Black holes and random matrices

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KITP October 12, 2017

A question

- What accounts for the finiteness of the black hole entropy–from the bulk point of view in AdS/CFT?
- The stakes are high here. Many approaches to understanding the bulk:
  - Eternal black hole $\leftrightarrow$ Thermofield double state
  - Ryu-Takayanagi
  - Geometry from entanglement
  - Tensor networks
  - ER = EPR
  - Bulk reconstruction and error correction
  - Complexity
  - Bit threads
  - ...

  suggest that any complete bulk description of quantum gravity (if one exists) must be able to describe these states.
  (e.g., what do the virtual indices in a tensor network represent?)

A diagnostic

- A simple diagnostic of a discrete spectrum [Maldacena]. Long time behavior of $\langle O(t)O(0)\rangle$. (O is a bulk (smeared boundary) operator)

$$\langle O(t)O(0)\rangle = \sum_{m,n} e^{-\beta E_m} |\langle m|O|n\rangle|^2 e^{i(E_m - E_n)t} / \sum_n e^{-\beta E_n}$$

- At long times the phases from the chaotic discrete spectrum cause $\langle O(t)O(0)\rangle$ to oscillate in an erratic way. It becomes exponentially small and no longer decreases.
  (See also [Dyson-Kleban-Lindesay-Susskind; Barbon-Rabinovici])
- To focus on the oscillating phases remove the matrix elements. Use a related diagnostic: [Papadodimas-Raju]

$$\sum_{m,n} e^{-\beta(E_m + E_n)} e^{i(E_m - E_n)t} = Z(\beta + it)Z(\beta - it) = Z(t)Z^*(t)$$

- The "spectral form factor"

Properties of $Z(t)Z^*(t)$

$$Z(t)Z^*(t) = \sum_{m,n} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)t}$$

- $Z(\beta,0)Z^*(\beta,0) = Z(\beta)^2$ $(= L^2 = e^{2S}$ for $\beta = 0)$
- Assume the levels are discrete (finite entropy) and non-degenerate (generic, implied by chaos)
- At long times, after a bit of time averaging (or J averaging in SYK), the oscillating phases go to zero and only the $n = m$ terms contribute.
- $Z(\beta)^2 \to Z(2\beta)$. $(= L = e^S$ for $\beta = 0)$
- $e^{2S} \to e^S$, an exponential change. How does this occur?

SYK as a toy model

- The Sachdev-Ye-Kitaev model can serve as a toy model to address these questions.

$$H = \sum_{abcd} J_{abcd}\psi_a\psi_b\psi_c\psi_d, \quad \langle J_{abcd}^2\rangle \sim J^2/N^3$$

- Maximally chaotic, discrete spectrum
- Has a sector dual to AdS_2 dilaton gravity
- Has a collective field description: $G(t,t') = \frac{1}{N}\psi_a(t)\psi_a(t'), \;\; \Sigma(t,t')$. Reminiscent of a bulk description:
  - $O(N)$ singlets
  - nonlocal
  - Nonperturbatively well defined (two replicas)

$$\langle Z(t)Z^*(t)\rangle = \int dG_{ab}d\Sigma_{ab}\exp(-N\ I(G_{ab},\Sigma_{ab}))$$

  - (Of course many differences with bulk descriptions...)

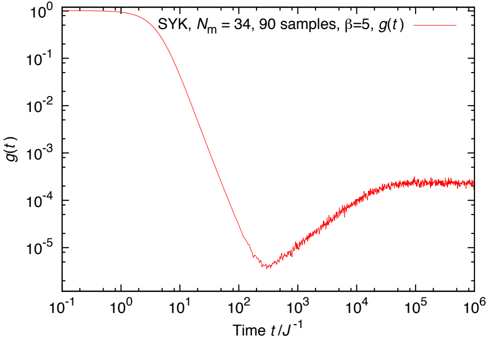
$ZZ^*(t)$ in SYK

- Finite dimensional Hilbert space, $D = L = 2^{N/2}$, amenable to numerics
- Guidance about what to look for

[Jordan Cotler, Guy Gur-Ari, Masanori Hanada, Joe Polchinski, Phil Saad, Stephen Shenker, Douglas Stanford, Alex Streicher, Masaki Tezuka]
([CGHPSSSST])

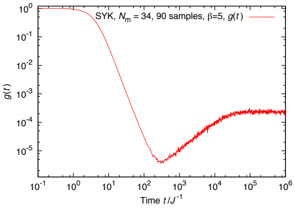
See also
[Garcia-Garcia–Verbaarschot]

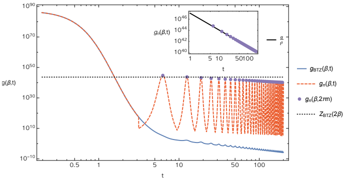
Results



- The Slope ↔ Semiclassical quantum gravity
- The Ramp and Plateau ↔ Random Matrix Theory ([see You-Ludwig-Xu])
- The Dip ↔ crossover time

Slope, contd.





