Quantum Waveform Estimation, Detection, and Noise Spectroscopy

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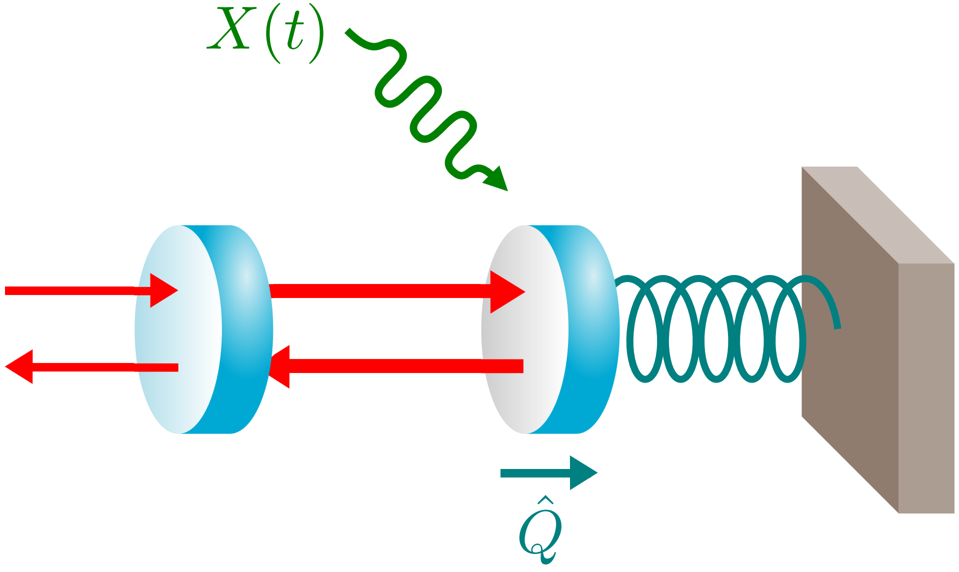
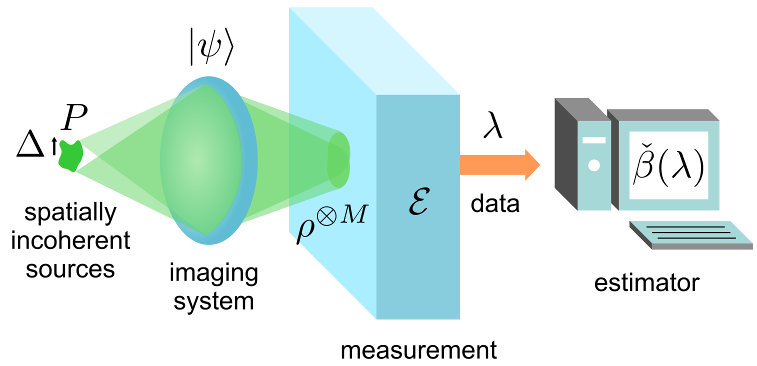
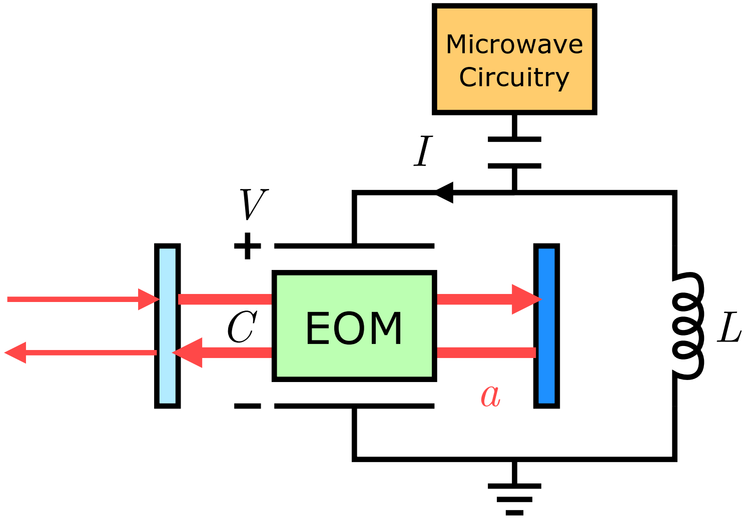
KITP, Oct 12, 2023



