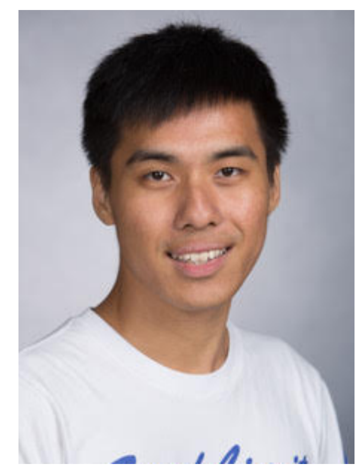
Renyi Entropy of Chaotic Eigenstates

Tarun Grover   Tsung-Cheng (Peter) Lu

(UCSD)



arXiv:1709.08784

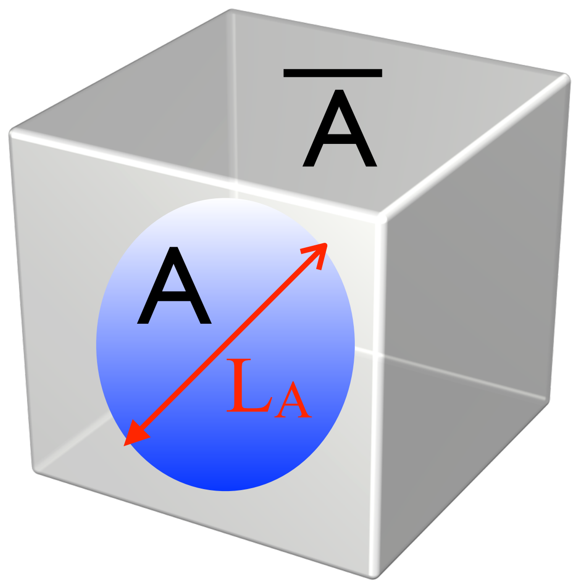
Alfred P. Sloan FOUNDATION

XSEDE
Extreme Science and Engineering
Discovery Environment

Jim Garrison (JQI, UMd)

arXiv:1503.00729

"Laws" of Entanglement Scaling



Ground States and other zero energy density states:

$S_n \sim L_A^{d-1}$ up to log corrections, "Area Law"

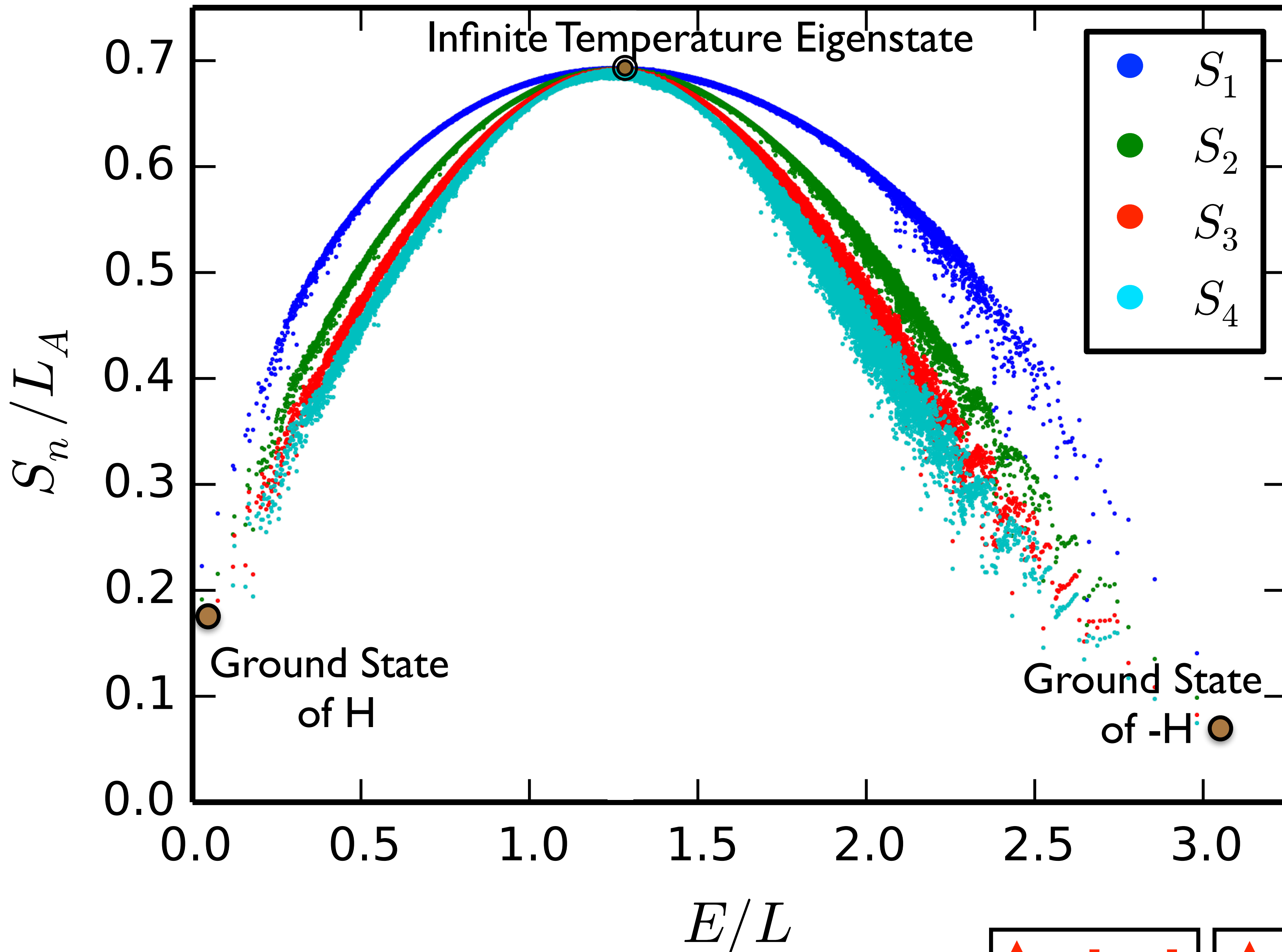
Finite energy density eigenstates: (assuming system does not many-body localize)

$S_n \sim L_A^d$, "Volume Law"

"What would be called a conjecture in computer science, would be declared a "Law" in physics" - Scott Aaronson (KITP 2013)

Renyi Entanglement Entropies of ALL eigenstates of a chaotic, local Hamiltonian

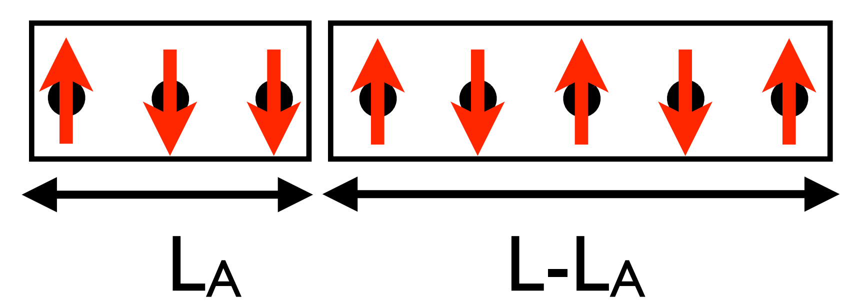
(Steve White's favorite slide)



$$S_n = \frac{1}{1-n}\log\left(\mathrm{tr}\,\rho_A^n\right)$$

(from J. Garrison, TG 2015)

$$H = \sum_i \left(-\sigma_i^z\sigma_{i+1}^z + h_x\sigma_i^x + h_z\sigma_i^z\right)$$



Ground State Entanglement and Universal Data

1+1-d CFT: $S \sim c \log(L_A/\varepsilon)$

2+1-d CFT or Topologically Ordered Phase: $S \sim L_A/\varepsilon - \gamma$

Fermi Surface in d+1 dimensions: $S \sim (k_F L_A)^{d-1} \log(k_F L_A)$

Holzhey, Wilczek, Larsen; Cardy, Calabrese; Casini, Huerta; Ryu, Takayanagi; Kitaev, Preskill; Wen, Levin; Swingle; Gioev, Klich, and many others.

Entanglement of Finite Energy Density States?

At finite energy density, temperature "T" or equivalently energy density "u" provide a new scale.

This allows the "volume law" entanglement to be independent of the ultraviolet cutoff.

For example, for the thermal state of a 1+1-d CFT,

$$\rho_{A,\mathrm{thermal}}(\beta) = \frac{\mathrm{tr}_{\overline{A}}\left(e^{-\beta H_{CFT}}\right)}{\mathrm{tr}\left(e^{-\beta H_{CFT}}\right)} \qquad S_1 = \frac{\pi c L_A}{3\beta} + \text{terms subleading in } L_A$$

"Volume Law" Coefficient (Definition)

$\lim_{V\to\infty} S_n^A / V_A$ while keeping $\underline{V_A/V}$ fixed as $V \to \infty$

<u>Example:</u> $$S_1 = \frac{\pi c L_A}{3\beta} + \text{terms subleading in } L_A$$

Volume law coefficient = $\frac{\pi c}{3\beta}$

This Talk:

Renyi Entropy of Eigenstates of Chaotic Hamiltonians.

What's their volume law coefficient?

What universal information they encode?

Can one construct approximate chaotic eigenstates?

Summary of Main Result

Let $|E\rangle$ be an eigenstate of a chaotic Hamiltonian.

