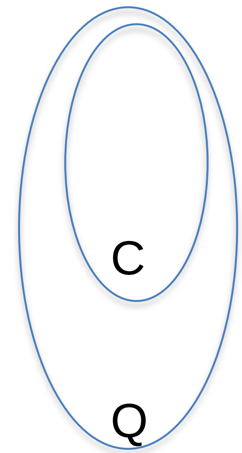
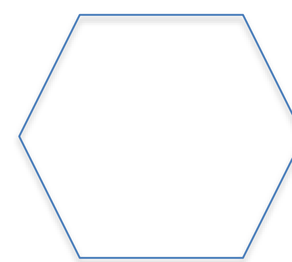


Quantum

Take entropies



or



C=Q

In some cases these are already equal

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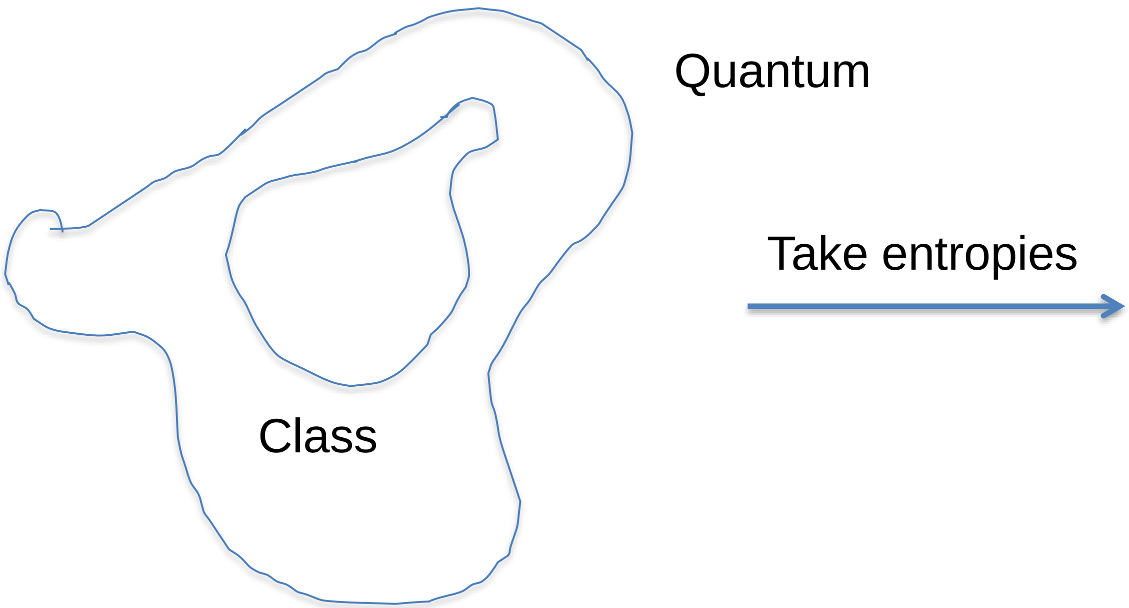
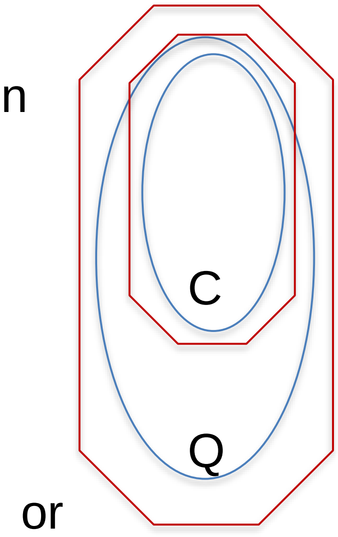
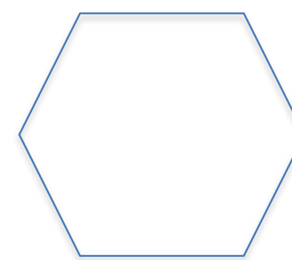
Entropy vectors