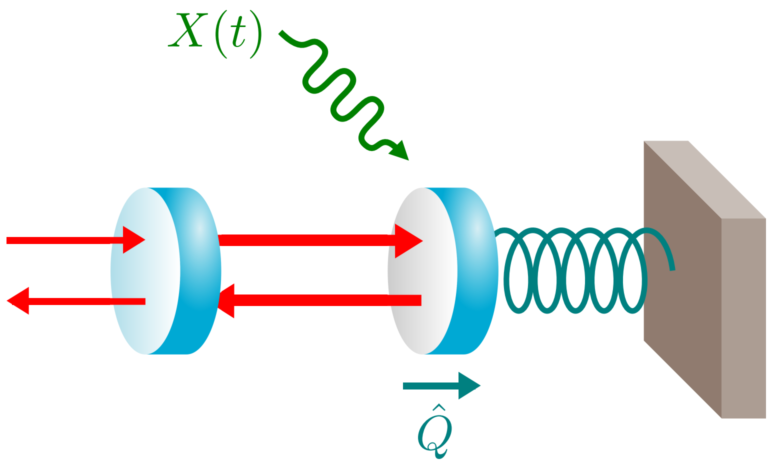
Quantum-inspired superresolution, electro-optics

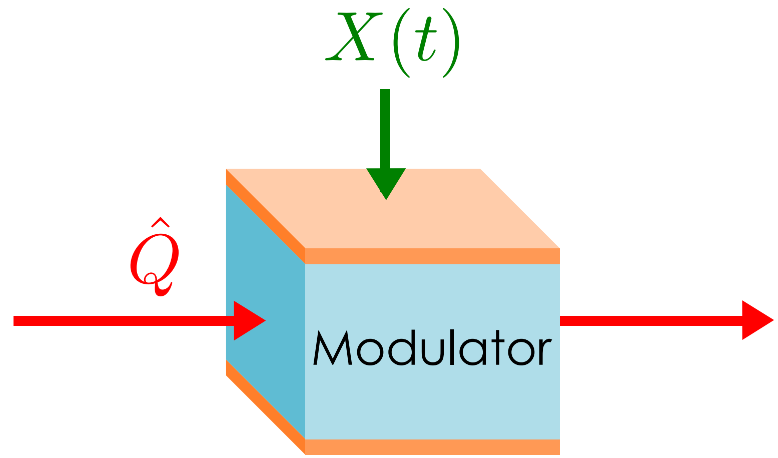
- **Quantum-inspired superresolution: Tsang, Nair, Lu, PRX (2016)**
  - 30+ experiments so far
  - https://online.kitp.ucsb.edu/online/qmetro23/tsang/
- **Tsang, "Cavity quantum electro-optics," PRA (2010,2011).**
  - Quantum transduction
  - Experiments: Rueda *et al.* (Schwefel, Erlangen), Optica (2016); Fan *et al.* (Tang, Yale), Sci. Adv. (2018); Sahu *et al.* (Fink, IST Austria), Science (2023), etc.





Sensing with a dynamical system





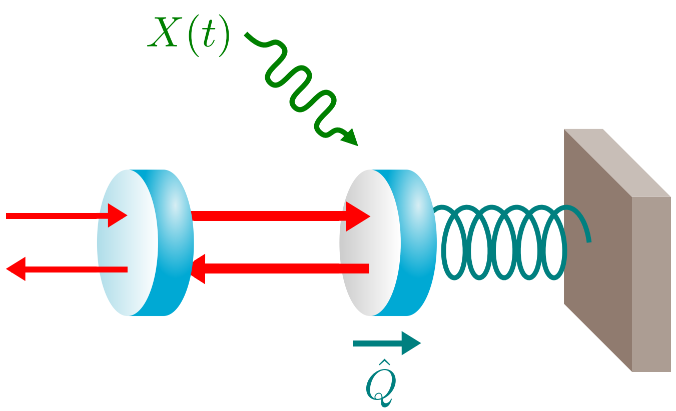
- **Dynamical system**: time-varying parameter (**waveform**) $X(t)$, continuous input field, continuous measurement of output field (many modes).
- Hamiltonian: $\boxed{H = H_0 - \hat{Q}X(t).}$
- Examples:

	Classical $X(t)$	**Generator** $\hat{Q}$	Measurement
Optomech. /LIGO	classical force	mechanical position	optical homodyne
Optical interferometer	phase modulation	photon flux	heterodyne, homodyne, etc.
Atomic magnetometer	magnetic field	atomic spin	optical

Quantum estimation and detection theory

- **Quantum limits for any measurement?**
  - Helstrom's book (1976)
- **Best measurement?**
  - Homodyne? Backaction evasion? Photon counting? In what optical modes?
- **Best data processing: classical statistics**
  - Maximum-likelihood? Bayesian? Filtering/smoothing? Likelihood-ratio test?
- Best input state?
- Best dynamics?