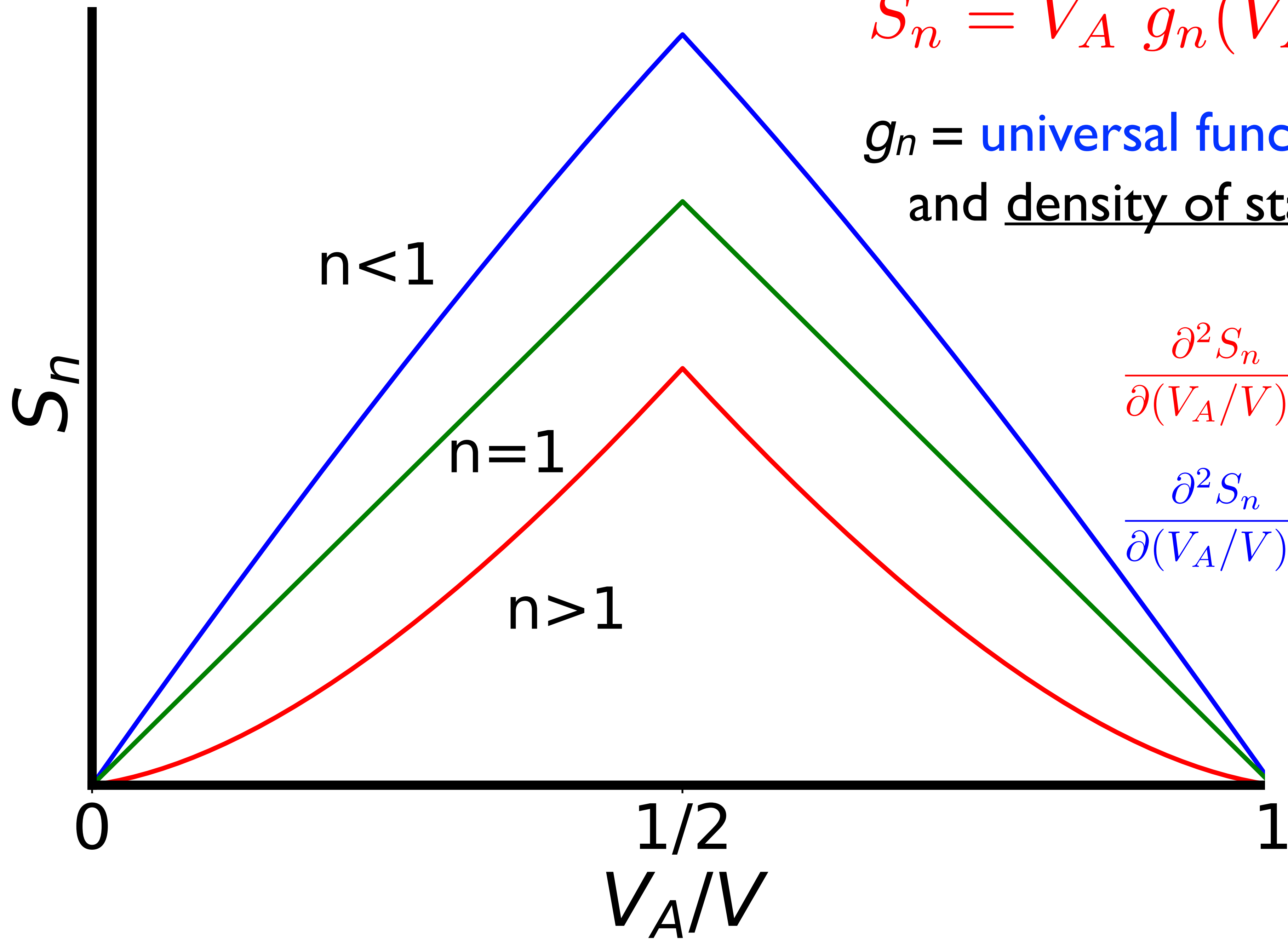
Consider $\rho_A = \mathrm{tr}_{\overline{A}} \, |E\rangle\langle E|$

$$S_n = \frac{1}{1-n} \log \left(\mathrm{tr}\, \rho_A^n \right)$$

Assuming ergodicity (to be be made precise soon), one finds…

$$S_n = V_A \; g_n(V_A/V, E/V)$$

g_n = universal function of V_A/V
and density of states at E/V

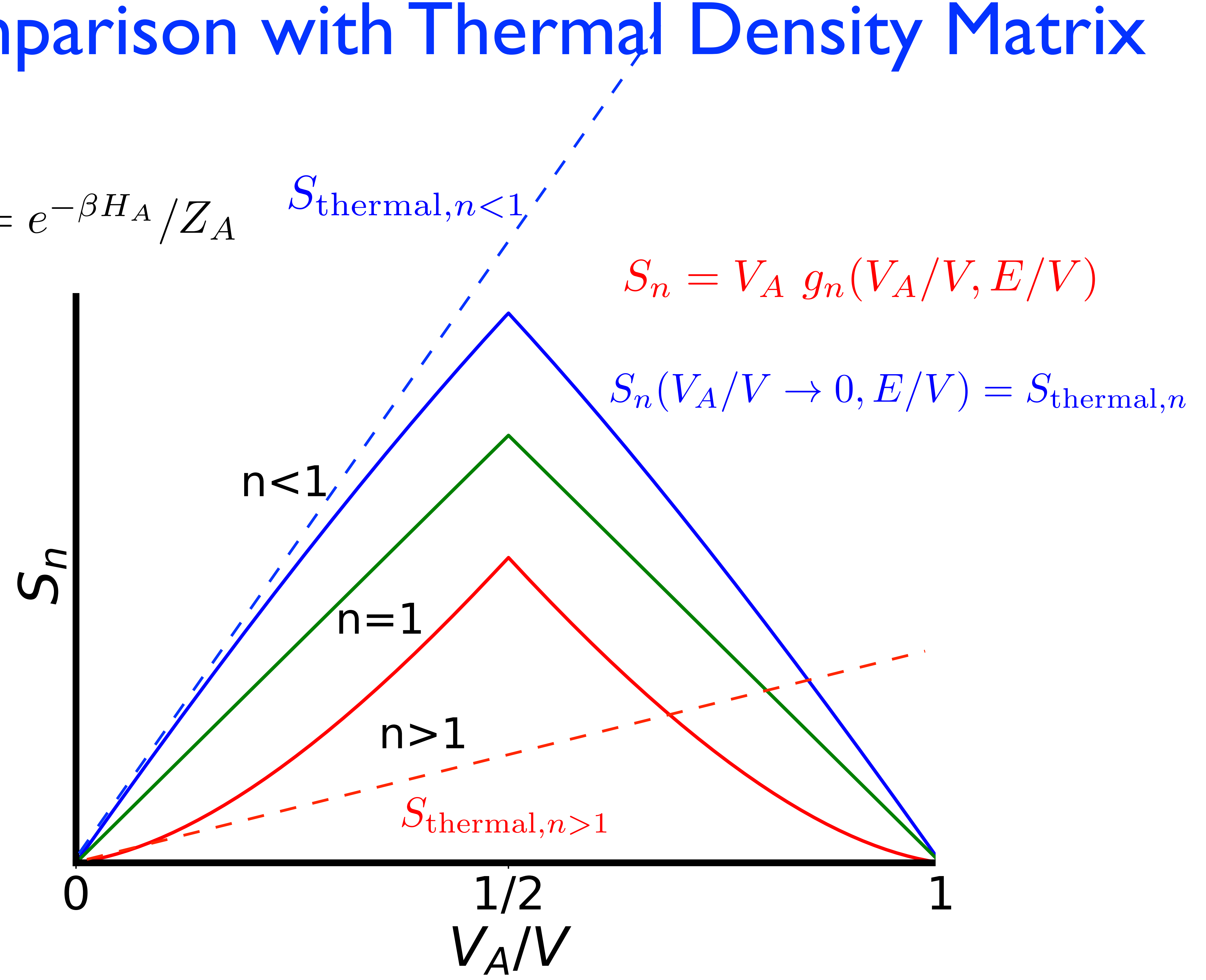


$$\frac{\partial^2 S_n}{\partial (V_A/V)^2} > 0 \quad \text{for } n > 1$$

$$\frac{\partial^2 S_n}{\partial (V_A/V)^2} < 0 \quad \text{for } n < 1$$

Comparison with Thermal Density Matrix

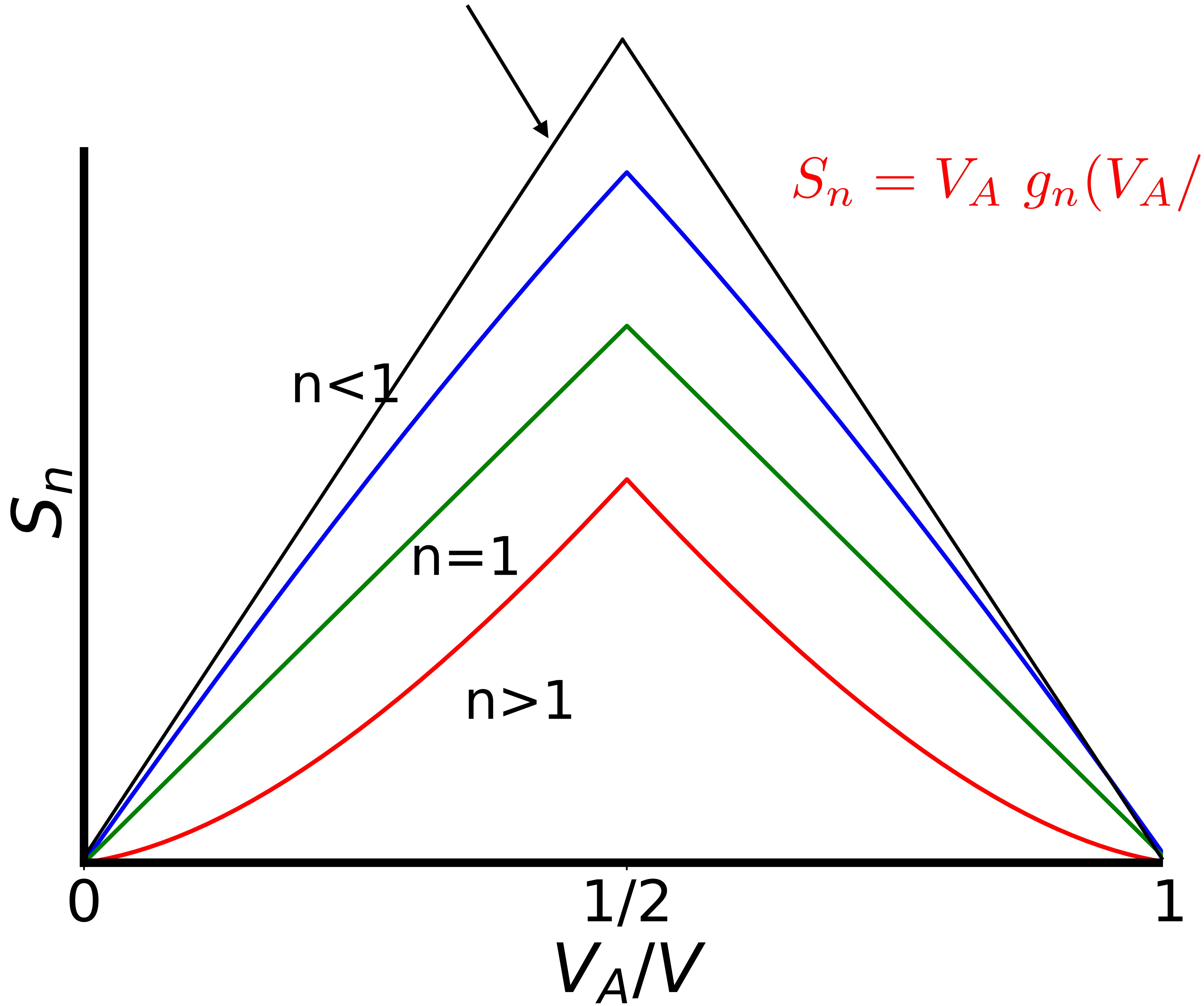
$$\rho_{\text{thermal}} = e^{-\beta H_A}/Z_A$$



Comparison with a "Typical State" in Hilbert Space

For a typical (Haar Random) State, S_n independent of n
and equals log(size of the Hilbert space of region A). "Page Curve" (Lubkin 1978, Page 1993).

$$S_n = V_A \, g_n(V_A/V, E/V)$$



Eigenstate Thermalization

Srednicki 1994,
Deutsch 1991

$$\langle E_\alpha|O|E_\beta\rangle = O(E)\delta_{\alpha\beta} + e^{-S(E)/2} f_O(E,\omega) R_{\alpha\beta}$$

$$E = \frac{E_\alpha + E_\beta}{2} \qquad \omega = E_\alpha - E_\beta$$

O(E) = microcanonical expectation value of *O*,

$f_O(E,\omega)$ smooth function,

R random complex variable with zero mean and unit variance.

