Rigorous RG: An efficient algorithm for low energy states of 1D quantum systems

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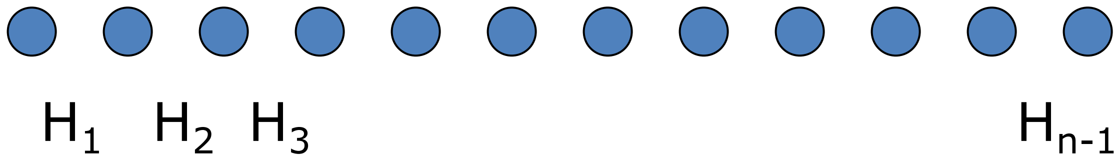
Local Hamiltonian in 1D

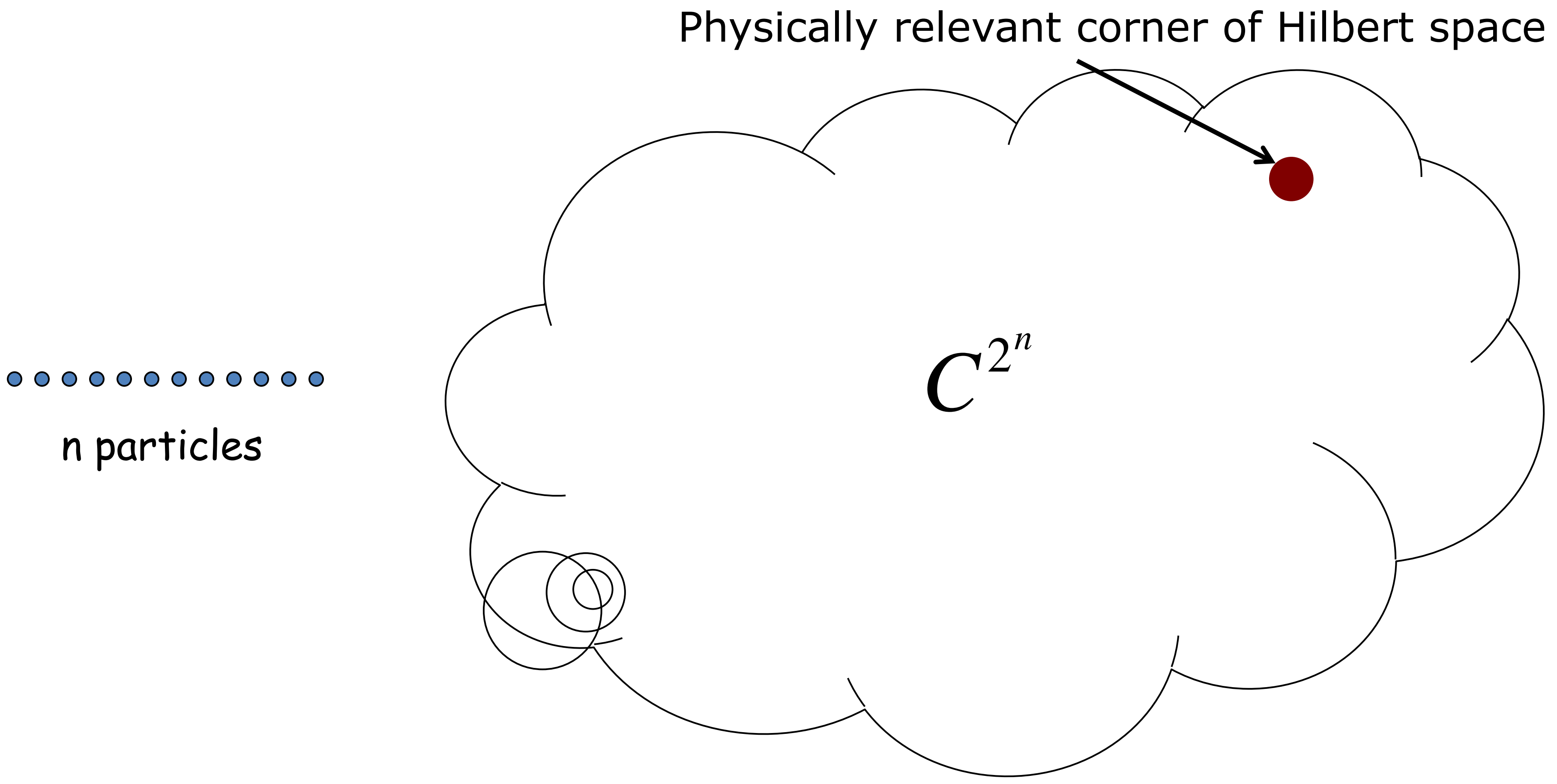


Assume each particle is a d-level system (qudit) with nearest neighbor interactions

Each term H_i is a $d^2 x d^2$ Hermitian matrix

$$H = H_1 + \cdots + H_m$$

- Do low energy states states of H have succinct classical description?
- Is there an efficient classical algorithm to find such a description?

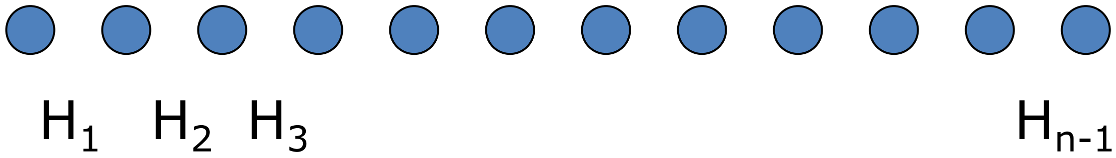


CMP: low energy states are special

- DMRG (Density Matrix Renormalization Group) [White ’92] has been remarkably successful in practice for 1D quantum systems. Quickly outputs compact representation of ground/low energy state.

