From TQFTs to CFTs: the tensor network approach

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Outline

- Tensor networks: symmetries
  - Symmetry protected topological phases
  - Matrix product operator algebras
  - Topological phases
  - Topological sectors and the Drinfield center

- Edge physics
  - Entanglement Hamiltonians
  - Strange correlators: from TQFT to CFT
  - Revisiting real-space renormalization group methods

References

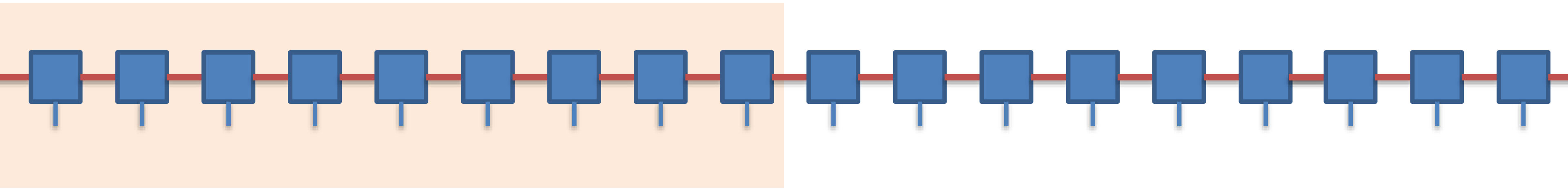
- Topological order in 2+1D
    - Tensor Fusion Category: Drinfeld, Turaev, Viro, Ocneanu, … '90s
    - Tensor Fusion Category in physics: Kitaev, Freedman, Wang, …: '98
    - String nets and Turaev-Viro state sums : Levin, Wen '04
    - SPT: Chen, Gu, Liu, Wen '11
    - SET: Barkeshli, Bonderson, Cheng, Wang '14

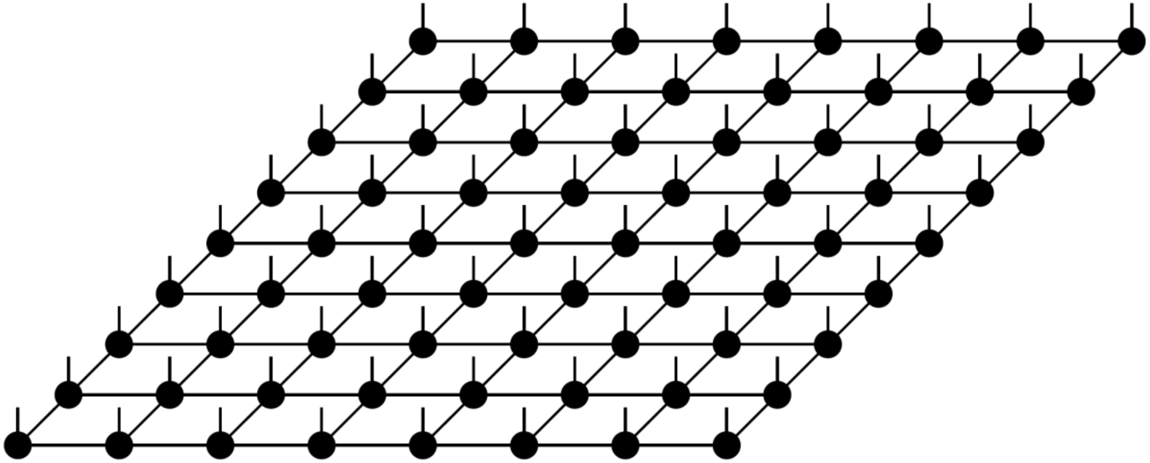
- Tensor Networks and topological order:
    - PEPS: FV, Wolf, Perez-Garcia, Cirac '06
    - MERA: Aguado, Vidal '08
    - Symmetries and TO: Gu, Levin, Wen '09
    - G-injectivity: Schuch, Cirac, Perez-Garcia '10
    - MPO-injectivity: Buershaper '14; Sahinoglu, Williamson, Bultinck, Marien, Haegeman, Schuch, FV '14

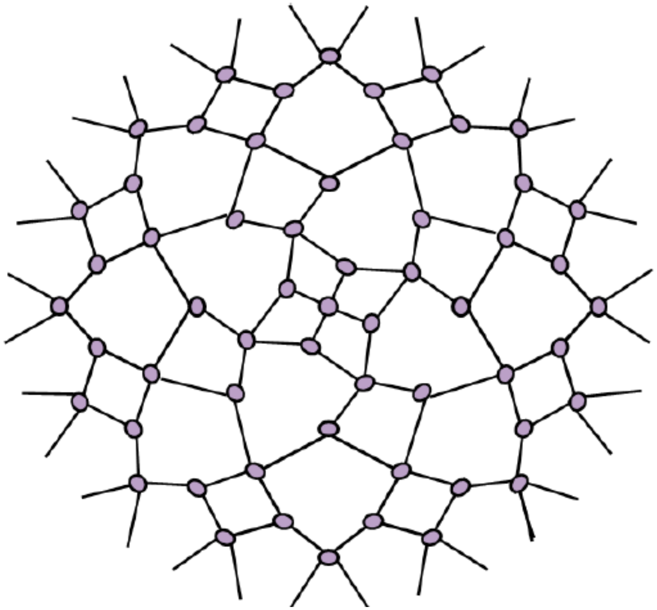
- From TQFT to CFT:
    - Fuchs, Runkel, Schweigert: '00-'07
    - Aasen, Mong, Fendley '16
    - Strange correlators: You, Bi, Rasmussen, Slagle, Xu '14; Scaffidi, Ringel '16

Tensor Networks

- MPS (Fannes, Nachtergaele, Werner '92; White '92), PEPS (FV, Cirac '04), MERA (Vidal '07)