Rigol, Dunjko, Olshanii 2008; Khatami, Pupillo, Srednicki, Rigol (2014).

First, consider a finite subsystem A of size V_A when the total system size $V >> V_A$.

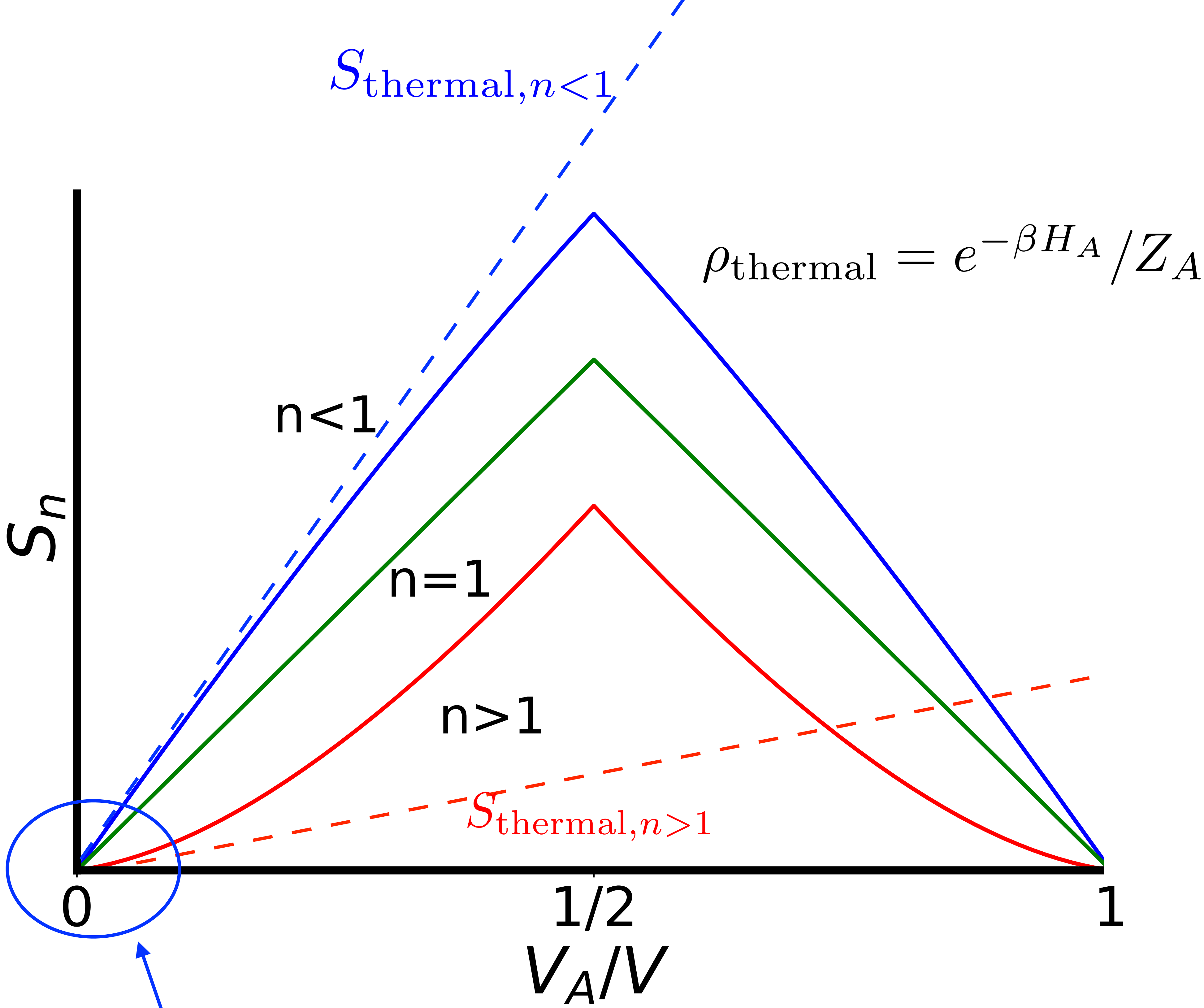
$$\langle E_\alpha | O | E_\beta \rangle = O(E)\delta_{\alpha\beta} + e^{-S(E)/2} f_O(E,\omega) R_{\alpha\beta}$$

If above equation holds for all operators O with support only in region A, then

$$\boxed{\rho_A(|\psi\rangle_\beta) = \rho_{A,\mathrm{th}}(\beta)} \qquad (V_A/V \to 0)$$

where $$\rho_{A,\mathrm{th}}(\beta) = \frac{\mathrm{tr}_{\overline{A}}\left(e^{-\beta H}\right)}{\mathrm{tr}\left(e^{-\beta H}\right)}$$

"thermal reduced density matrix"

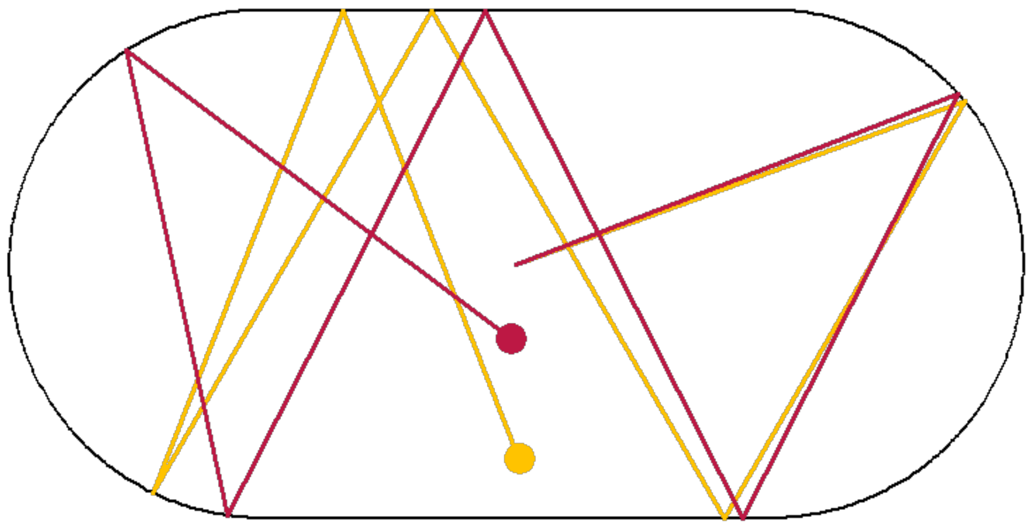


ETH for $V_A/V << 1$ predicts that $S_n(V_A/V \to 0, E/V) = S_{\text{thermal},n}$

What about the limit $V_A \to \infty, V \to \infty$ such that V_A/V is non-zero?

Berry's Conjecture for Chaotic Quantum Billiard Balls

Berry 1977



$$|E\rangle = \int d^3p \; \alpha(\vec{p}) \, \delta(p^2/2m - E) \, |\vec{p}\rangle$$

$\alpha(\vec{p})$ = random gaussian complex numbers

Berry's Conjecture for a Many-body Hard-Sphere System

Srednicki 1994

$$|E\rangle = \int d^{3N}p \; \alpha(\{\vec{p}\})\, \delta(\textstyle\sum_i p_i^2/2m - E)\, |\{\vec{p}\}\rangle$$

$\alpha(\{\vec{p}\})$ = random gaussian complex numbers

Leads to ETH equation

$$\langle E_\alpha|O|E_\beta\rangle = O(E)\delta_{\alpha\beta} + e^{-S(E)/2} f_O(E,\omega) R_{\alpha\beta}$$

Many-body Berry Eigenstates in General

Consider an integrable system perturbed by an infinitesimal integrability-breaking term

$$H = H_0 + \epsilon H_1$$

e.g., $$H = \sum_{i=1}^{L} -Z_i Z_{i+1} - h_z Z_i + \epsilon X_i$$

$$\lim_{\epsilon \to 0} \lim_{V \to \infty} |E\rangle = \sum_{\alpha} C_{\alpha} |s_{\alpha}\rangle$$

"Spontaneous Integrability Breaking"

$$P(\{C_{\alpha}\}) \sim \delta(1 - \sum_{\alpha} |C_{\alpha}|^2)$$

Ansatz recovers ETH

Non-perturbative Generalization

$$H|\psi\rangle = E|\psi\rangle$$

$$H = H_A + H_{\overline{A}} + H_{A\overline{A}}$$

$$|\psi\rangle = \sum_{i,j} C_{ij} \left| E_i^A, E_j^{\overline{A}} \right\rangle$$

"Ergodic Bipartition" Conjecture:

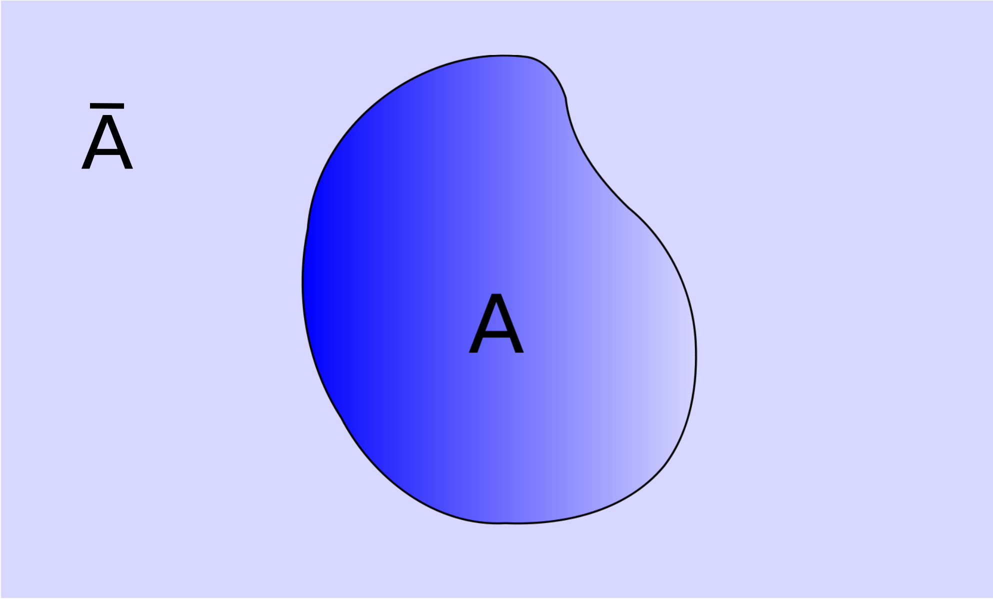
$$P(\{C_{ij}\}) \propto \delta(1 - \sum_{ij} |C_{ij}|^2) \prod_{i,j} \delta(E_i^A + E_j^A - E)$$

Recovers ETH for all bulk quantitites (i.e. operators away from the boundary of subsytem A).