- Doesn’t always work. Artificial hard examples known [Eisert ’06]

Local Hamiltonian in 1D



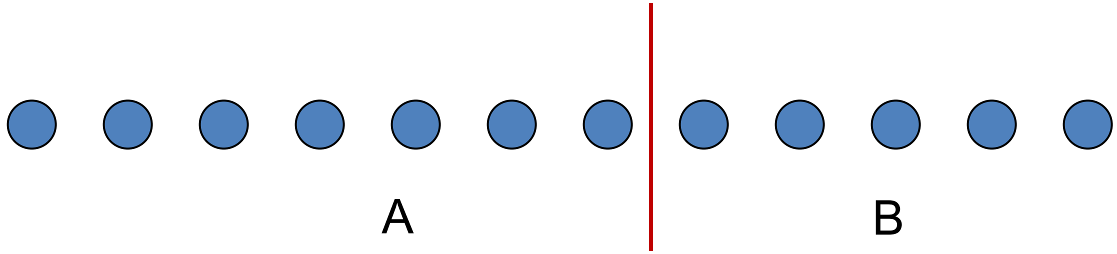
- Do low energy states states of H have succinct classical description?

[Hastings '06] (Close approximations to) Ground states of gapped 1D Hamiltonians ($E_1 = E_0 + c$) have compact description as an MPS with poly(n) bond dimension.

- Is there an efficient classical algorithm to find such a description?

[Landau, V, Vidick '14] Polynomial time algorithm to find ground states of gapped 1D Hamiltonians.

Area Law/Succinct Description for Ground State



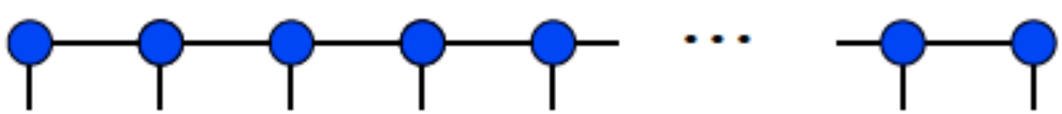
$$|\psi\rangle = \sum c_i |a_i\rangle \otimes |b_i\rangle, \quad c_i \geq 0$$

Schmidt decomposition: $\{|a_i>\}$, $\{|b_i>\}$ orthonormal sets

Schmidt rank = number of non-zero terms

If $|\psi>$ has low Schmidt rank then if $|a_i>$ and $|b_i>$ vectors compactly representable then so is $|\psi>$

If $|\psi>$ has low Schmidt rank across all cuts then $|\psi>$ has a compact representation (in the form of a tensor network called an MPS).



Two classes of Hamiltonians:

1. Gapless Hamiltonians with a low density of low-energy states

   r=poly(n) dimensional space of eigenstates with energy in the range $[E_0, E_0 + c]$

2. Degenerate gapped ground space:

   H has smallest eigenvalue E_0 with associated eigenspace of dimension r=poly(n), and $E_1 = E_0 + c$

   Constant degeneracy: [Huang; Chubb, Flammia 2014]

Do Low Energy States have compact descriptions?

- Will consider the case of a degenerate ground space of polynomial dimension

- All proofs bounding entanglement across cuts show that for every cut there is some ground state with low complexity.

- Want to show: there is a ground state with low complexity across all cuts.
  Or better still, all ground states have low complexity across all cuts.

Results:

- Polynomial time algorithm for computing basis for poly(n)-degenerate ground space for 1D gapped Hamiltonian.

  Implies area law + succinct MPS description for poly(n)-degenerate ground spaces.

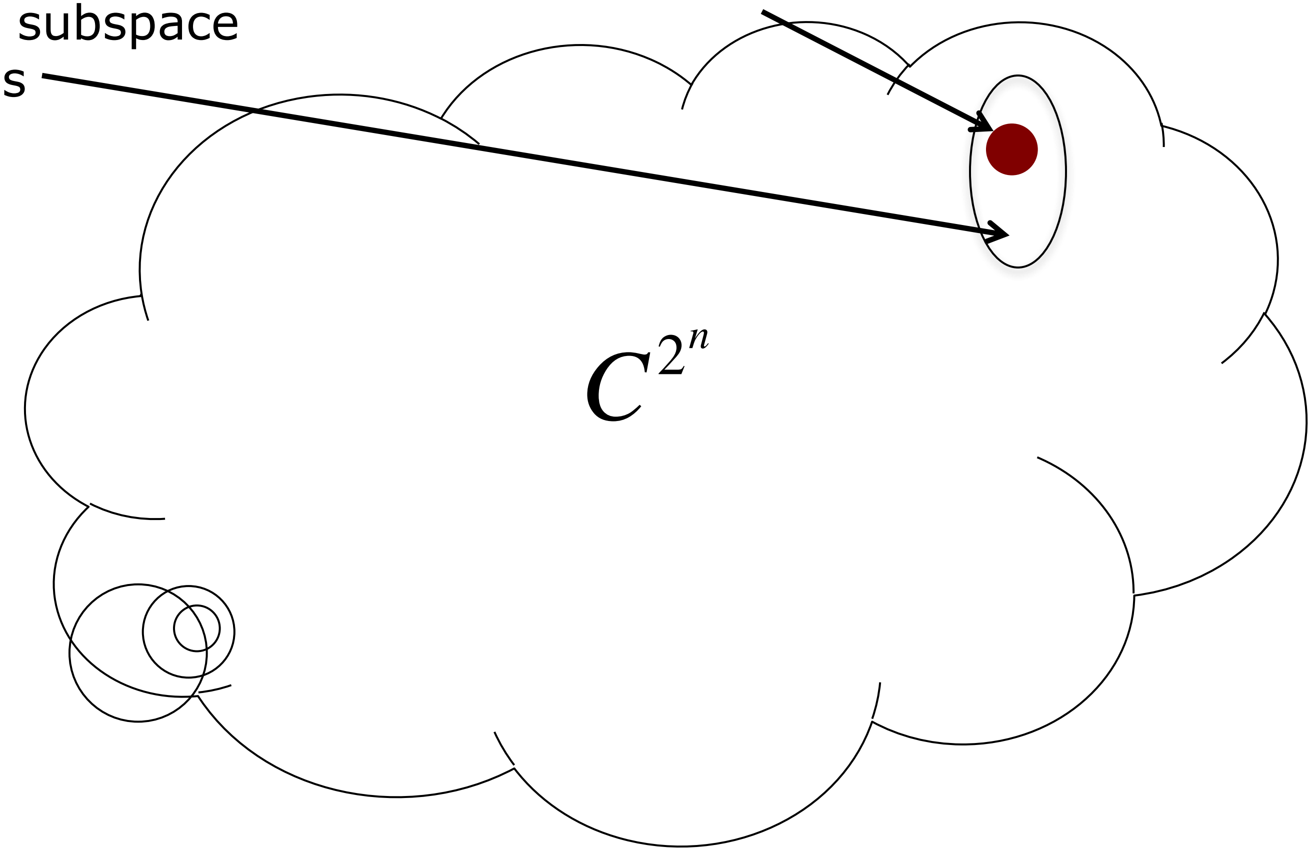
- $n^{O(\log n)}$ algorithm for computing MPS descriptions of low energy states of 1D systems in gapless case (poly(n) eigenstates of energy $\varepsilon_0 + c$).