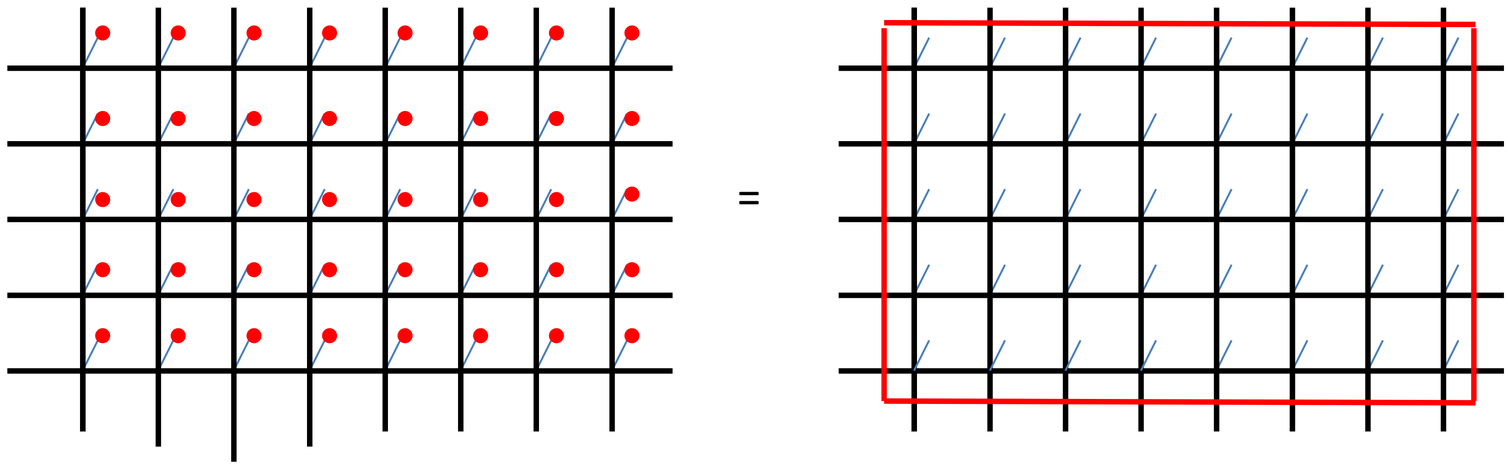
Tensor Networks

- Mantra of tensor networks:
  - Model entanglement structure of many body wavefunctions
  - Different phases of matter can be distinguished by the different ways in which symmetries can act on those entanglement degrees of freedom
    - Landau’s paradigm: phase transition = symmetry breaking

- Those symmetries can act in a local or nonlocal (anomalous) way on the auxiliary Hilbert space, but in both cases we can represent them using matrix product operators

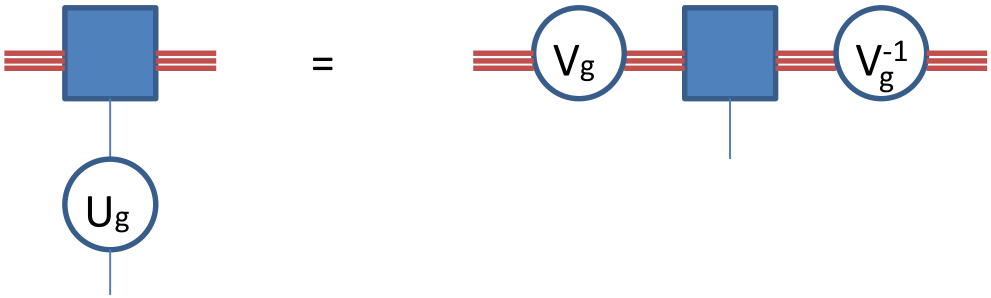
Entanglement Hamiltonians and edge physics

- PEPS / MERA yield a natural tensor product structure for the entanglement Hamiltonian and for describing physics at the edge
- The entanglement and edge Hamiltonian inherit all symmetries of the physical system (but can exhibit more symmetries!)



Example 1: MPS

- In case of gapped 1D quantum spin systems with a global symmetry, the entanglement degrees of freedom transform according to a projective representation of the global symmetry:



$$V_g . V_h = e^{i\omega(g,h)} V_{g.h}$$

  - Second cohomology group $H^2(G,U(1))$ classifies the different projective representations, and 2 wavefunctions transforming according to a different representation cannot adiabatically be connected without crossing a phase transition: SPT classification of 1D quantum spin chains
  - Similar argument for reflection, time reversal, …

Perez-Garcia, Wolf, Sanz, FV, Cirac '08
Pollmann, Turner, Berg, Oshikawa '10; Chen, Gu, Wen '11; Schuch, Perez-Garcia, Cirac '11

- Caveat: argument only works when the MPS is injective

Graded MPS: Majorana modes

- In case of 1D fermionic systems: Z_2 graded tensor product
  - Leads to 2 different classes of "irreducible" MPS, namely the ones carrying even and odd parity (2 types of simple Z_2 graded algebras):

$$|\psi\rangle_e = \sum_{\{i\}} \mathrm{tr}(\mathcal{P} A^{i_1} A^{i_2} \ldots A^{i_N}) |i_1\rangle |i_2\rangle \ldots |i_N\rangle. \qquad \mathcal{P} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$A^i = \begin{pmatrix} B^i & 0 \\ 0 & C^i \end{pmatrix} \quad \text{if } |i| = 0, \qquad A^i = \begin{pmatrix} 0 & D^i \\ F^i & 0 \end{pmatrix} \quad \text{if } |i| = 1,$$

$$|\psi\rangle_o = \sum_{\{i\}} \mathrm{tr}(Y A^{i_1} A^{i_2} \ldots A^{i_N}) |i_1\rangle |i_2\rangle \ldots |i_N\rangle. \qquad Y = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

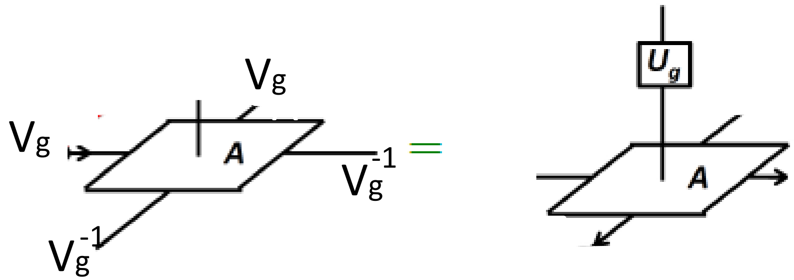
$$A^i = \begin{pmatrix} B^i & 0 \\ 0 & B^i \end{pmatrix} = \mathbb{1} \otimes B^i \quad \text{if } |i| = 0, \qquad A^i = \begin{pmatrix} 0 & B^i \\ -B^i & 0 \end{pmatrix} = y \otimes B^i \quad \text{if } |i| = 1,$$

  - The second type represents Majorana physics, and corresponds to "cat" states protected by superselection rules
  - Combined with time-reversal: yields Z_8 classification of Kitaev and Fidkowski

$$\bar{A}^{ij} = \mathcal{T}^{-1} A^{ij} \mathcal{T}, \qquad \mathcal{T}\bar{\mathcal{T}} = (-1)^{\kappa} \mathbb{1}, \qquad \mathcal{T}\mathcal{P} = (-1)^{\mu} \mathcal{P}\mathcal{T}, \qquad \mathcal{T}\bar{Y} = (-1)^{\mu} Y \mathcal{T}$$