Relation to Ergodicity

Average Reduced Density Matrix for "Many-body Berry" States and "Ergodic Bipartition" States

$$\overline{\rho_A} = \frac{1}{N}\sum_{\alpha} |s_\alpha\rangle \langle s_\alpha| \, e^{S_{M\bar{A}}(E-E_\alpha)}$$



Physical meaning: for a given energy E_A in region A, all states in its complement equally likely ("Ergodicity").

Exactly same as postulated via Canonical typicality arguments in Dymarsky, Lashkari, Liu's "Subsystem ETH" (2016).

Analogous result for systems with U(1) symmetry at infinite T (Garrison, TG (2015)).

Why not directly work with $\overline{\rho_A} = \frac{1}{N}\sum_{\alpha} |s_\alpha\rangle\langle s_\alpha| e^{S_{M\bar{A}}(E-E_\alpha)}$

to calculate Renyi entropies?

Three kind of averaged Renyi entropies:

(a) $S_n^A(\overline{\rho_A}) = \frac{1}{1-n}\log\left(\mathrm{tr}\left((\overline{\rho_A})^n\right)\right)$

(b) $S_n^A(\overline{\mathrm{tr}\,\rho_A^n}) = \frac{1}{1-n}\log\left(\overline{\mathrm{tr}\,\rho_A^n}\right)$

(c) $S_{n,\mathrm{avg}}^A = \frac{1}{1-n}\overline{\log\left(\mathrm{tr}\left(\rho_A^n\right)\right)}$

(c) most relevant. (c) = (b) upto terms exponentially small in system size (recall: No Fannes' inequality for Renyis). Studying (b) requires wavefunction.

Renyi Entropies

$$H = H_A + H_{\overline{A}} + H_{A\overline{A}}$$

Let density of States of H_A at energy $E_A = e^{S_A^M(E_A)}$

Similarly, density of States of $H_{\overline{A}}$ at energy $E_{\overline{A}} = e^{S_{\overline{A}}^M(E_{\overline{A}})}$

$$S_2 = -\log \mathrm{Tr}\,\overline{\rho_A^2} \;\; = -\log\left[\frac{\sum_{E_A} e^{2S_A^M(E_A)+S_{\overline{A}}^M(E-E_A)} + e^{S_A^M(E_A)+2S_{\overline{A}}^M(E-E_A)}}{\left[\sum_{E_A} e^{S_A^M(E_A)+S_{\overline{A}}^M(E-E_A)}\right]^2}\right]$$

Renyi Entropies

$$S_n^A = \frac{1}{1-n}\log\left[\frac{\sum_{E_A} e^{S_A^M(E_A)+nS_{\bar{A}}^M(E-E_A)}}{\left[\sum_{E_A} e^{S_A^M(E_A)+S_{\bar{A}}^M(E-E_A)}\right]^n}\right]$$

at the leading order as $V \to \infty$, $V_A \to \infty$ while V_A/V is fixed.

Renyi Entropies in Thermodynamic limit

$$S_n^A = \frac{V}{1-n}\left[f s(\epsilon_A) + n(1-f)s(\tfrac{\epsilon-\epsilon_A f}{1-f}) - ns(\epsilon)\right]$$

where ϵ_A satisfies $$\left.\frac{\partial s}{\partial \epsilon}\right|_{\epsilon_A} = n\left.\frac{\partial s}{\partial \epsilon}\right|_{\frac{\epsilon-\epsilon_A f}{1-f}}$$

V = total volume, $f = V_A/V$, s = thermal entropy density,

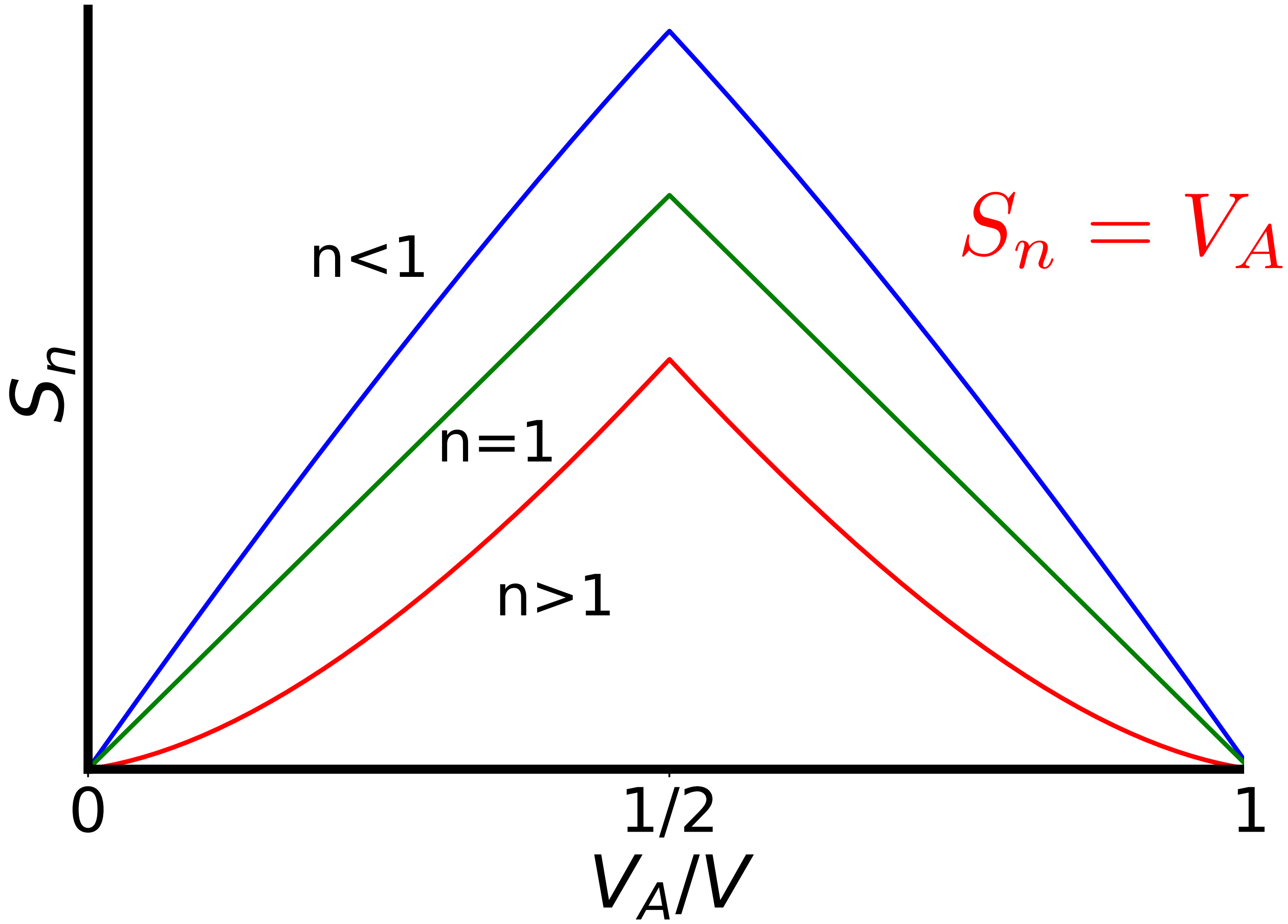
$\epsilon = E/V$ = energy density of the eigenstate

Only when $n = 1$ (von Neumann entropy), $\epsilon_A = \epsilon$, and then S^A/V_A is <u>independent</u> of $f = V_A/V$ ($\Rightarrow$ no curvature i.e. "Page curve")

Curvature dependence of Renyi Entropies

Using above equations, one can prove that

$$\frac{\partial^2 S_n}{\partial (V_A/V)^2} > 0 \quad \text{for } n > 1 \qquad \frac{\partial^2 S_n}{\partial (V_A/V)^2} < 0 \quad \text{for } n < 1$$



$$S_n = V_A \; g_n(V_A/V, E/V)$$

$$g_n(V_A/V, E/V) = \frac{1}{(1-n)f}\left[f\,s(u_A^*) + n(1-f)s(u_{\bar{A}}^*) - ns(u)\right]$$

$$\text{where } \epsilon_A \text{ satisfies } \left.\frac{\partial s}{\partial \epsilon}\right|_{\epsilon_A} = n\left.\frac{\partial s}{\partial \epsilon}\right|_{\frac{\epsilon-\epsilon_A f}{1-f}}$$