  Implies area law (up to log correction) + succinct MPS description.

What we would like!

Subspace T of low energy states

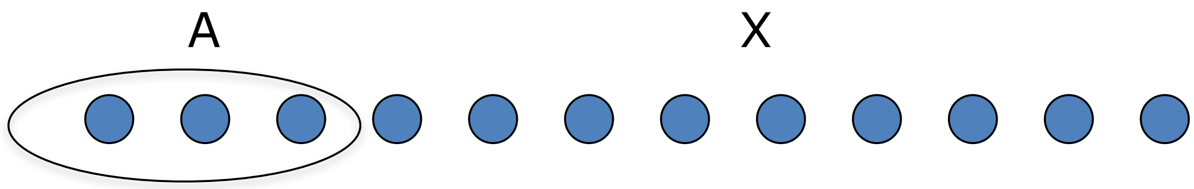
Quickly identify a small subspace in which the solution is guaranteed to lie



H = $H_1 + H_2 + \ldots H_{n-1}$

Once this is done, can compute eigenvectors quickly!

Local Approach



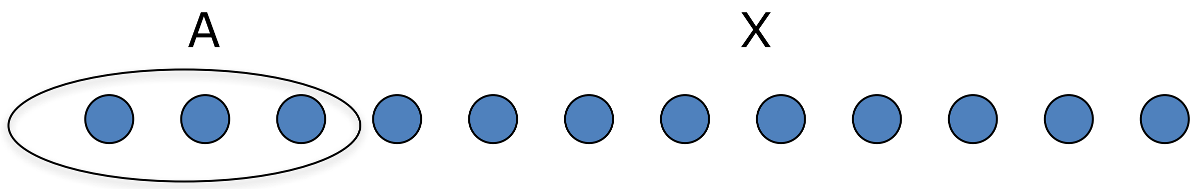
Suppose we are trying to find unique ground state $|\psi>$

What does a partial solution on the first few particles look like?

If $|\psi\rangle = \sum c_i |a_i\rangle \otimes |b_i\rangle, \quad c_i \geq 0$

then partial solution looks like S = span{ $|a_i>$ }

Local Approach



T = Target subspace of low energy states

Ideally: Identify a subspace $S \subseteq T$ such that $T \subseteq S \otimes X$

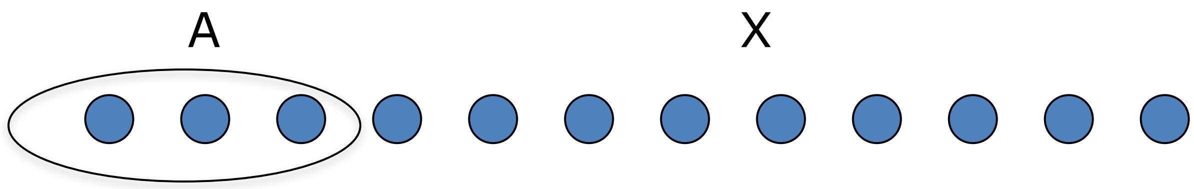
Definition: We will say that S is a δ −viable set if

$$P_T P_{S \otimes X} P_T \geq (1 - \delta) P_T$$

δ = error. $(1 - \delta)$ = overlap

Want: dim(S) small, δ small.

Local Approach



Definition: We will say that S is a δ −viable set if

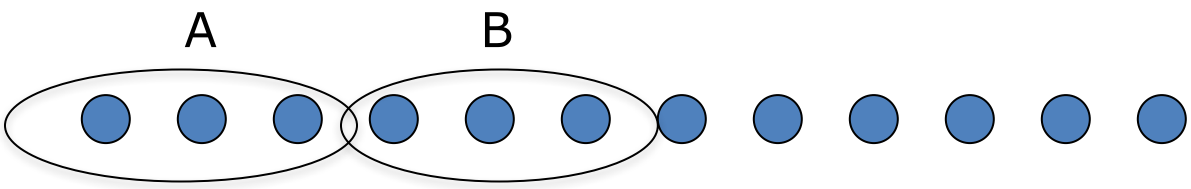
$$P_T P_{S \otimes X} P_T \geq (1- \delta) P_T$$

Interpretation: A unit ball of $S \otimes X$ when projected onto T contains a ball of radius $(1- \delta)$

Viable sets

- Think of S interchangeably as subspace or as a a compactly represented basis for the subspace.
- δ = error.   (1- δ) = overlap
- No efficient test for whether a set is viable or to estimate δ