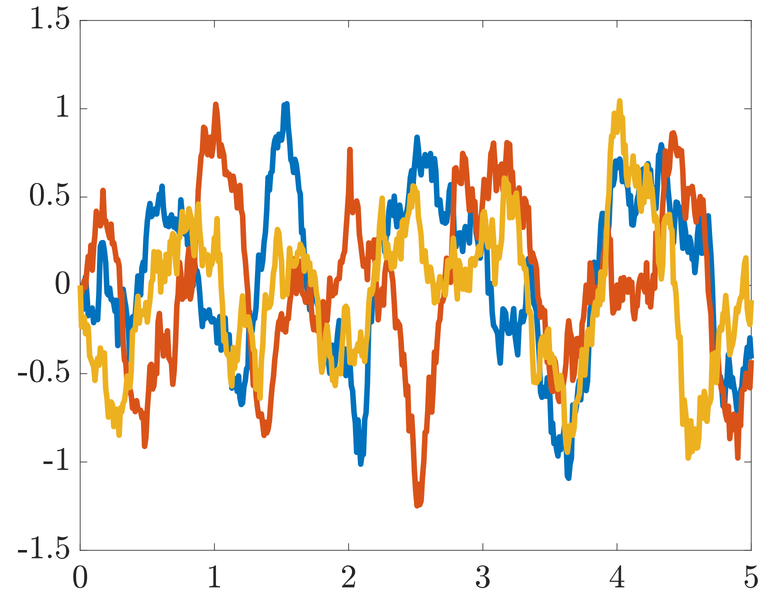
Statistical tasks



1. **Waveform estimation** [Tsang, Wiseman, Caves, PRL (2011), etc.]
   - Estimate unkown $X(t)$.
2. **Noise spectroscopy** [Ng *et al.*, PRA (2016); Tsang, PRA (2023)]
   - $X(t)$ is stochastic, estimate **parameters of its power spectral density (PSD)** $S_X(\omega|\theta)$.
3. **Waveform detection** [Tsang & Nair, PRA (2012); Tsang, PRA (2023)]
   - **Hypothesis testing**: Null hypothesis: $X(t) = 0$. Alternative hypothesis: $X(t) =$ something.

Waveform estimation

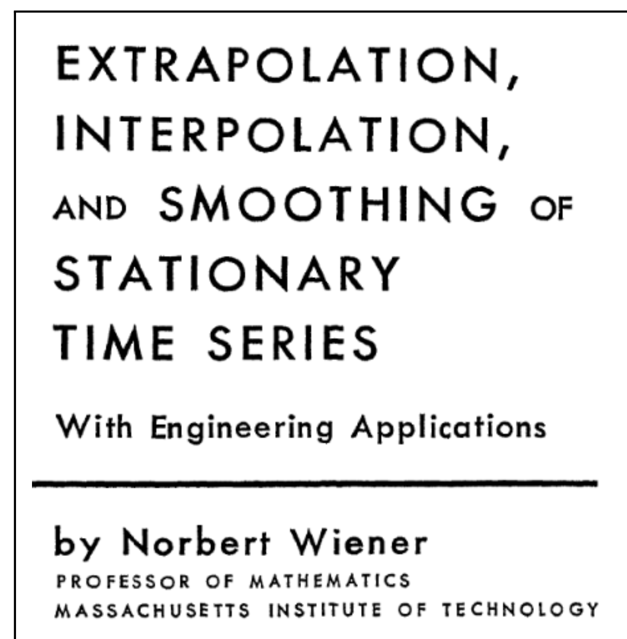
- Waveform $X(t)$: **many many** parameters.
- ~~Maximum-likelihood, unbiased, Cramér-Rao bound, etc.~~:
  - Rely on asymptotics: many observations all conditioned on a **static** parameter.
  - Doesn't work for waveform: $X(t)$ doesn't repeat.



- Classical (e.g., radar): **Bayesian** [Kolmogorov (1941), Wiener (1949), Van Trees' books (1968–)], $X(t)$ is **random process**:

$$\mathrm{MSE} = \mathbb{E}_X \left\{ \mathbb{E}_{Y|X} \left[\left\| \check{X}(Y) - X \right\|^2 \right] \right\}. \quad (1)$$

- $\check{X}(Y)$ is estimator given **observation** Y.
- $\mathbb{E}_X$ is expectation using a **prior**.
- (Norm for waveform: $\|f\|^2 \equiv \frac{1}{T} \int_0^T |f(t)|^2 dt$)
- (minimax: worst-case error)