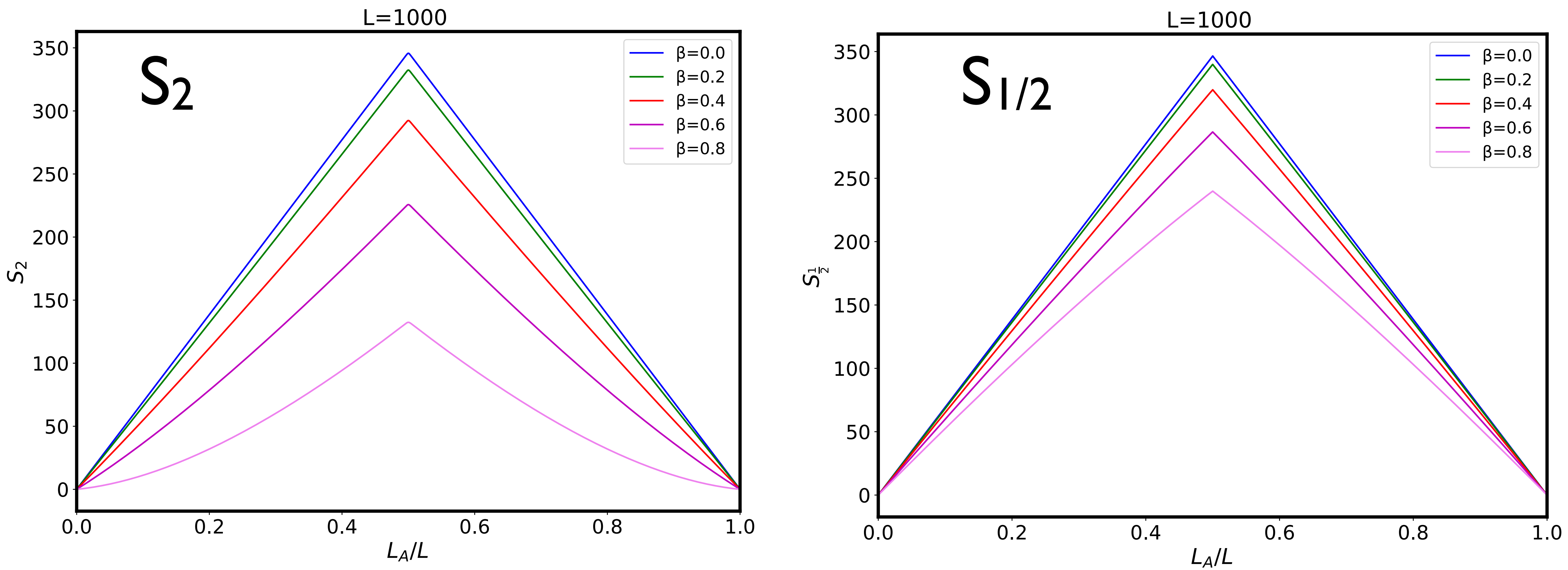
Example: H_0 with Gaussian density of states.

$$S_n = Vf\left[\log(2) - \frac{n}{2[1+(n-1)f]}\beta^2\right]$$

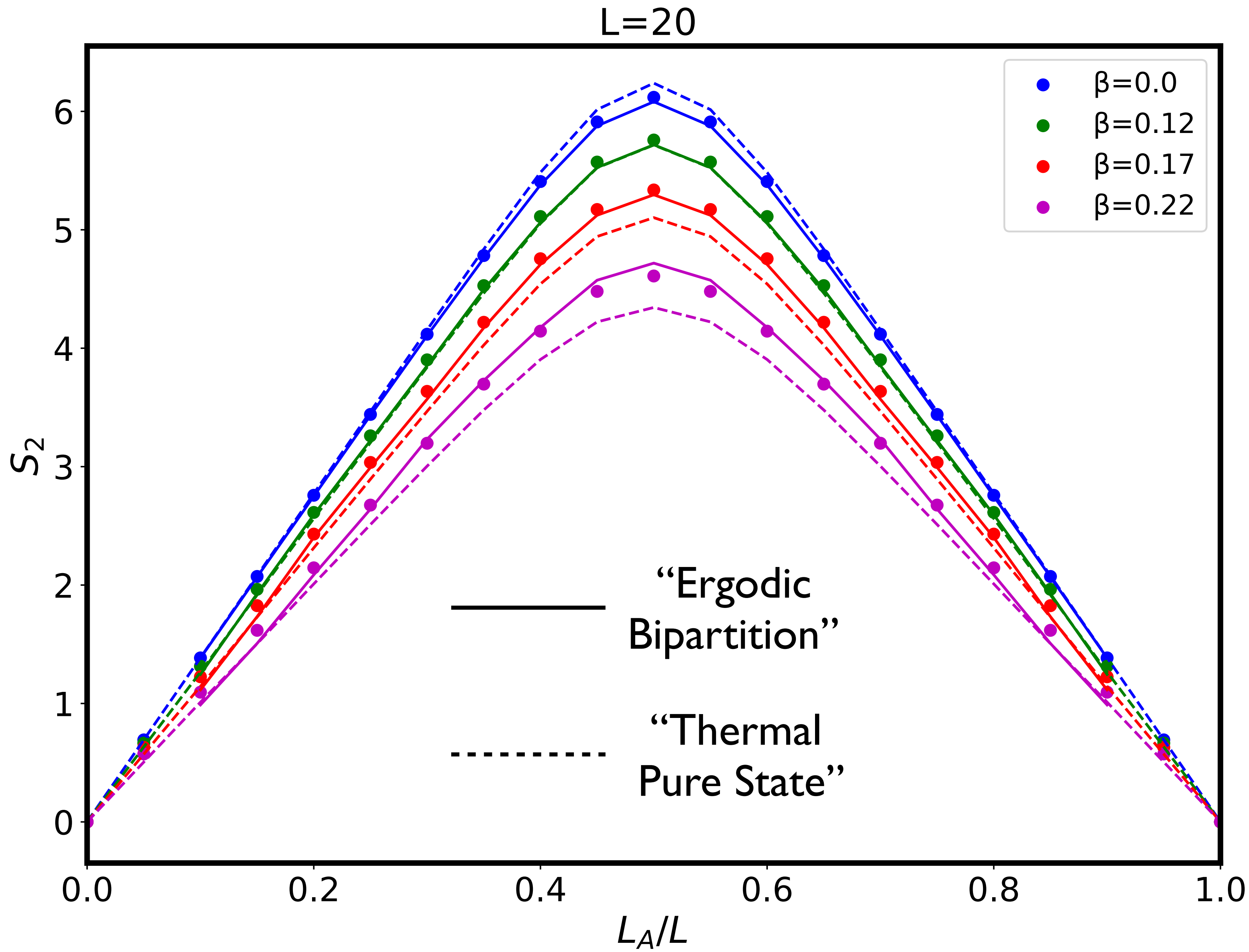
$$(f = V_A/V)$$

Convex for n > 1,
Concave for n < 1.

No Page Curve for S_n, $n \neq 1$



Comparison of theory with Numerics



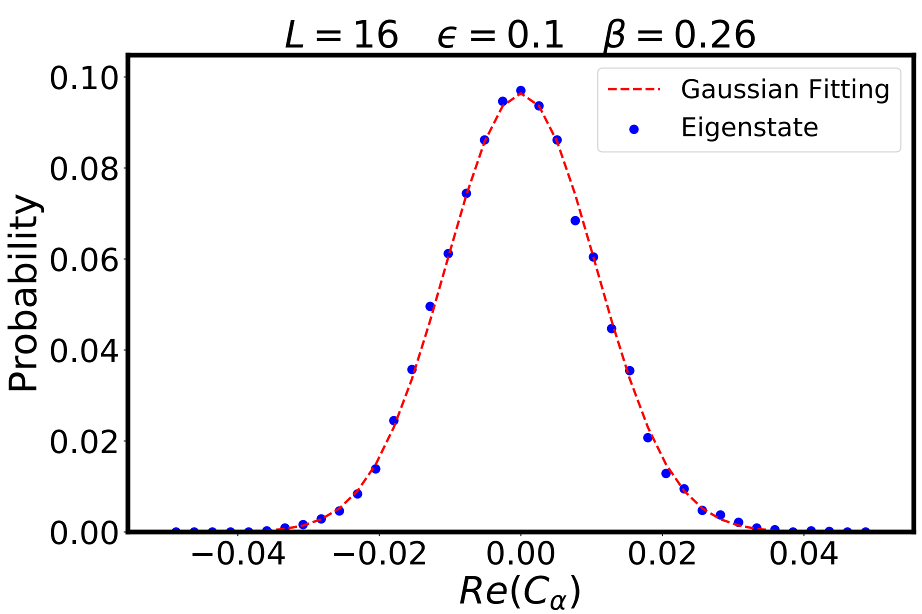
"Thermal Pure State" = $e^{-\beta H/2}|\text{Haar random state}\rangle$

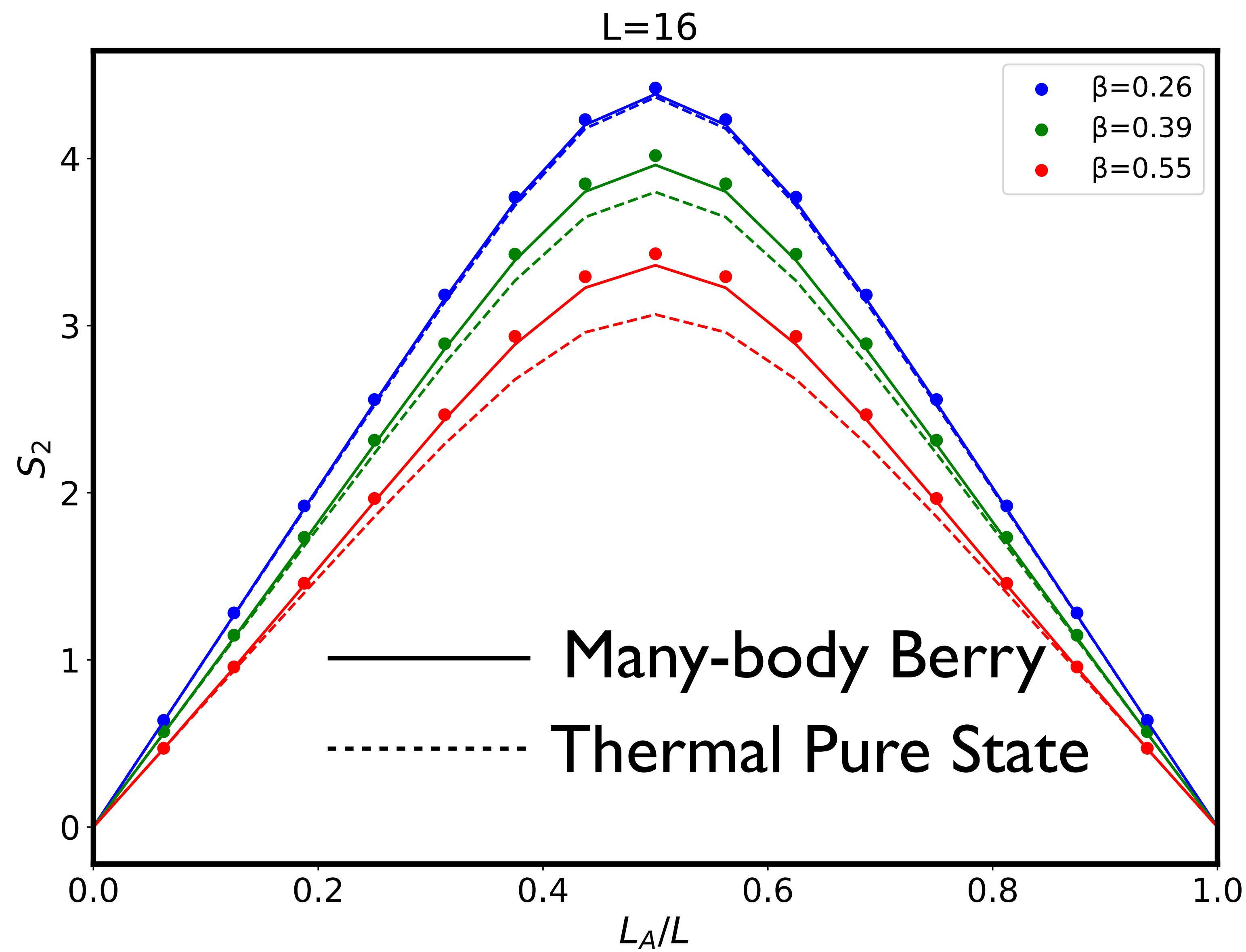
(Fujita et al 2017)