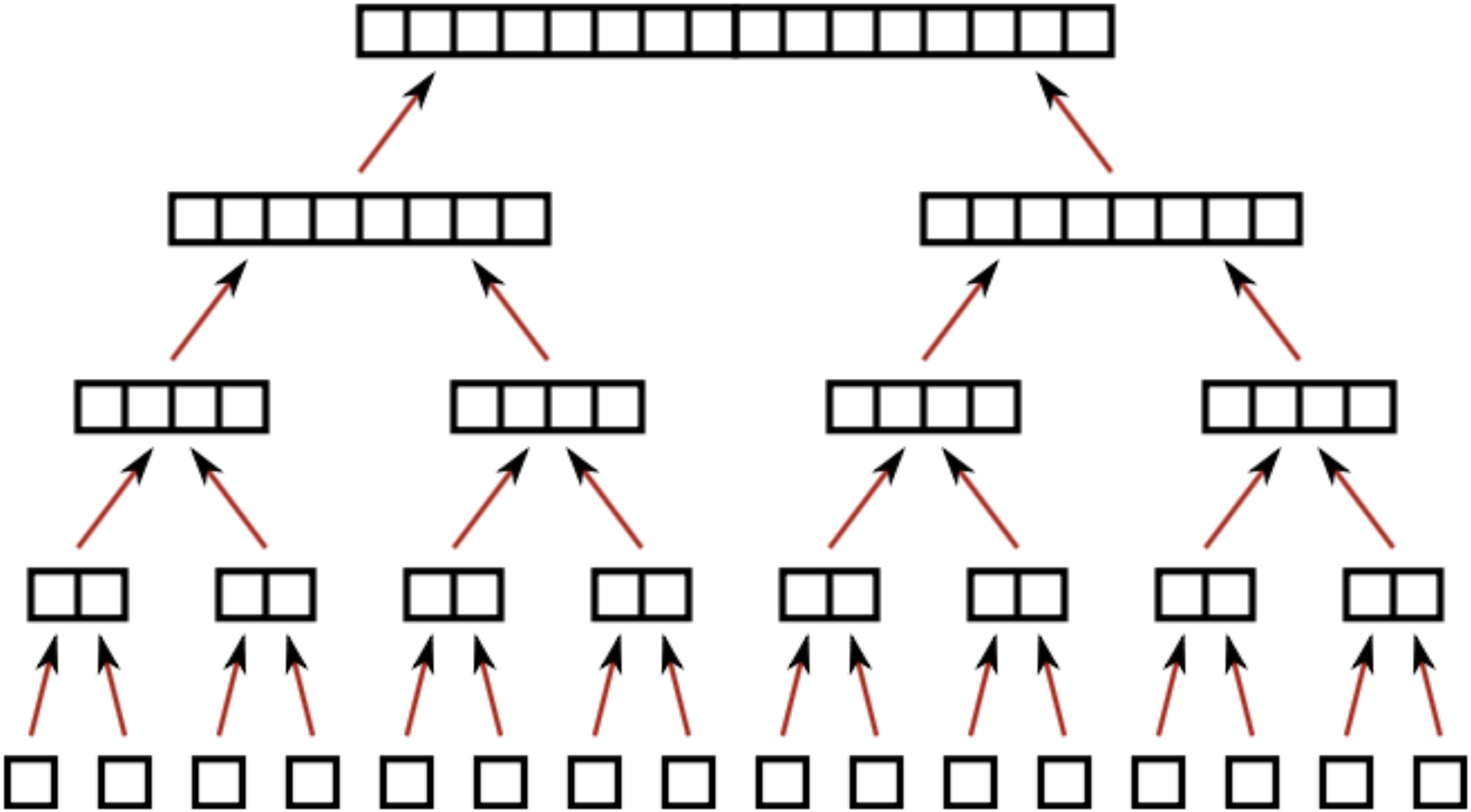
Algorithm design primitives for viable sets:



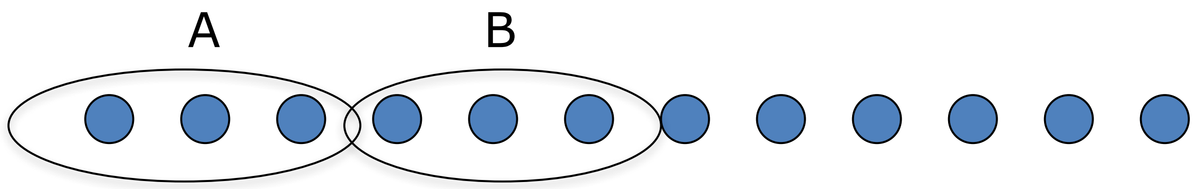
1. Tensoring

   If S_1 is δ_1-viable and S_2 is δ_2-viable then $S_1 \otimes S_2$ is $(\delta_1+\delta_2)$-viable

   $\mathrm{Dim}(S_1 \otimes S_2) = \mathrm{Dim}(S_1)\ \mathrm{Dim}(S_2)$



Algorithm design primitives for viable sets:

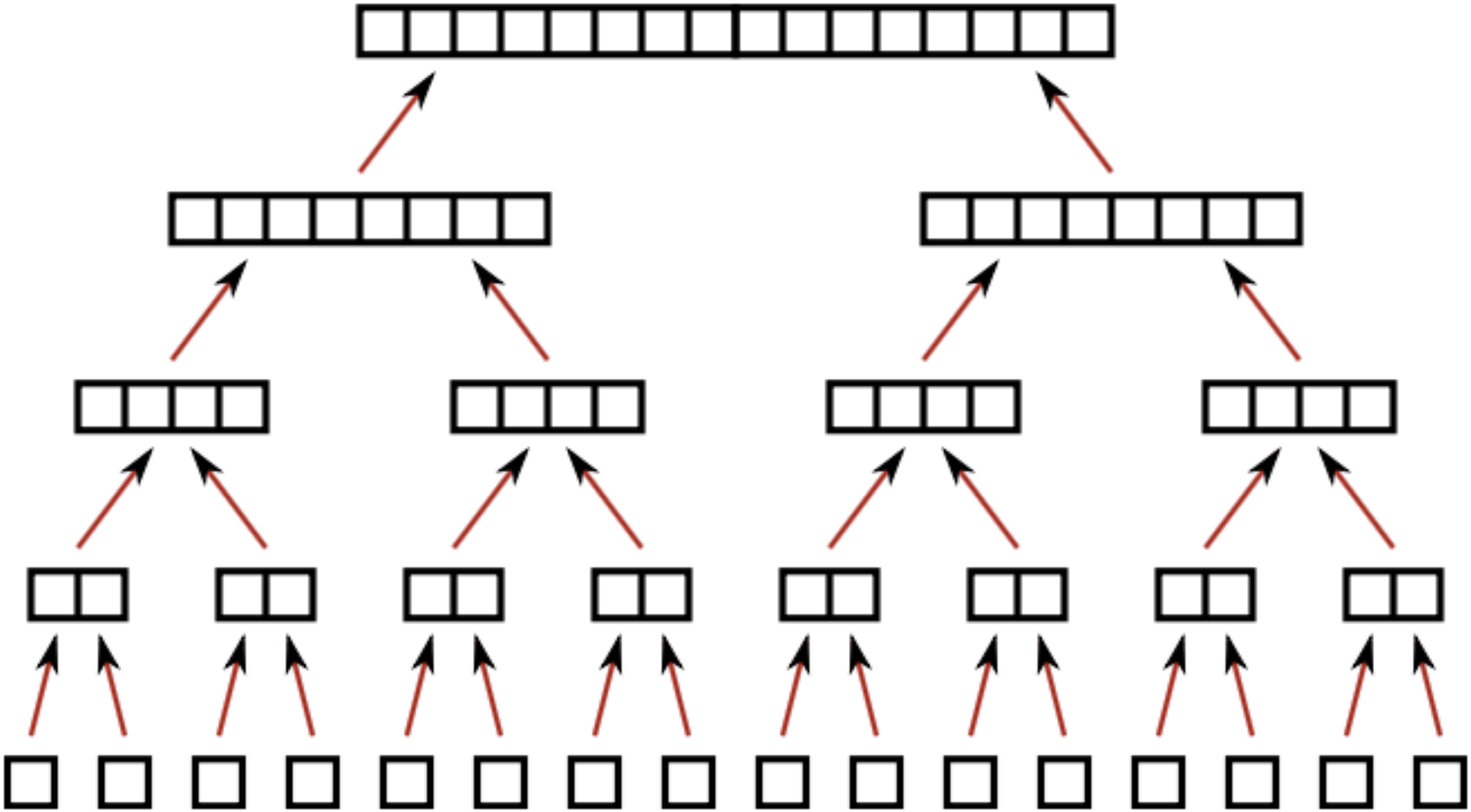


2. Random projection

If S is δ-viable i.e. overlap = 1- δ, and Dim(S) = s,

Let R be a random subspace of S of dimension r.

Then R has overlap ~ (1- δ) r/s



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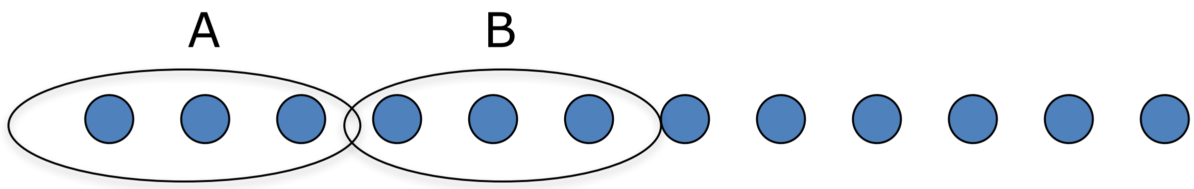
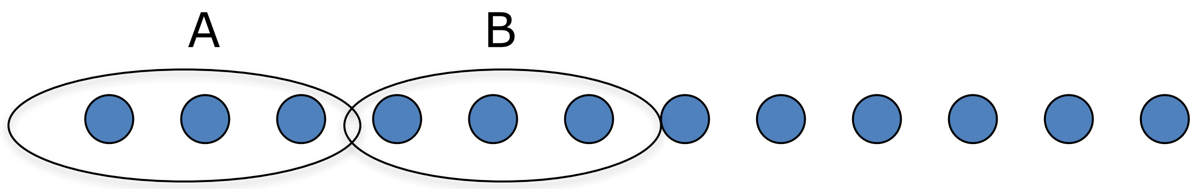


3. Error Reduction

If S is δ-viable and Dim(S) = s. Then error reduction yields S' which is $\Delta/(1-\delta)^2$-viable and with $\mathrm{Dim}(S') = D^2 s$.

Error reduction carried out by applying D-Δ AGSP, which satisfies $D^{16}\Delta << 1$