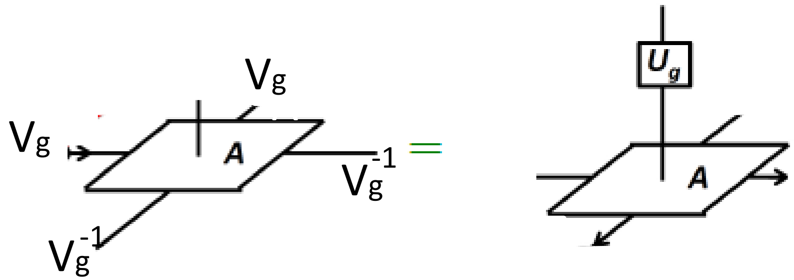
Bultinck, Williamson, FV '16; Kapustin, Turzillo, You '16

Symmetries in PEPS

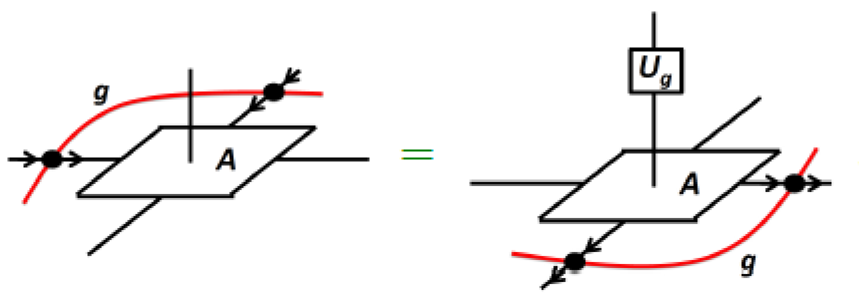


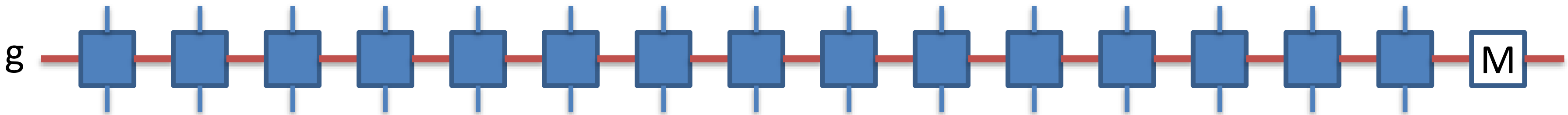
- Not good enough: the V_g's do not have to form a representation

Symmetries in PEPS



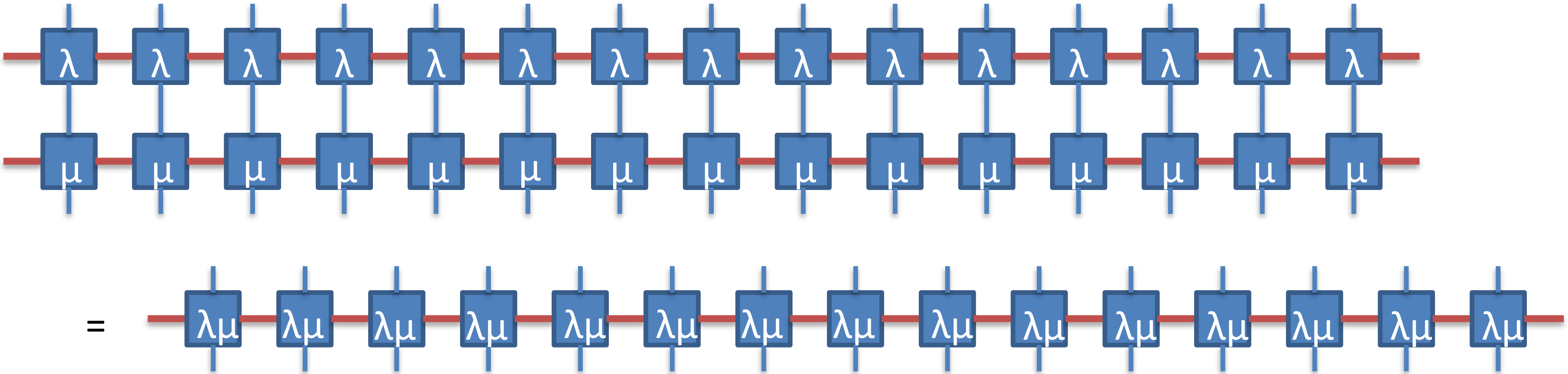
- Not good enough: the V_g's do not have to form a representation
- Correct way: pulling through of matrix product operators which form representation of group:



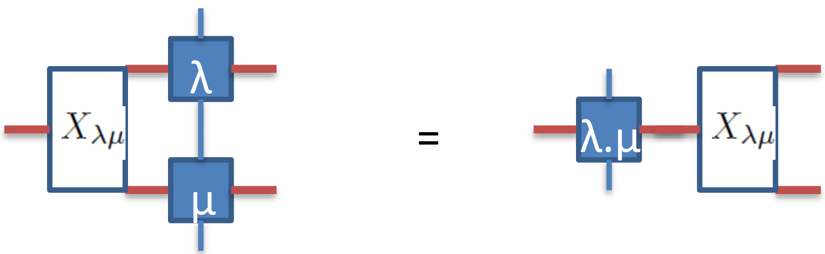


Williamson et al. '16

- MPOs form representation of the group: $O_\lambda . O_\mu = O_{\lambda.\mu}$



- Fundamental theorem of MPS: zipper condition



- Associativity leads to existence of F-symbols which satisfy the 3-cocycle condition
- MPO representation of a group are equivalent to Chen, Gu, Liu and Wen's classification of SPT phases in terms of $\mathcal{H}^3(G, U(1))$

PEPS and SPT phases

- Nontrivial SPT phases can hence be represented by PEPS with a nontrivial MPO symmetry
  - If the 3-cocycle is nontrivial, then this symmetry is represented in a NONLOCAL way on the entanglement degrees of freedom, and hence also on the edge