Demonstration of Many-body Berry Conjecture

$$H = \sum_{i=1}^{L} Z_i + \epsilon \times \text{Random Matrix}$$

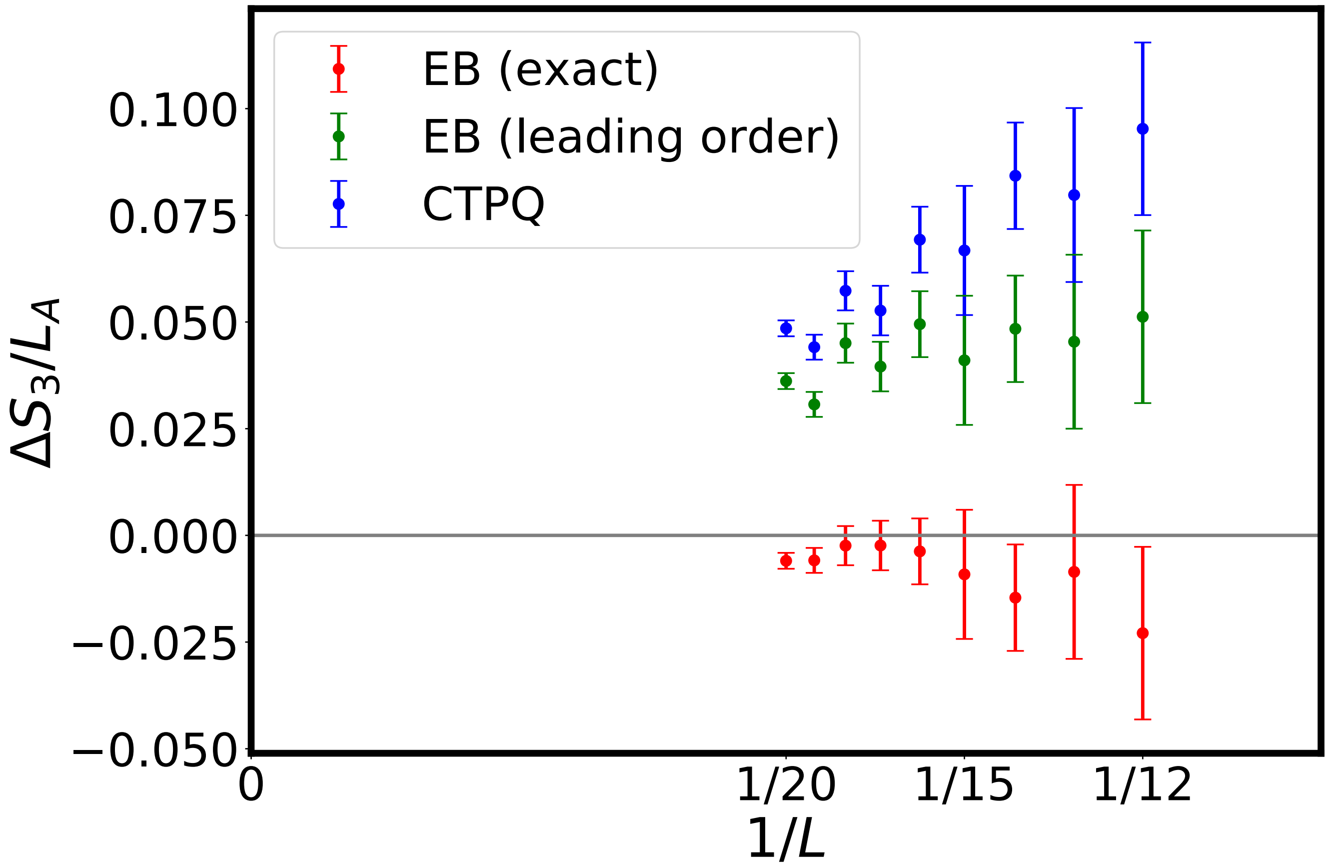
$$\lim_{\epsilon \to 0} \lim_{V \to \infty} |E\rangle = \sum_{\alpha} C_\alpha |s_\alpha\rangle$$





Finite Size Scaling: Exact Vs Asymptotic

$$H = \sum_i \left(-\sigma_i^z \sigma_{i+1}^z + h_x \sigma_i^x + h_z \sigma_i^z\right)$$



Exact:

$$\overline{S_3} = -\frac{1}{2}\log\left[\frac{\sum_{E_A} e^{S_A^M(E_A)+3S_{\bar{A}}^M(E-E_A)} + 3e^{2S_A^M(E_A)+2S_{\bar{A}}^M(E-E_A)} + e^{S_A^M(E_A)+S_{\bar{A}}^M(E-E_A)} + e^{3S_A^M(E_A)+S_{\bar{A}}^M(E-E_A)}}{\left[\sum_{E_A} e^{S_A^M(E_A)+S_{\bar{A}}^M(E-E_A)}\right]^3}\right]$$

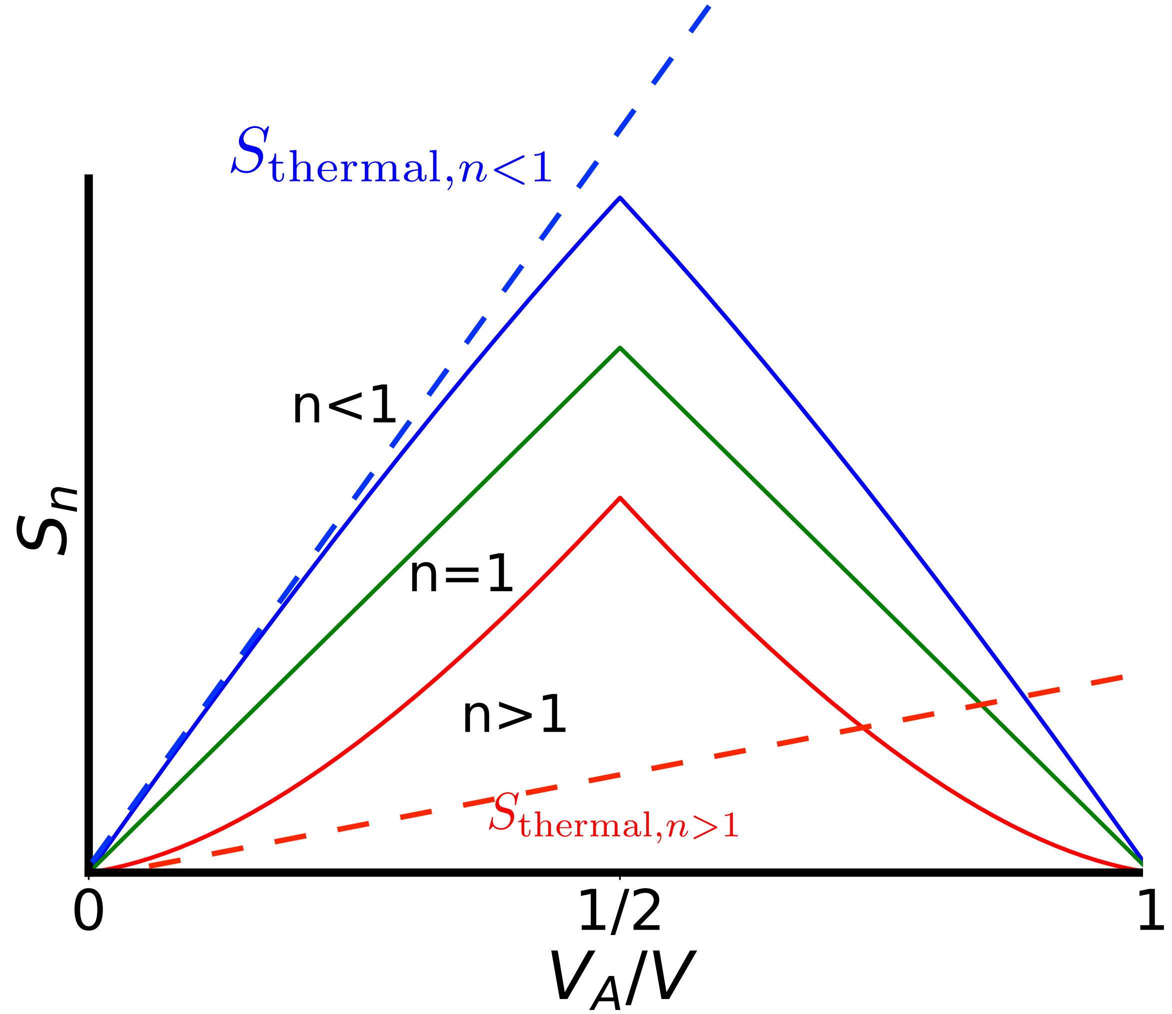
Asymptotic:

$$S_n^A = \frac{V}{1-n}\left[f s(\epsilon_A) + n(1-f)s\left(\frac{\epsilon-\epsilon_A f}{1-f}\right) - ns(\epsilon)\right] \text{ where } \epsilon_A \text{ satisfies } \left.\frac{\partial s}{\partial \epsilon}\right|_{\epsilon_A} = n\left.\frac{\partial s}{\partial \epsilon}\right|_{\frac{\epsilon-\epsilon_A f}{1-f}}$$

Consequences

Renyi entropies can tell the difference between a pure state and a thermal state even when $V_A << V$.

$\rho_{\text{thermal}} = e^{-\beta H_A}/Z_A$



Consequences for decoding information from Hawking radiation?

Consequences

Prediction for Renyi entropy of eigenstates of chaotic CFTs (e.g. holographic CFTs).

In a CFT$_{d+1}$, the entropy density $s(u) = c\, u^{\alpha}$ where u is the energy density, and $\boldsymbol{\alpha} = d/(d+1)$.