Merge Process

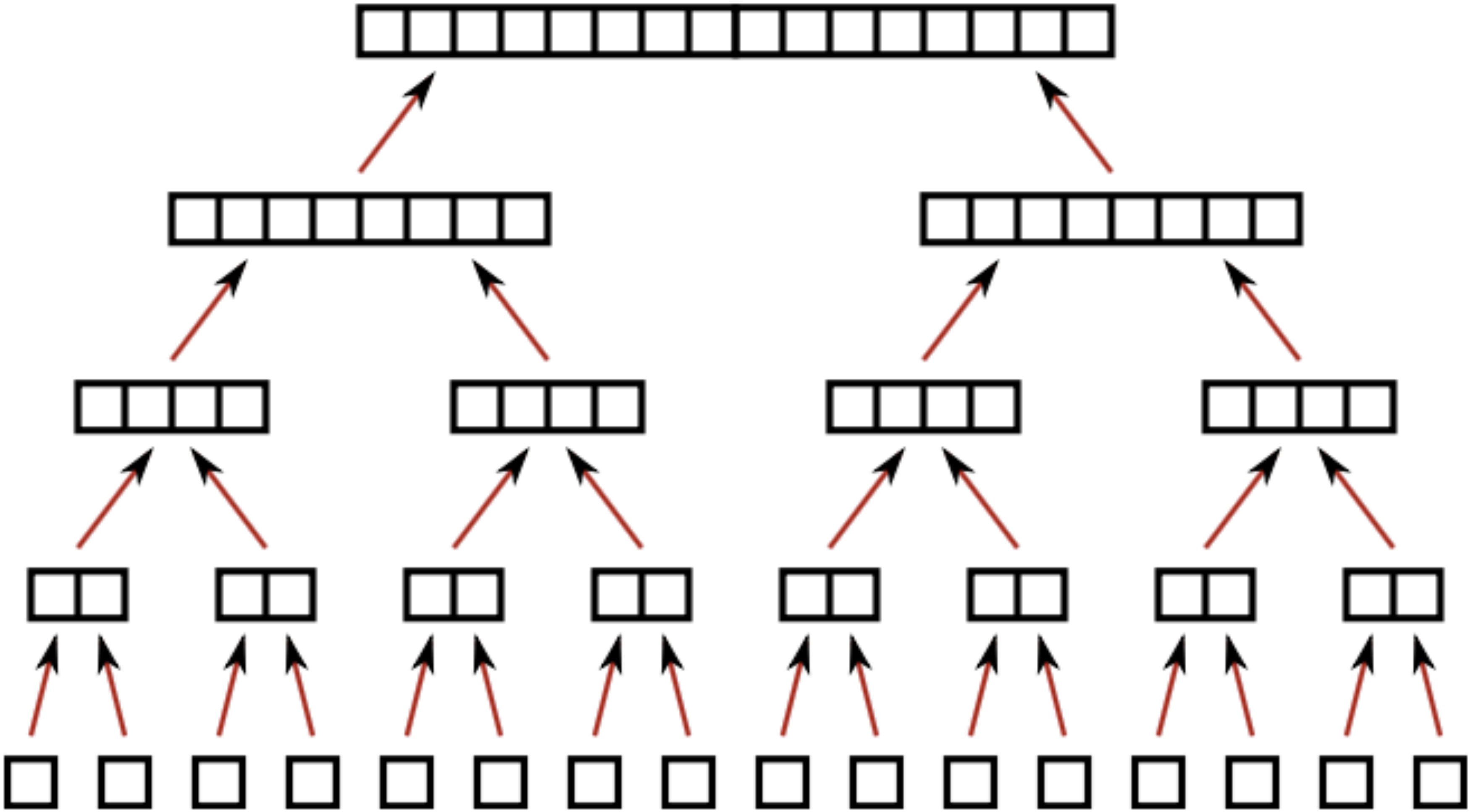


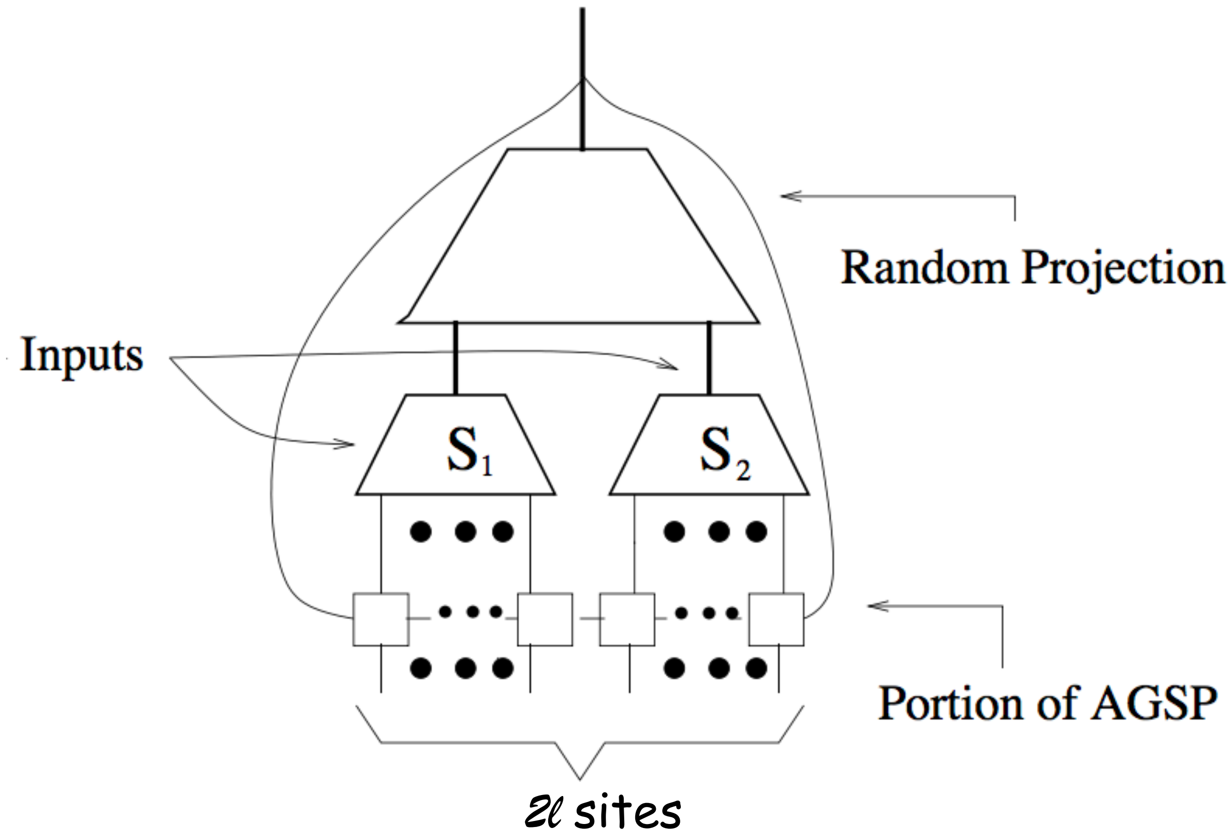
Step 1: Tensor S_1, S_2 δ-viable, each of Dim = s,
to get $S_1 \otimes S_2$ 2δ-viable and of Dim = s^2

Step 2: Random projection to Dim = r
to get $\sim(1- r/s^2)$-viable

Step 3: Error reduction to Dim = s = $D^2 r$ to get
$\Delta/\ (1- \delta)2$ -viable

$$\Delta s^4/r^2 << s^4/r^2\ r^8/s^8 = r^6/s^4 << 1$$





Random Projection
Inputs
S_1
S_2
Portion of AGSP
2ℓ sites

Results:

- Polynomial time (n^{1/c^2}) algorithm for computing basis for r = poly(n)-degenerate ground space for 1D gapped Hamiltonian.