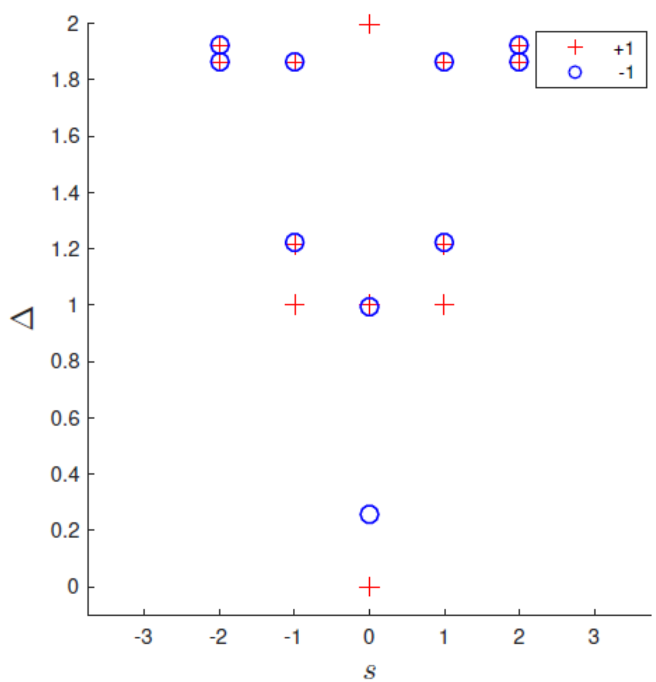
- Xie Chen's theorem (X. Chen '11):

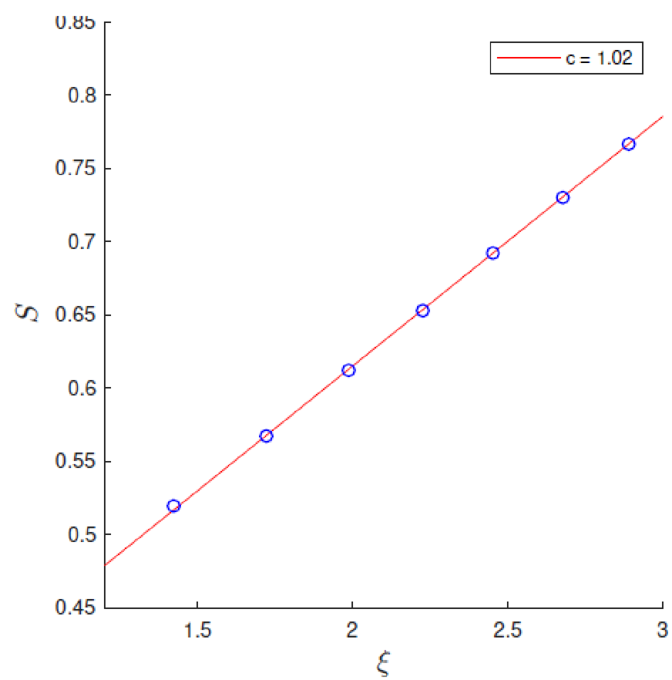
  - A local 1D Hamiltonian with a discrete MPO symmetry corresponding to a nontrivial 3-cocycle must be either gapless or symmetry broken
    - Argument goes by proving that an injective MPS cannot exhibit a nontrivial MPO symmetry
  - Discrete analogue of gaplessness of WZW model

- Consequence: the entanglement Hamiltonian and edge of an SPT PEPS will be gapless !

Example: Z_2 SPT phase

- CZX model (Chen et al.) is the ground state of a frustration free model, hence the entanglement spectrum is bimodal.
- We considered a PEPS wich is a perturbation of that model such that symmetries are preserved, but with a correlation length and hence dynamics, and calculate its entanglement spectrum
  - Numerics indicate that the entanglement Hamiltonian is indeed described by a CFT (compactified boson c=1,R=2)





Bultinck, Vanhove, Haegeman, FV ‘17

Grading the MPO algebra: SET phases

- Just as in case of MPS, we can relax the requirement of injectivity, and define the group elements as sums of injective (irreducible) MPOs

$$O_g O_h = O_{gh} \qquad O_g = \sum_x O_{gx}$$

- Example:
  - Suppose I can find set of MPOs satisfying

$$O_\sigma . O_\sigma = I + O_\psi$$
$$O_\sigma . O_\psi = O_\psi . O_\sigma = O_\sigma$$
$$O_\psi . O_\psi = I$$

  - Then we can form a graded MPO algebra of Z_2 by defining

$$O_1 = I + O_\psi \qquad O_2 = O_\sigma$$

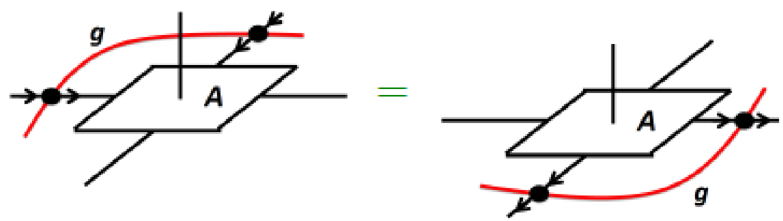
- Consistency (associativity) conditions of those sets of MPOs leads to the concept of graded tensor fusion categories and symmetry enhanced topological order (SET)

Barkeshli, Bonderson, Cheng, Wang '14
Williamson, Bultinck, FV '17

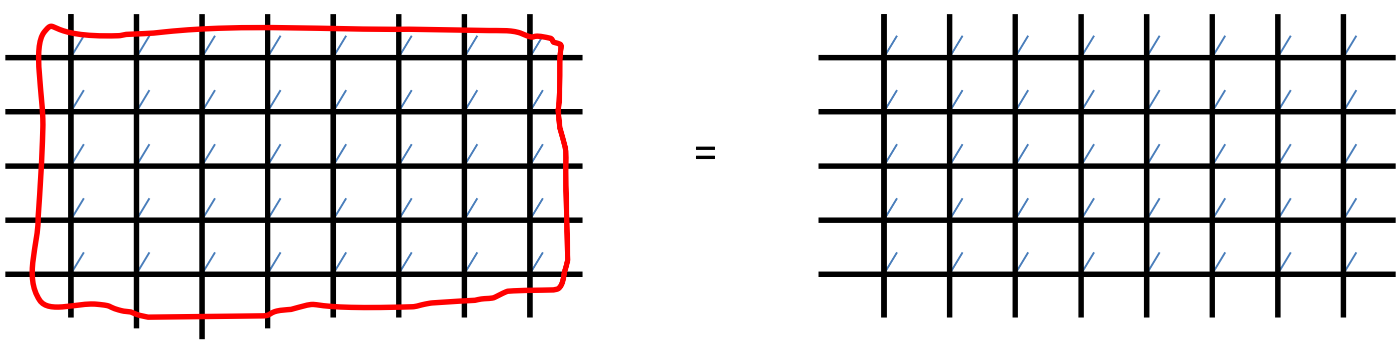
- Topological phases are obtained by having just 1 group element:

$$O^2 = O \qquad O = \sum_x w_x O_x \qquad \sum_{xy} w_x w_y O_x O_y = \sum_z w_z O_z$$

- The point is now that the theory of PEPS imposes that all O_x have to satisfy the pulling through equation without an action on the physical level