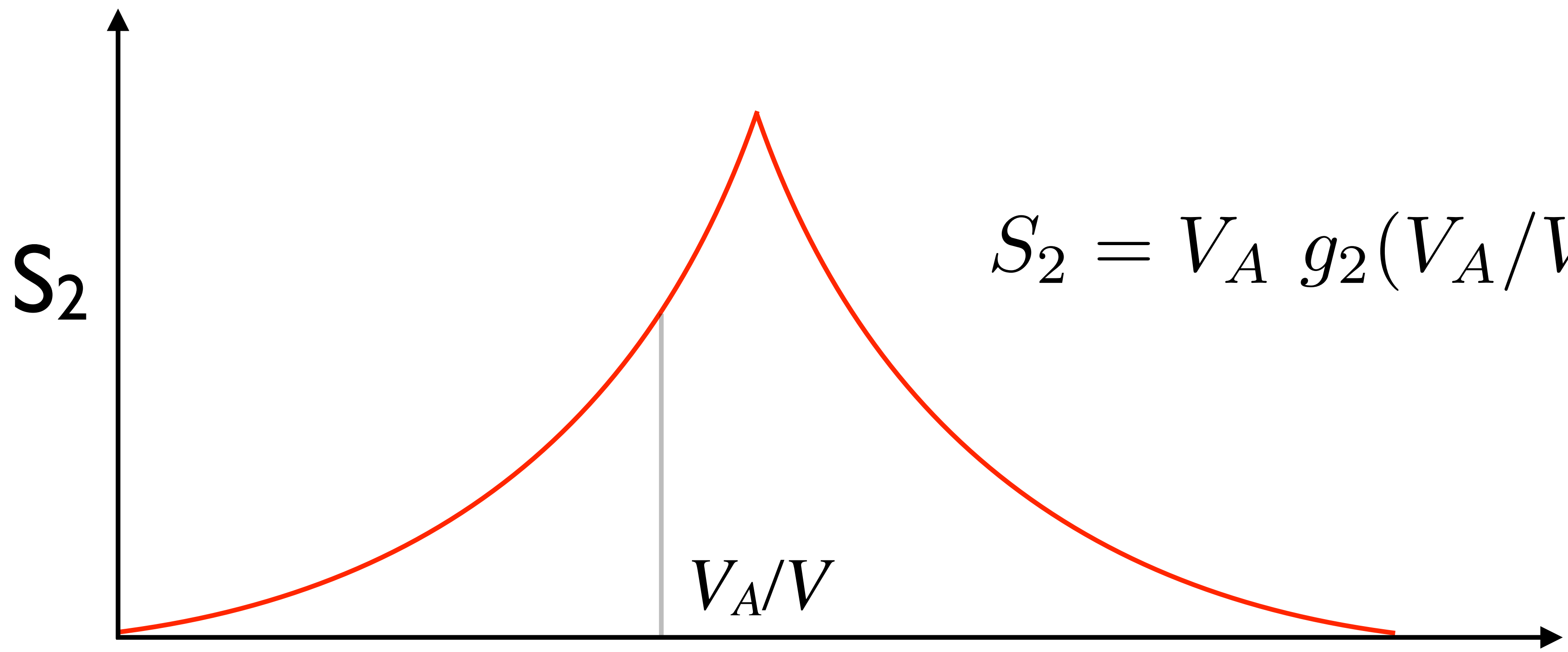
Solving the saddle point equations,

$$S_n = \frac{n}{n-1} c\, u^{\alpha}\, V \left[\{(1-f) + f n^{1/(\alpha-1)}\}^{1-\alpha} - 1 \right]$$

Can this be checked for large central charge CFTs?
(S_1 already matches up, as worked out by Hartman and collaborators).

Consequences

The dependence of Renyi entropy on V_A/V allows one to extract free energy at <u>all temperatures</u> from a <u>single</u> Renyi entropy.

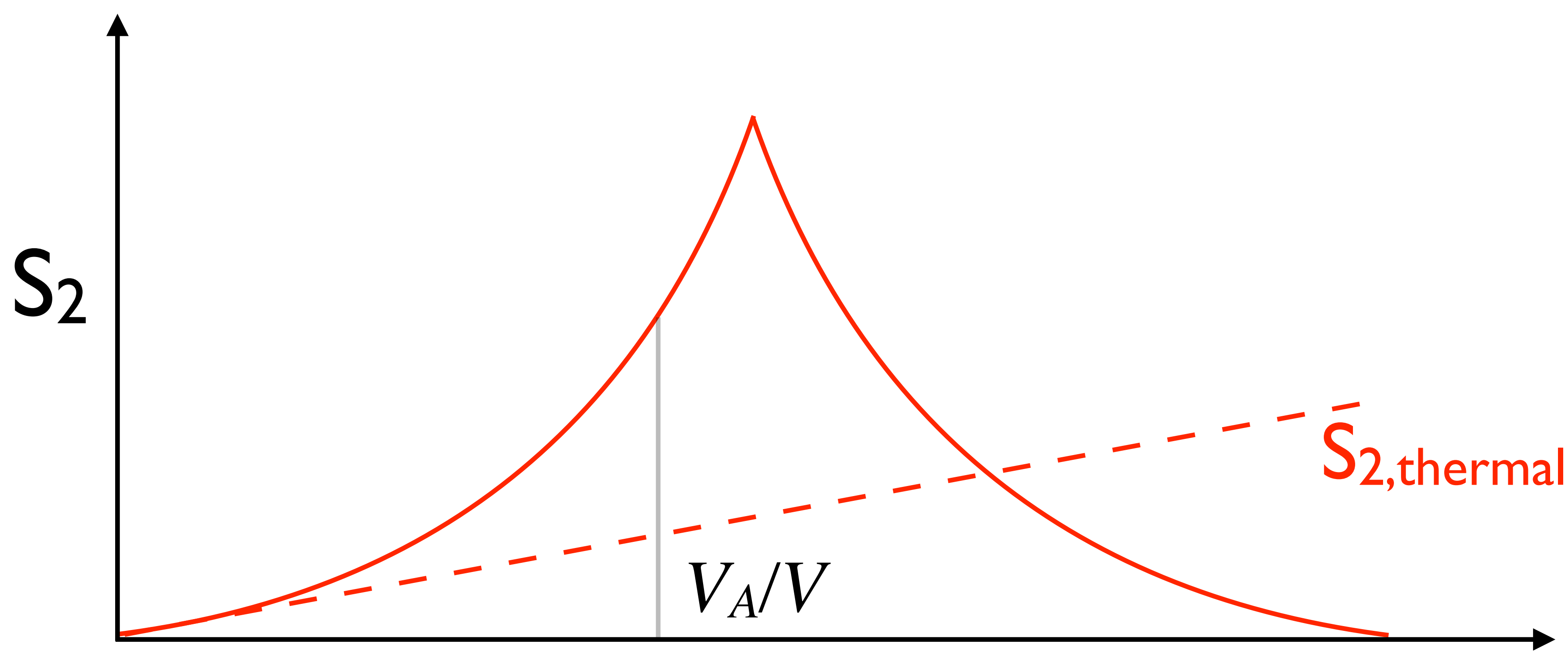


$$S_2 = V_A \; g_2(V_A/V, E/V)$$

Different values of V_A/V encode free energy data at <u>different</u> temperatures.

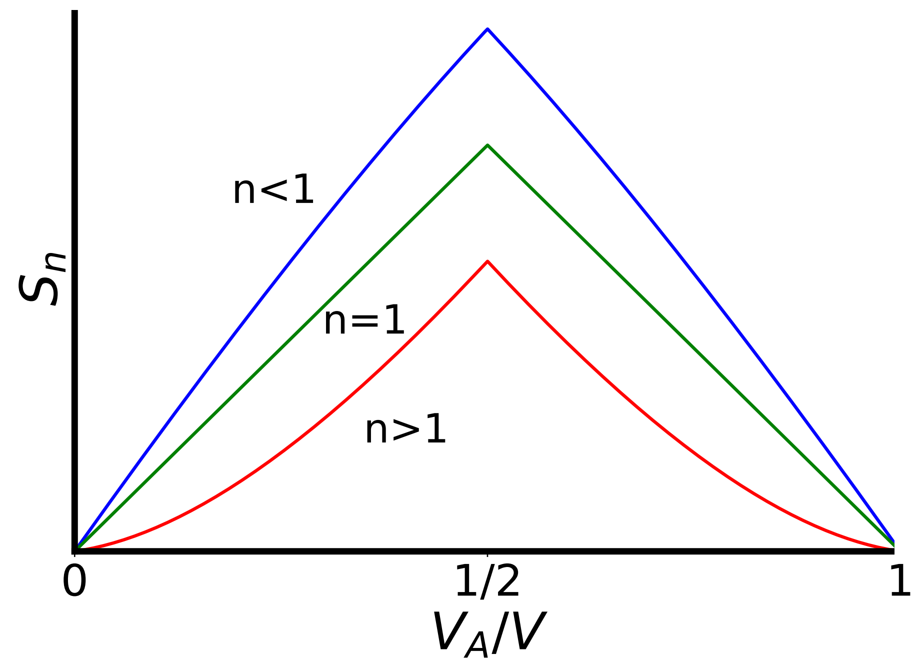
Consequences

In contrast, if one restricts to $V_A/V << 1$, then one needs S_n for <u>ALL</u> n to get the free energy at all temperatures.



$V_A/V << 1$: $$S_n(|\psi\rangle_\beta) = \frac{n}{n-1} V_A \beta \left(f(n\beta) - f(\beta) \right)$$