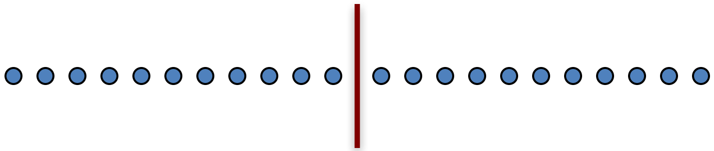
  Implies area law: $O(\log r + \frac{log^3 d}{c})$

- $n^{O(\log\ n)}$ algorithm for computing MPS descriptions of low energy states of 1D systems in gapless case (r = poly(n) eigenstates of energy $\varepsilon_0 + c$).

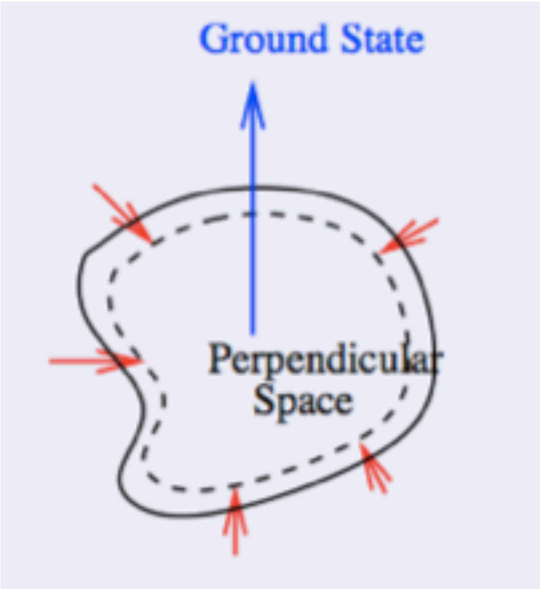
  Implies area law (up to log correction):

$$O(\log r + \frac{log^3 d}{c}) log n$$

AGSP: Approximate Ground State Projector



An AGSP is an operator K that is not "too complex" and approximately projects onto the ground state:



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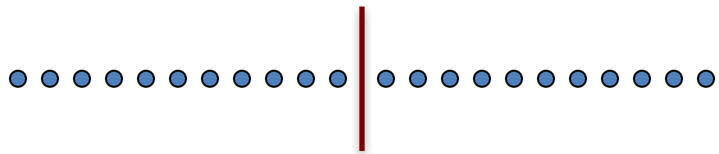
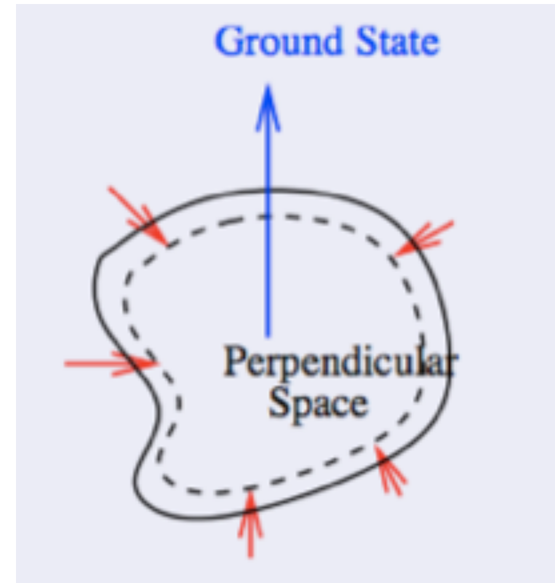
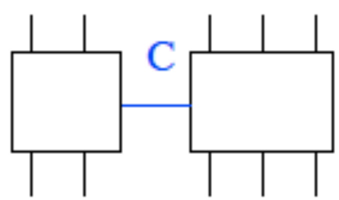




An AGSP is an operator K that is not "too complex" and approximately projects onto the ground state:

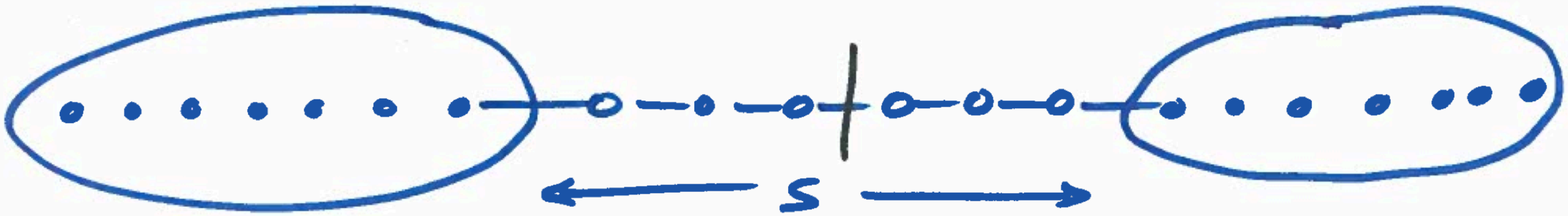
- K|GS> = |GS>
- Shrinks orthogonal space by $\Delta < 1$
- Has low entanglement rank D: $D\Delta << 1$

An operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the form $\sum_1^C A_i \otimes B_i$ will be said to have entanglement rank C.