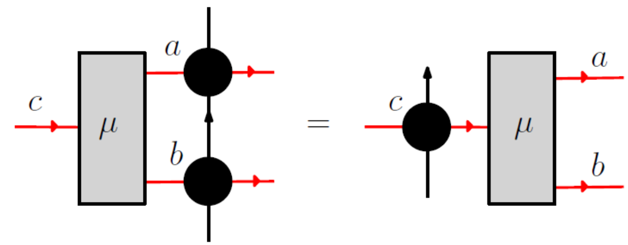


- This immediately implies a topological correction to the area law for the entanglement entropy: $S(A) = c.\partial A - \log\sqrt{\sum_i d_i^2}$ (Kitaev, Preskill '06; Levin Wen '06)

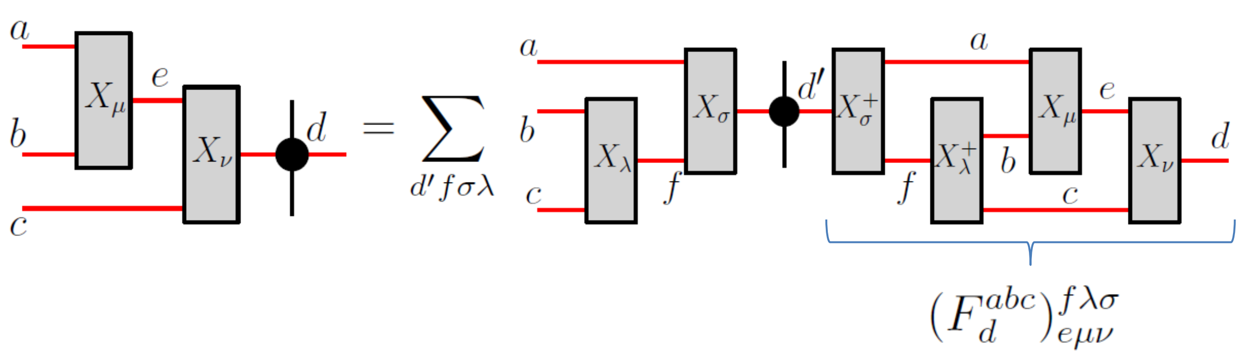


- The equality $\sum_{xy} w_x w_y O_x O_y = \sum_z w_z O_z$ combined with the fundamental theorem of MPS implies the existence of integers N_{ab}^c and the existence of fusion tensors such that

$$O_a . O_b = \sum_c N_{ab}^c O_c$$



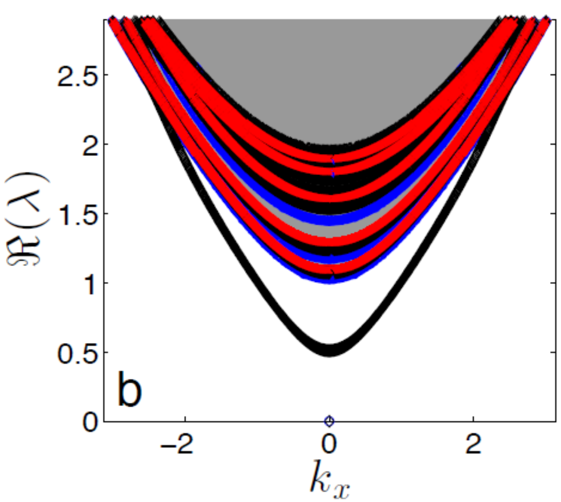
- Associativity implies the existence of F-symbols, and those on their turn have to satisfy the pentagon equation

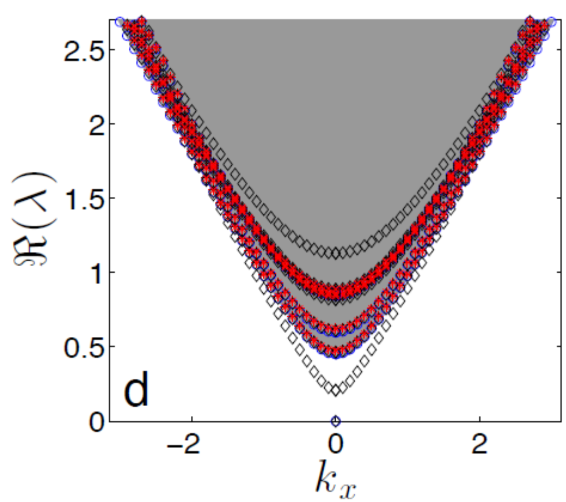


- Just like what happens in the case of Yang-Baxter equations, we can find solutions to all those equations by using the fundamental representation
  - Turns out to coincide with Turaev-Viro state sums or Levin-Wen string nets

Bultinck, Marien, Sahinoglu, Williamson, Haegeman, FV '17

- Closed algebras of MPOs form representations of tensor fusion categories, labeled by the triple $(N_{ab}^{c}, F_{d}^{abc}, \varkappa_a)$; in other words: the only consistent MPO symmetries of PEPS are precisely the ones studied in tensor fusion category
  - Topological order is completely characterized by the LOCAL symmetries of the entanglement degrees of freedom
  - But PEPS yields more than just the symmetries: there are plenty of variational parameters left open which give rise to correlation lengths and dynamics / dispersion relations / interesting edge physics
    - We can start studying topological phase transitions, …



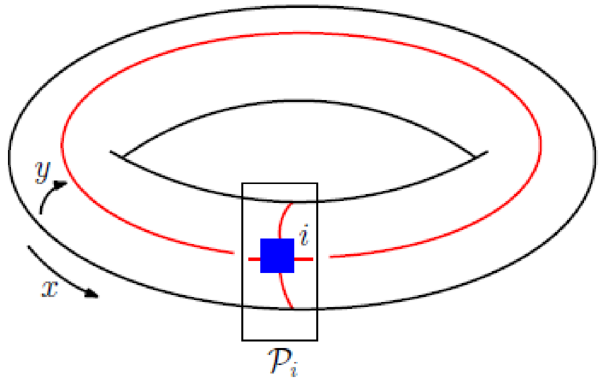


Condensation of the boson in the perturbed Fibonacci string net; critical point corresponds to minimal rational model (9,10)