Consequences



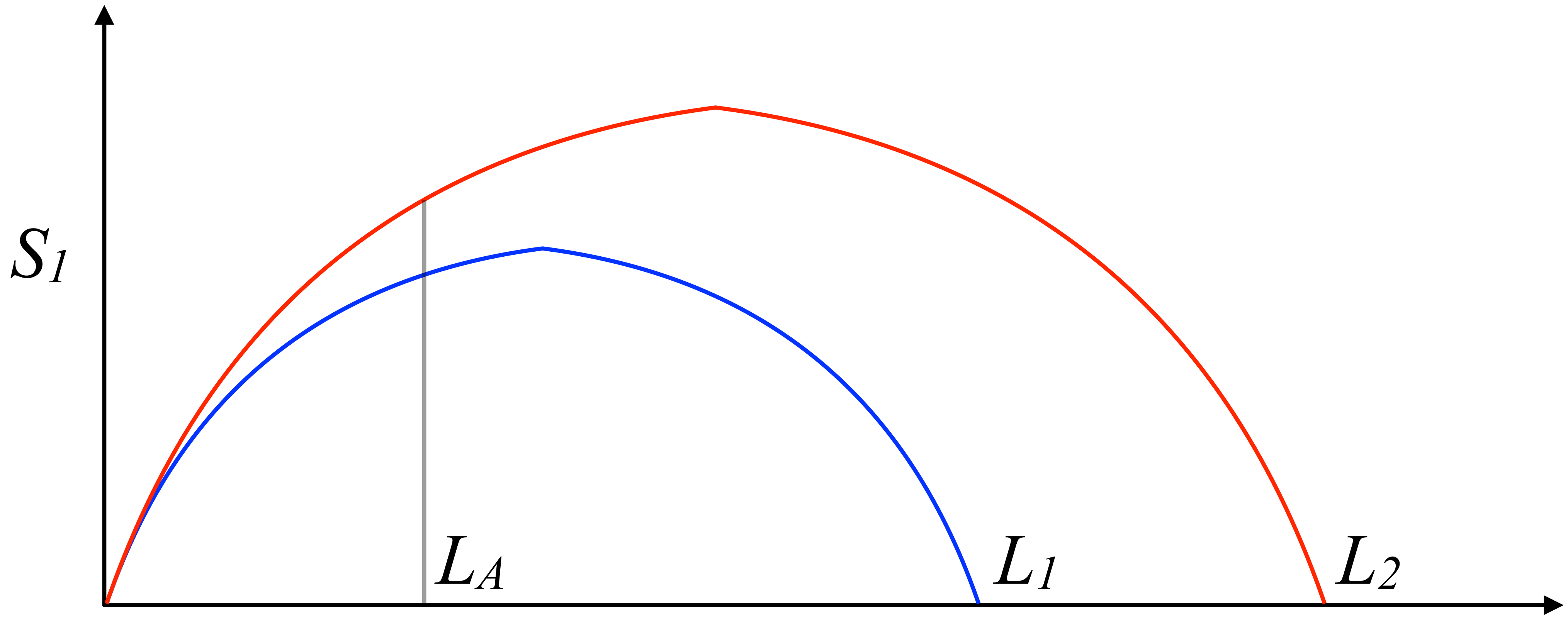
von Neumann entropy S_1, at the leading order,
is additive: $S_1 = V_A\, s_{thermal}(\beta)$.

In contrast S_n, for $n \neq 1$, is not additive.

In fact, for $n > 1$, S_n *is not even subadditive:* $S_{n,A} + S_{n,B} < S_{n,A \cup B}$

Why positive curvature of S_n for $n > 1$ is interesting.

Consider increasing the total system size of a translationally invariant Hamiltonian.



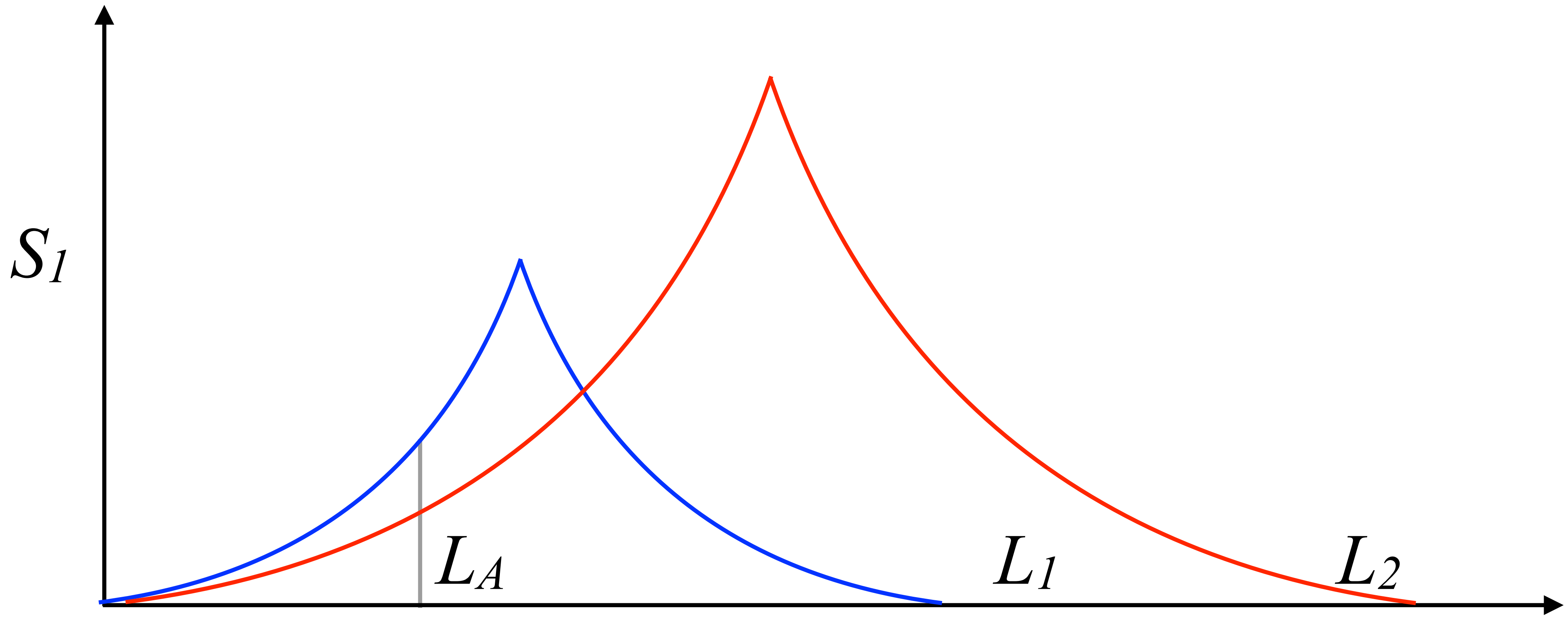
Strong subadditivity implies that S_1 is non-convex

$$\implies \frac{\partial S_1^A}{\partial L} \geq 0$$

Increasing the "heat-bath" size increases entanglement of a subsystem.

Why positive curvature of S_n for $n > 1$ is interesting.

Consider increasing the total system size of a translationally invariant Hamiltonian.



In contrast,

$$\Longrightarrow \frac{\partial S_2^A}{\partial L} < 0$$

Increasing the "heat-bath" size <u>decreases</u> S_2 of a subsystem.

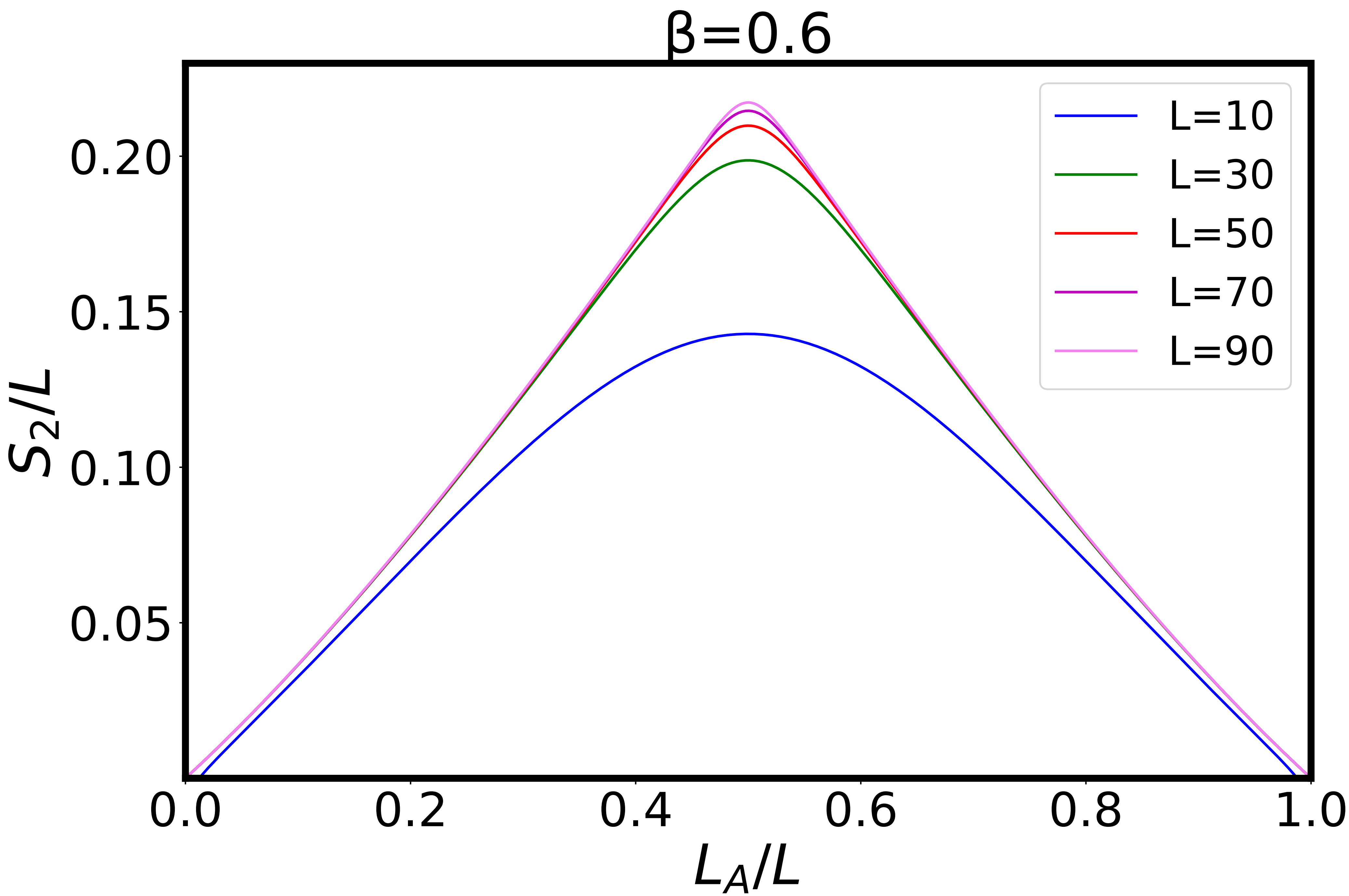
Summary and Questions

- Ergodicity based arguments seemingly explain several universal features of entanglement scaling. Numerical evidence seems good. Specifically:

a. von Neumann entropy density for an eigenstates equals thermal entropy density as long as $V_A < V/2$ ("finite T Page Curve"). One doesn't need $V_A << V$.

b. Renyi entropies S_n have a universal dependence on the subsystem to system ratio V_A/V and the density of states. For n > 1 (n<1), the Renyi entropy densities (= S_n/V_A) are bigger (smaller) than those for the corresponding thermal state.

- Holographic/large-c checks for the chaotic CFT Renyi expressions? (alert: we are dealing with pure eigenstates).
- Implications for black hole physics? Renyi entropies as a diagnosis of non-thermal correlations in Hawking radiation?
- Quantum dynamics using Berry's conjecture?
- Consequences for experimentally measured Renyis under quantum quench?
- Towards random matrix like theory <u>with locality</u> built-in.


β=0.6
L=10
L=30
L=50
L=70
L=90
0.20
0.15
0.10
0.05
S_2/L
0.0
0.2
0.4
0.6
0.8
1.0
L_A/L