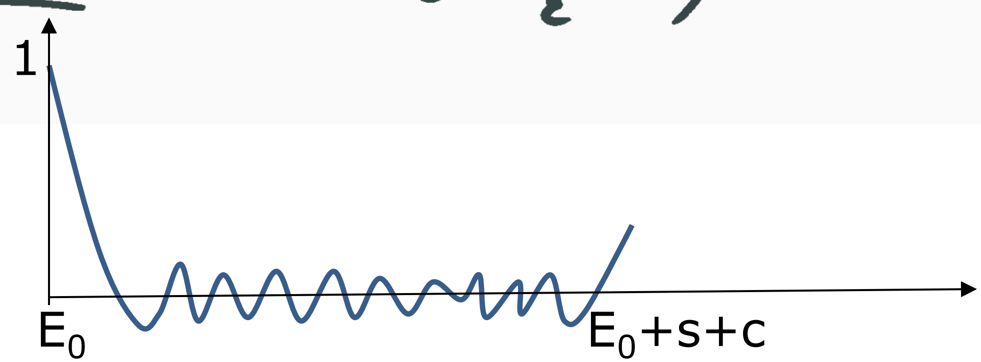
AGSP Construction

[Arad, Kitaev, Landau, V '13]



$$H' = H_L + H_1 + H_2 + \cdots + H_s + H_R$$

Theorem: $s = O\left(\frac{\log^2 d}{\varepsilon}\right)$, then $S_{1D} = O\left(\frac{\log^3 d}{\varepsilon}\right)$



$K = C_l(H')$, where C_l is a scaled Chebyshev polynomial

Must modify AGSP construction to ensure:

1. Bond dimension away from special cuts is O(poly(n))

2. Computationally efficient

Discussion

Implementation of heuristic version by
[Roberts, Vidick, Motrunich]:

- Uses heuristic version of AGSP, trotter expansion of e^{-kH/t}

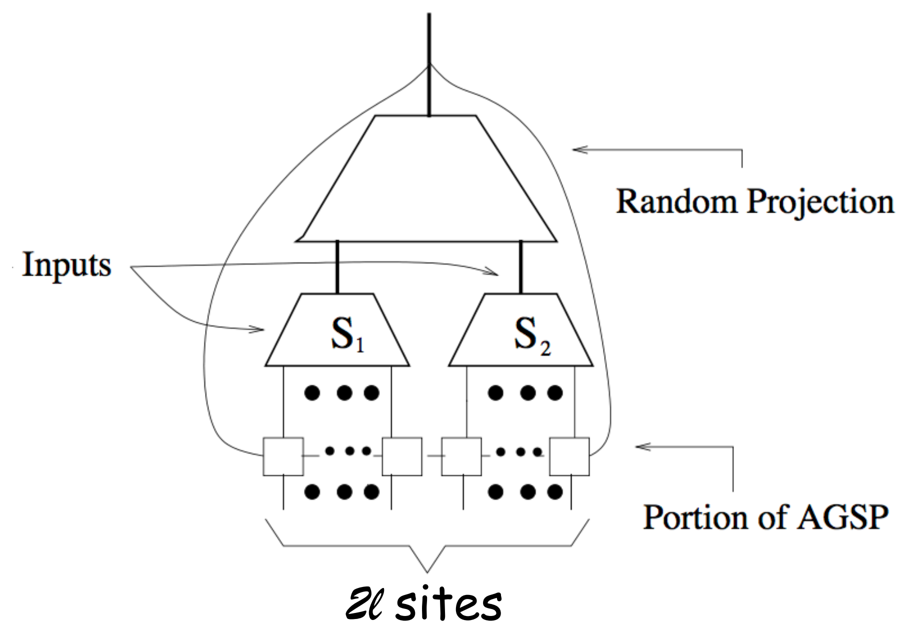
- Random sampling replaced by taking bottom s minimal eigenvectors of restricted Hamiltonian

- 5-10 times slower than DMRG

- RRG appears to do better when the ground state degeneracy is large or when reconstructing low energy space (Bravyi-Gosset model, XY model with random couplings)

- For frustration-free Hamiltonians with unique ground state, the algorithm works in ~O(nM(n)) time. If a conjecture about bond trimming is correct, then the running time can be reduced to ~O(n).

- Renormalization



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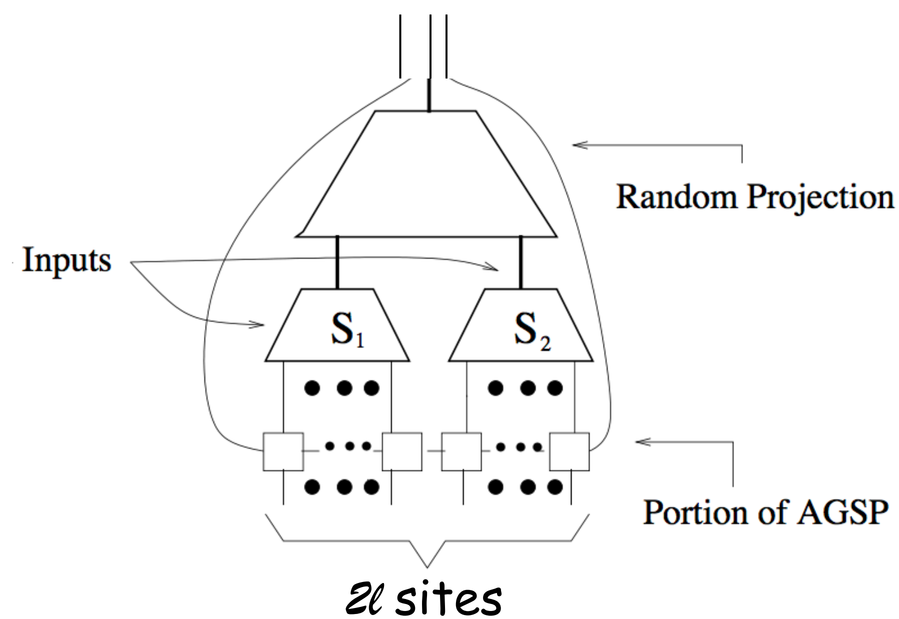

- Random projection step acts on a bond in the tensor network.
- Consider a toy model for quantum states where we think of an underlying physically manifested tensor network. Suppose the bonds of the tensor network are subjected to a noise process, described by random projections. What is the computational complexity of this model? BQP? BPP?
  i.e. does it support fault-tolerance?