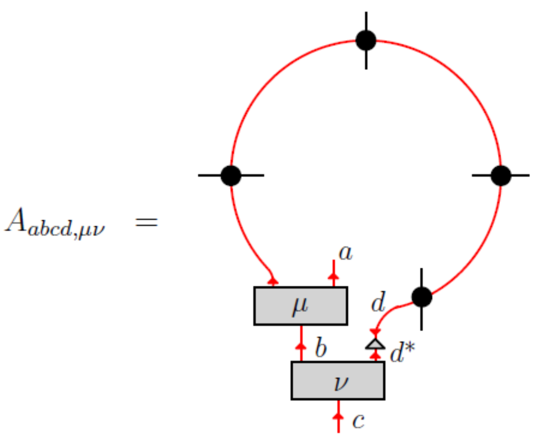
Marien, Haegeman, Fendley, FV '16

Topological sectors in PEPS

- Ground state degeneracy depends on genus: different ground states obtained by putting MPOs on nontrivial loops (acting purely on the entanglement degrees of freedom)



- All topological sectors can be obtained by considering the idempotents of an MPO algebra with 2 extra fusion tensors (a C* algebra):

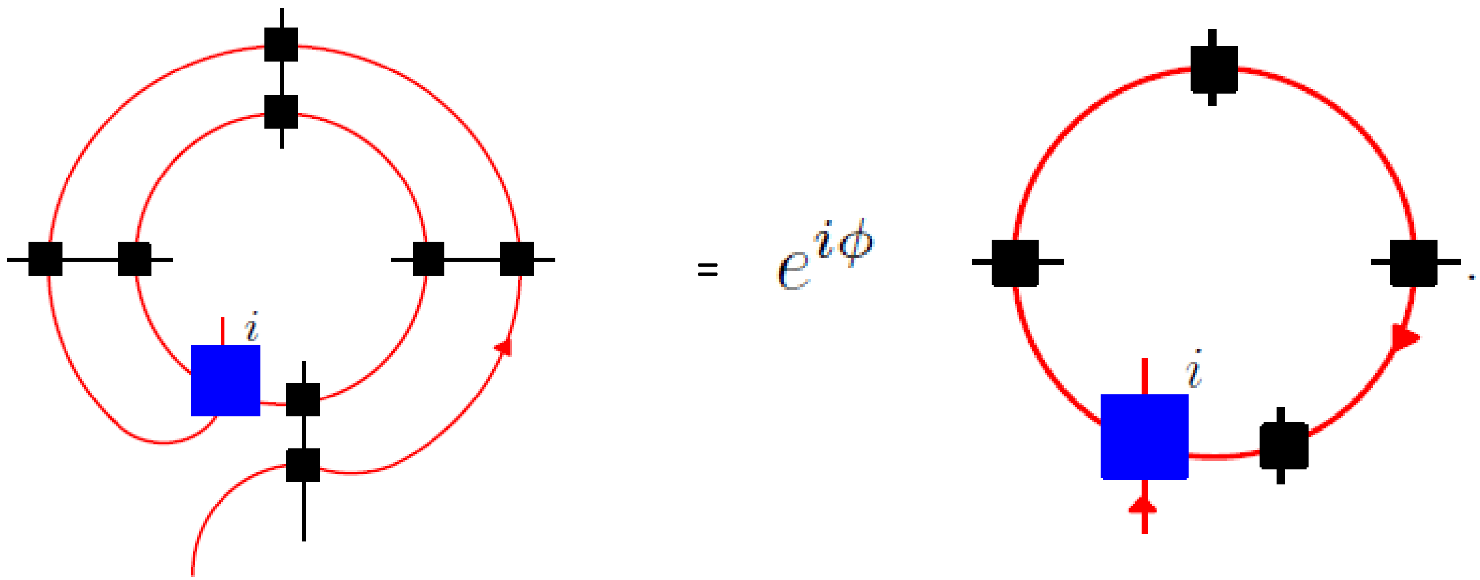


$$\mathcal{P}_i = \sum_{abd,\mu\nu} c^i_{abd,\mu\nu} A_{abad,\mu\nu}$$

- Construction parallels construction of Drinfeld center and Ocneanu's tube algebra
- Similar for SET phases!

Bultinck, Marien, Sahinoglu, Williamson, Haegeman, FV '17

- This C* algebra contains all information about the anyon excitations in the theory: those anyons realize the Drinfeld center of the input category
  - Topological spin, braiding, fusion all have clear meanings in terms of MPO algebras

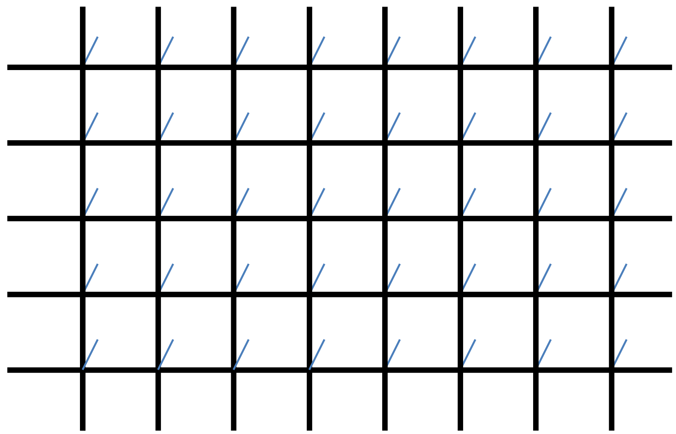


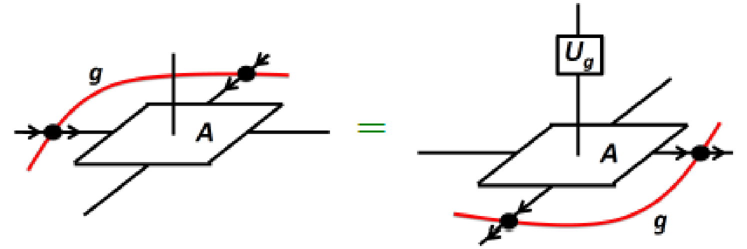
Strange correlators

- We have already seen that entanglement Hamiltonians (or edge physics) of (perturbed) topological phases gives rise to CFTs
  - Is there a more direct way of making this connection?
  - We have fusion rules, topological spins, topological sectors, … for the emerging anyons; how to map this to the corresponding data in CFTs?
  - Can we make mapping from 3D TQFT to 2D CFT explicit in terms of tensor networks?