Use algorithm to give outer approximation

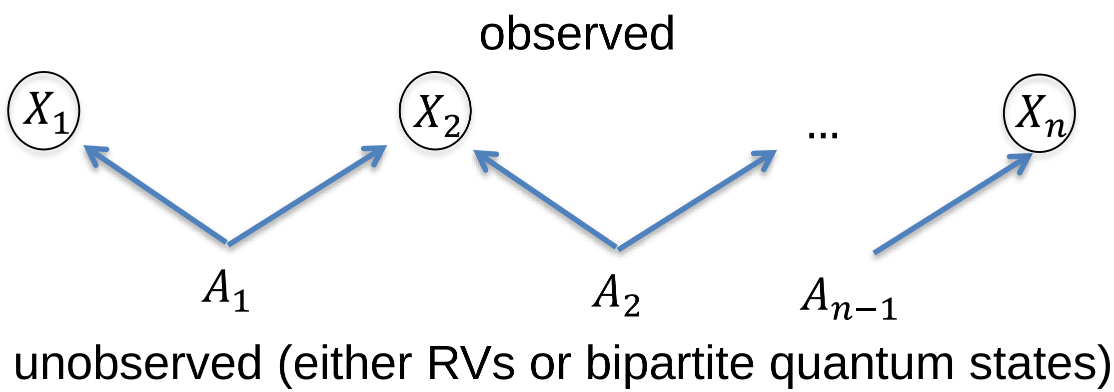






In some cases these are already equal

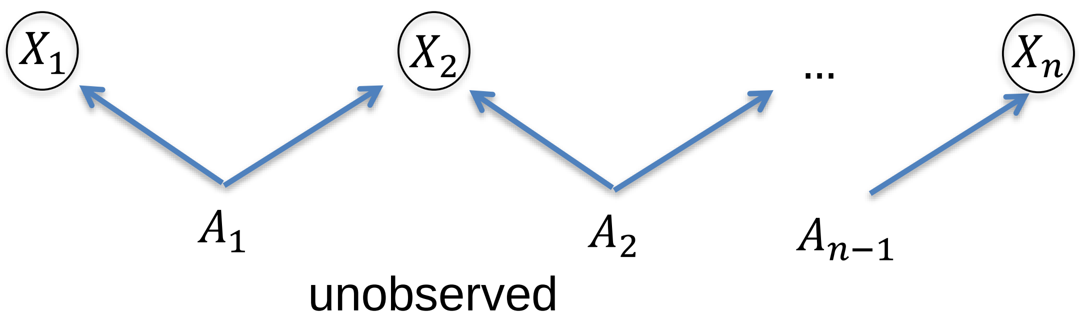
Example: Line-like causal structures (no post-selection)



arXiv:1603.02553

Example: Line-like causal structures (no post-selection)

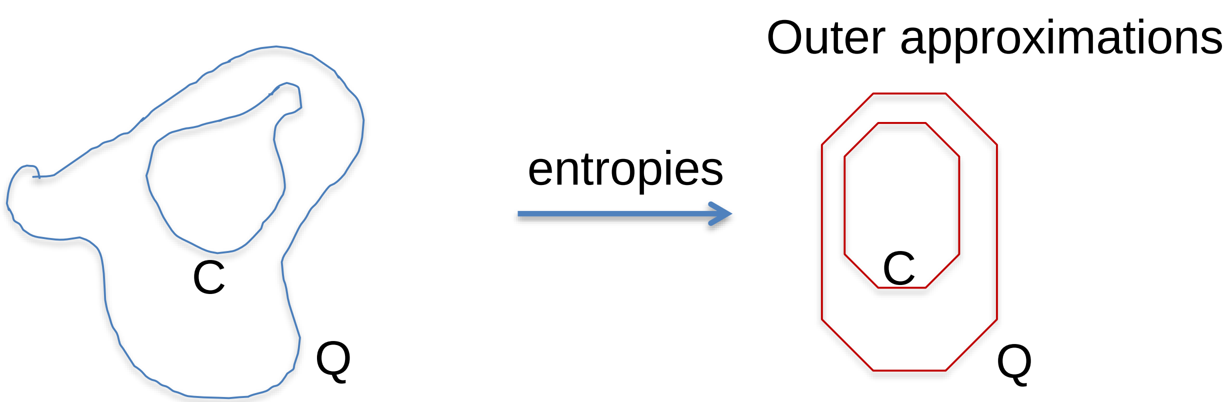
- In other words, for all members of this family (i.e. for all n), any entropy vector that can be obtained using (hidden) quantum states can be obtained classically
- This holds, in spite of the existence of non-local correlations for all $n \geq 4$.