EXTRAPOLATION, INTERPOLATION, AND SMOOTHING OF STATIONARY TIME SERIES

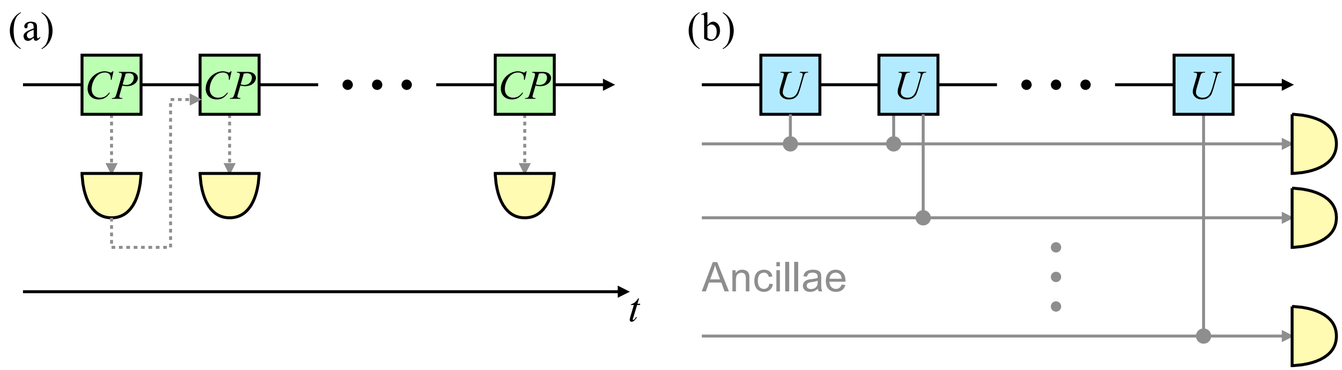
With Engineering Applications

by Norbert Wiener

PROFESSOR OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Quantum measurement theory

- Introduction by Chantasri
- **Larger Hilbert space + principle of deferred measurement**:



$$P_{Y|X}(y|x) = \mathrm{tr}\left[E(y)\rho_x\right], \qquad \rho_X = U_X |\psi\rangle\langle\psi| U_X^\dagger, \tag{2}$$

$$U_X = \mathcal{T}\exp\left[-i\int_0^T H_X(t)dt\right], \qquad H_X = H_0 - \hat{Q}X(t). \tag{3}$$

- $|\psi\rangle$ is the initial state of **everything** (input light, mechanical object, environment)
- H_0 is **the rest of all dynamics** (optics, mechanics, interactions with probe and environment).
- State at final time: ρ_X. Final measurement: POVM E.

Bayesian Quantum Bound [Tsang, Wiseman, Caves, PRL (2011)]

- Tasteful assumptions:
  1. Quantum: stationary (not bad for optomech/LIGO).
  2. prior P_X: Gaussian, stationary, with PSD $S_X(\omega)$.
  3. Long observation time (relative to correlation time of $X(t)$)
- **For any POVM E, any biased/unbiased estimator**, Bayesian quantum Cramér-Rao:

$$\boxed{\mathrm{MSE} \geq \mathrm{MSE}_{\mathsf{quantum}} \equiv \int_{-\infty}^{\infty} \frac{1}{4S_Q(\omega) + 1/S_X(\omega)} \frac{d\omega}{2\pi}.} \tag{4}$$

$$S_Q(\omega) = \text{PSD of } \Delta\hat{Q}(t) \equiv \hat{Q}(t) - \langle\psi|\, \hat{Q}(t)\, |\psi\rangle \text{ in } \textbf{Heisenberg picture}. \tag{5}$$

- Compare with vanilla QCRB:

$$U_\phi = e^{i\hat{Q}X}, \qquad \Delta X^2 \geq \frac{1}{4\langle\psi|\, \Delta\hat{Q}^2\, |\psi\rangle}. \tag{6}$$

- Optomech: $S_Q(\omega)$ is PSD of mechanical position. Optical phase: $S_Q(\omega)$ is PSD of photon flux.
- $1/S_X(\omega)$ is prior information.