- YZ You, Z Bi, A Rasmussen, K Slagle, Cenke Xu '14 (see also T Scaffidi, Z Ringel '16) introduced a concept which allows for such a mapping: the strange correlator
  - Their motivation: detection of SPT phases
  - Construction: take overlap of a nontrivial SPT phase with a product state with the same symmetry

Strange Correlators





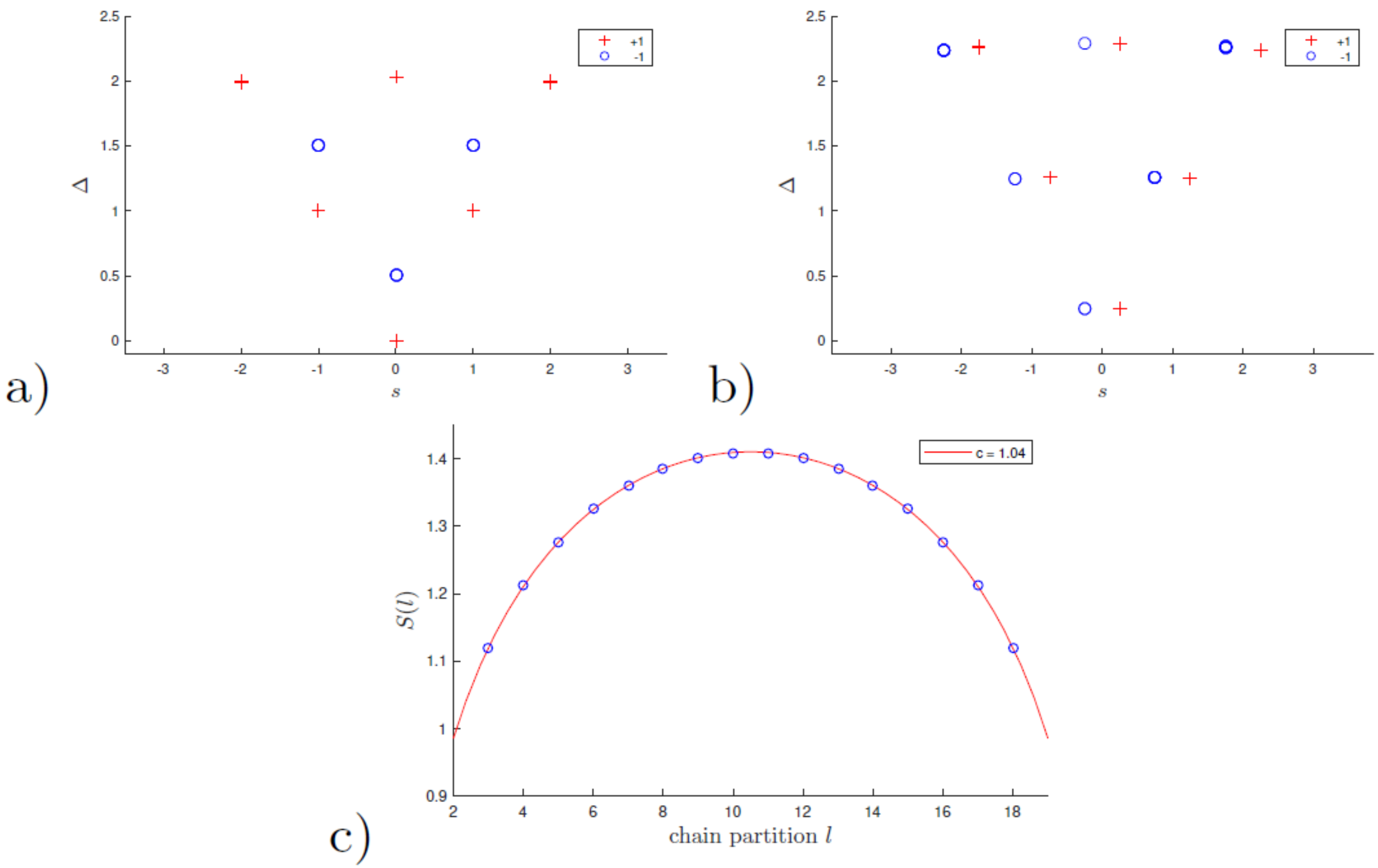
$$Z = \left(\langle x|\right)^{\otimes N} |\psi\rangle$$

$$U_g|x\rangle = |x\rangle$$

- The overlap between the SPT wavefunction and the product state with the same symmetry yields a partition function of a “classical” spin system

- The corresponding transfer matrix inherits all MPO symmetries of the SPT phase, hence Chen’s theorem implies that this partition function will be either gapless or symmetry breaking for nontrivial SPTs
  - As those are the 2 only possibilities, the critical phase will not have to be fine-tuned!
  - You also do not have to know the MPOs; their existence is enough

- Example: eigenvalues of transfer matrix of perturbed CZX model: free boson (c=1, R= $\sqrt{2}$) with periodic and twisted boundary conditions



- Defect line represents a semion topological defect
- Equivalent to $SU(2)_1$ Wess-Zumino-Witten model; Z_2 symmetry action is equivalent to exchanging g <-> -g which is anomalous

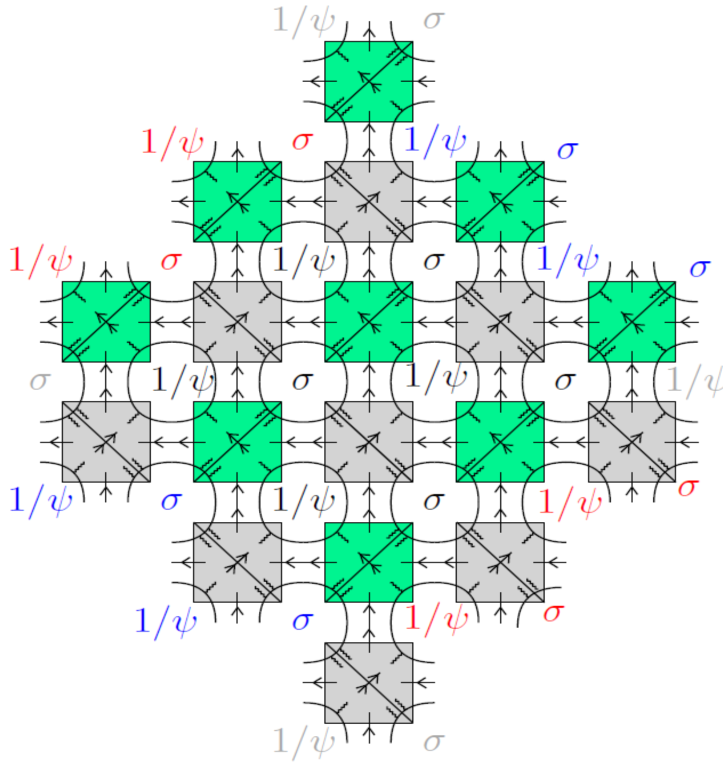
- By taking overlap between PEPS with nontrivial MPO symmetry and a trivial product state, we hence get the partition function of a CFT
  - All conformal boundary conditions can readily be implemented by putting MPO strings on the original PEPS
  - The different topological sectors correspond to the different heighest weight vectors in the CFT towers