Slope is determined by semiclassical quantum gravity – nonuniversal.

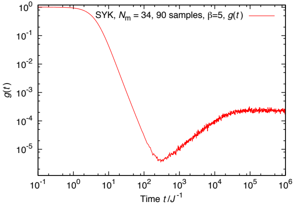
In SYK slope $\sim 1/t^3$. From sharp edge in DOS. One loop exact Schwarzian result: $\rho(E) \sim e^{S_0}(E - E_0)^{1/2}$. ([Bagrets-Altland-Kamenev; CGBPSSSST; Stanford-Witten])

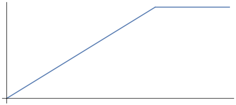
In BTZ summing over geometries gives oscillating slope with power law envelope: nonperturbatively small oscillations in the density of states [Dyer–Gur-Ari]

In AdS_5 "graviton gas" $\rightarrow$ constant slope ([CGBPSSSST])

In each case the dip time is $\sim e^{aS}$ (with different a), exponentially shorter than the plateau time

The Ramp and Plateau





The Ramp and Plateau are signatures of Random Matrix Statistics, believed to be **universal** in quantum chaotic systems

$\langle ZZ^*(t)\rangle$ is essentially the Fourier transform of $\rho^{(2)}(E,E')$, the pair correlation function

$$\rho^{(2)}(E,E') \sim 1 - \frac{\sin^2(L(E-E'))}{(L(E-E'))^2}$$

[Dyson; Gaudin; Mehta]

The decrease before the plateau is due to repulsive anticorrelation of levels ("The correlation hole")

Conjecture that this pattern is universal in quantum black holes

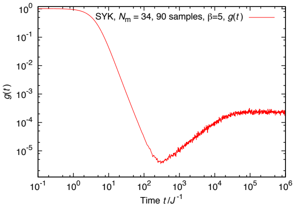
N versus L

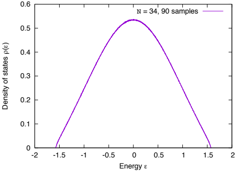
- $\rho^{(2)}(E,E') \sim 1 - \frac{\sin^2(L(E-E'))}{(L(E-E'))^2}$
- $t \ll t_p, \;\; \rho^{(2)}(E,E') \sim 1 - \frac{1}{L^2(E-E')^2}$, "spectral rigidity"
- $\frac{1}{L^2}$ perturbative in RMT, $\frac{1}{L^2} \sim e^{-cN}$, nonperturbative in $\frac{1}{N}$, SYK.
- $\sin^2(L(E-E')) \to \exp(-2L(E-E'))$, Altshuler-Andreev instanton
- $\sim \exp(-e^{cN})$ in SYK (!)
- These effects must be realized in the G, Σ formulation. A research program...

Onset of RMT behavior

[Hrant Gharibyan (Stanford), Masanori Hanada (Kyoto), SS, Masaki Tezuka (Kyoto)]

[In progress]





At how large an energy eigenvalue separation does spectral rigidity end?

At what time t_r does the ramp begin?

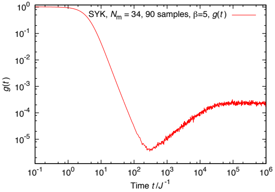
The Thouless time [Garcia-Garcia–Verbaarschot]

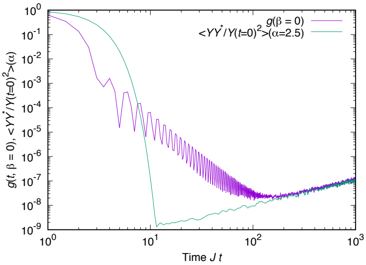
The dip is just a crossover: edge versus bulk dynamics, $t_r \neq t_d$

Follow the ramp below the slope: use Gaussian filter [Stanford]

$$Y(\alpha, t)Y^*(\alpha, t) = \sum_{m,n} e^{-\alpha(E_n^2+E_m^2)} e^{+i(E_m-E_n)t}$$

YY^*





Dip time $t_d \sim 200,\ N = 34$

Onset of ramp $t_r \lesssim 10,\ N = 34$

(The ramp is an exponentially subleading effect in ZZ^* and correlation functions before the dip)

An upper bound. Very little variation in N for $N \leq 34$

log N? scrambling?

Maybe; no.

Simplify problem by looking at nearest neighbor qubit chain, random couplings, n qubits. Scrambling time $\sim n$, easier to study.