Other cases

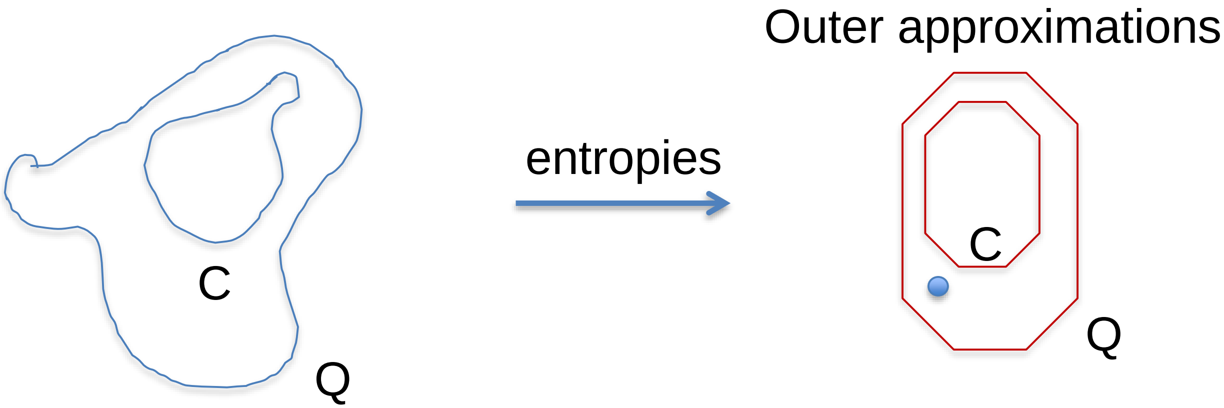
There is separation at level of correlations

- Was this just bad luck?
- We studied other cases taking “interesting” examples from Henson, Lal, Pusey.
- Some cases were as previously (no entropic separation). Others had a separation in outer approximations



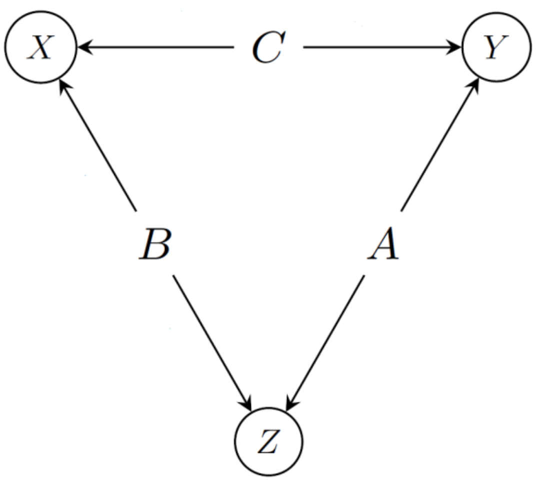
Other cases

- However, we don't know whether this is a real separation: we weren't able to find distributions in the gap.



Concrete example with potential gap

- Triangle causal structure, simplest non-trivial causal structure



- Derived a new tighter outer approximation in the classical case using non-Shannon inequalities
- Known outer approximation in the quantum case is less constrained
- Known non-classical correlations do not lie outside the classical entropic boundary
- Post-selection not possible here
- We also have an inner approximation for this case ($-I(X:Y:Z) \geq 0$ where $I(X:Y:Z) := I(X:Y) - I(X:Y|Z)$)

Summary of entropic techniques

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Case	Entropic C-Q sep.	Sep. in best known approx.	Example
No post-selection	Sometimes no	No	Line-like
	Sometimes unknown	Yes	Triangle
Post-selection	Usually	Yes	Info. causality

- If non-Shannon inequalities are useful, we get a separation in the approximations

Open questions

- Does taking entropy always destroy classical-quantum separation at the level of observed variables (i.e. without post-selection)?
- Are there non-vN inequalities? Do any non-Shannon inequalities fail for vN entropy?
- What other methods can distinguish quantum and classical causal structures?
  - Generic, reasonably tight, simple to compute
  - Note that there are other proposals including
    - Polynomial Bell inequalities [Rosset et al]
    - Techniques via algebraic geometry [Lee & Spekkens]
    - Inflation technique [Wolfe et al]