- This PEPS strange correlator construction seems to be an explicit representation forthe theory of Fuchs, Runkel and Schweigert in which they describe CFTs using TQFT data

- It is also related to the recent work of Aasen, Mong and Fendley (‘16), where conformal sectors in classical spin systems where constructed using concepts of tensor fusion category

Classical Ising model from SET

- Partition function of classical Ising can exactly be written as a strange correlator of the Ising SET, including all defects (duality defects)
  - The product state by the requirement of the Z_2 symmetry

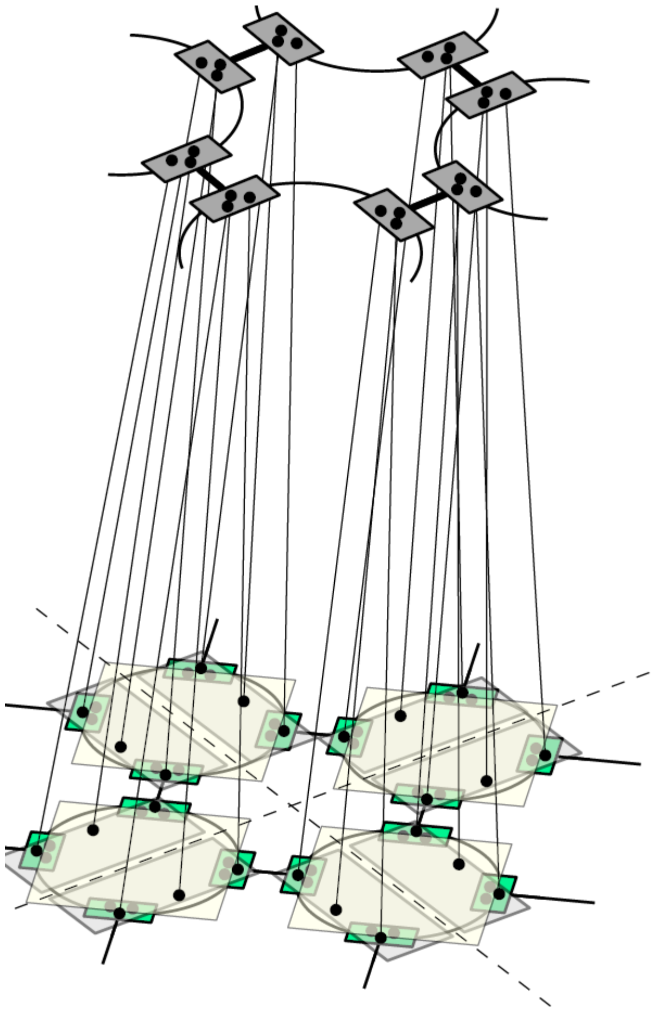


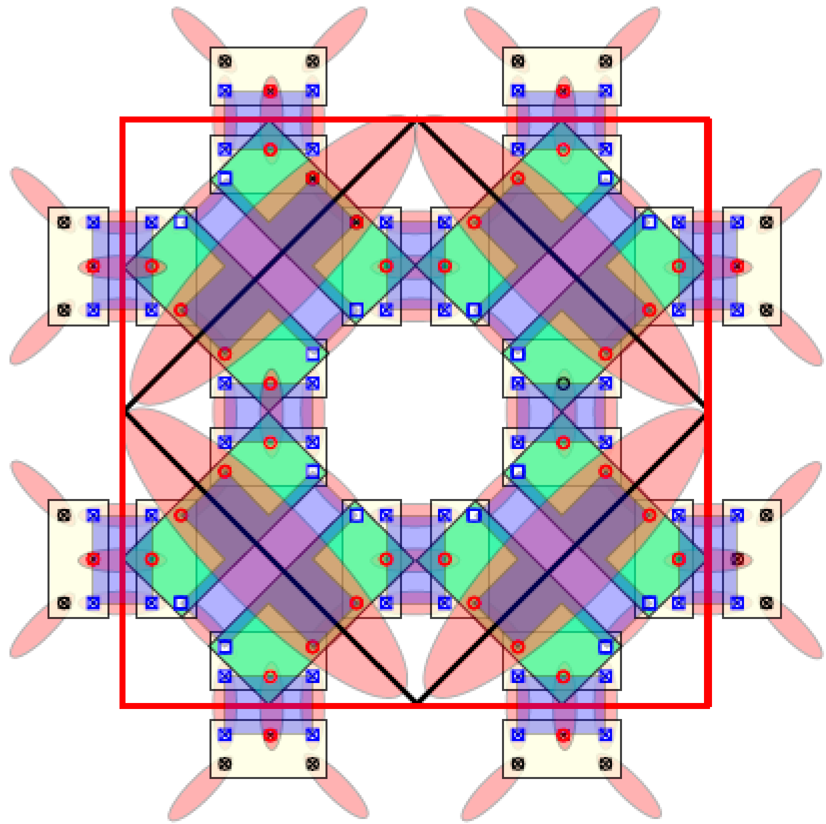
  - By including MPOs, we can realize all 9 topological/conformal sectors
  - All information about scaling exponents, primary fields, etc. is encoded algebraically in the MPO tube-algebras

Real-Space RG revisited

- This picture of a critical spin system as an overlap of 2 zero-correlation length quantum spin systems yields a new way of looking at RG:
  - The string net is a renormalization group fixed point: we can always remove/add sites without changing the topological nature (see e.g. Koenig, Reichardt, Vidal ‘09); this gives us exact “disentanglers”
  - A real space RG scheme is then obtained by applying those isometries to the product state; this way the topological/conformal sectors are always exactly conserved, and the exact critical exponents are preserved
  - Different real-space RG schemes use different ways of throwing away information such that the bond dimension does not blow up

Bal, Williamson, Bultinck, Haegeman, FV ‘17





- Kadanoff blocking = mean field ansatz: product state remains product state
- TRG: local SVD truncation of the bonds (simple update PEPS)
- TNR: keep larger unit cells, and remove entanglement in loops (but still s.u.)
- PEPS: approximate flow of product state using full information of environment

Conclusion

- Quantum phases of matter can be characterized by the symmetries of the entanglement degrees of freedom
- PEPS yield a very natural framework for studying those entanglement degrees of freedom, as they exhibit a natural tensor product structure
- Framework of tensor fusion categories emerges naturally from study of matrix product operator algebras, leading to wavefunctions representing SPT, SET and topological order, both for the spin case and the fermionic case
- Through the concept of strange correlators, PEPS provide an explicit link between 3D TQFTs and 2D CFTs,
  - Which CFTs can be described like that?
  - What about supersymmetric CFTs?