Brownian circuits

- Scrambling describes the growth of a simple operator [Roberts-Stanford-Susskind; Lieb-Robinson]
- Generic. Also happens in Brownian circuit
- $e^{-iHt} \to e^{-iH_m \Delta t} e^{-iH_{m-1} \Delta t} \dots e^{-iH_1 \Delta t}$
- H_m drawn from an ensemble
- Unitary gates $U = U_m U_{m-1} \dots U_1$ (random quantum circuit)
- Can analyze dynamics including scrambling analytically [Oliveira-Dahlsten-Plenio; Lashkari-Stanford-Hastings-Osborne-Hayden; Harrow-Low; Brandao-Harrow-Horodecki; Brown-Fawzi ...]
- Theory of approximate unitary k designs. Approximations to Haar ensemble that accurately compute monomials of k U's , k $U^\dagger$'s [Denkert et al. ...]

Markov chain

- Study $U\rho U^\dagger$, $\rho = \sum_p \gamma_p \sigma_p$, σ_p a string of Paulis
- (Need k copies for k design)
- Defines a Markov process on on Pauli strings e.g., $\mathrm{I\ I\ I\ Z\ Z\ I\ I\ X\ I\ I} \ldots$
- Two qubit Haar random gates and $k = 2$: $\mathrm{I\ I} \to \mathrm{I\ I}$; $\mathrm{AB} \to$ 15 other possibilities, uniformly [Harrow-Low]
- Initial condition for an OTOC: $\mathrm{Z\ I\ I\ I\ I\ I\ I\ I\ I\ I}$
- Time to randomize last qubit $\sim n$, scrambling time [Nahum-Vijay-Haah; Keyserlingk-Rakovszky-Pollmann-Sondhi]

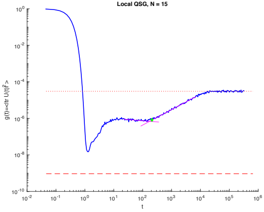
Markov chain, contd.

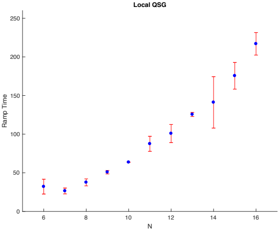
- For spectral statistics study $\langle tr(U^k) tr((U^\dagger)^k)\rangle$, $k = 1, 2 \ldots$
- RMT statistics $\langle tr(U^k) tr((U^\dagger)^k)\rangle \to$ Haar average value
- For $k = 2$ (two design) slowest terms are like $U_{aa} U^*_{aa} U_{aa} U^*_{aa}$ (no sum)
- Study $U|a\rangle\langle a|U^\dagger$ where $|a\rangle = |00\ldots00\rangle$
- $|00\ldots00\rangle\langle 00\ldots00| = (\frac{1}{2})^n (\mathrm{I} + \mathrm{Z})^{\otimes n}$
- $\mathrm{Z\ I\ Z\ Z\ I\ I\ Z\ Z\ I} \ldots$ Easy to equilibrate

Markov chain, contd.

- Z I Z Z I I Z Z I . . .
- Length of longest run of I s in typical string $\sim \log n$. Equilibrates in $\sim \log n$
- n rare strings I I I I Z I I . . .. Contribution decays like ne^{-t}. Order one at $t \sim \log n$.
- At long times the system relaxes at a rate determined by the gap of the Markov chain. The gap is independent of n [Brandao-Harrow-Horodecki]
- Equilibration time $\sim \log n$, shorter than scrambling !
- Correlation functions of very complicated operators [Roberts-Yoshida; Cotler-Hunter-Jones-Liu-Yoshida]
- For nonlocal pair interactions (2 local, analogous to SYK), get a time of order $\log \log n$, (for non Haar random gates goes back to $\log n$)

Hamiltonian systems (geometrically local)





n geometrically local qubits

$H = \sum_i J_i^{\alpha\beta} \sigma_i^{\alpha} \sigma_{i+1}^{\beta}$, J random

Gaussian density of states $\rightarrow$
slope $\sim \exp(-Nt^2)$, rapid decay

Scrambling time $\sim n$

$t_r \sim n^2$

Slower than scrambling ??

Diffusion

- Crucial difference between Hamiltonian and random quantum circuit systems – conserved quantities (energy)
- $t_r \sim n^2$ is the time for something to diffuse across the n qubit chain.
- For a single particle described by a random hopping H (a banded matrix) the ramp time is just the time to for the particle to diffuse across the system, the Thouless time [Altshuler-Shklovskii, Efetov...].
- Same here, for energy?
- A new tool: Random quantum circuit with a conserved quantity S_z. Analytically tractable [Khemani-Vishwanath-Huse]. Small diffusive corrections to OTOCs.
- Plan: compute spectral form factor quantities using this circuit. Large effect because signal is so small.

The Thouless time for black holes

- Assume geometrically local d dimensional Hamiltonians have t_r governed by diffusion, $t_r \sim n^{2/d}$
- q-local systems like SYK correspond to $d \to \infty$. Then $n^{2/d} \to \log n$ [Susskind]
- $t_r \sim \log n$ but with a different coefficient than the scrambling time.
- Important because the black hole evaporation time is of order $S \sim n \gg \log n$.
- So these phenomena would appear in small black holes as well, although as an exponentially subleading effect
- We need to know what they mean in quantum gravity...