Optimal measurement & data processing

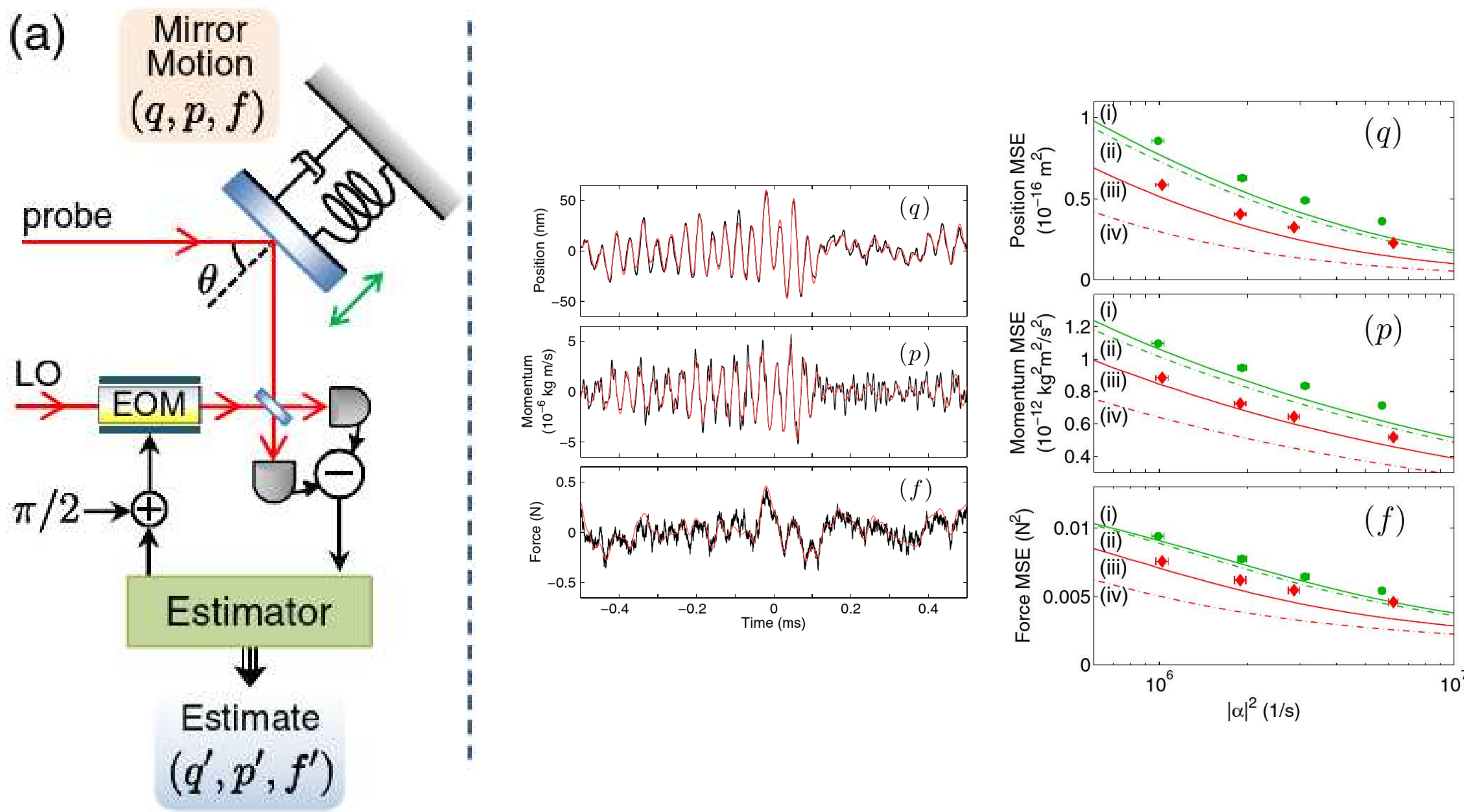
- □ Assume coherent or squeezed input light (stationary, Gaussian).
- □ Optomech: **Measurement backaction noise** (radiation-pressure noise, ponderomotive squeezing) **negligible** or **removed**.
  - Kimble *et al.*, PRD (2001).
  - Tsang & Caves, PRL (2010); PRX (2012).
    - ▹ "Quantum noise cancellation," "quantum-mechanics-free subsystems"
    - ▹ see also Polzik (2001–), Hammerer (2009–), Heurs, etc.
- □ **Optical homodyne**
- □ **Classical data processing**: minimum MSE estimator = conditional expectation = "**smoothing**"

$$\boxed{\mathrm{MSE}_{\text{smoothing}} = \int_{-\infty}^{\infty} \frac{1}{1/S_{\text{noise}}(\omega) + 1/S_X(\omega)} \frac{d\omega}{2\pi},} \quad \boxed{S_{\text{noise}}(\omega) \geq \frac{1}{4S_Q(\omega)}.} \tag{7}$$

- Kolmogorov, Wiener, Van Trees' books, etc.
- Magnetometry, linear Gaussian: Petersen & Mølmer, PRA (2006).
- General quantum theory: Tsang, PRL (2009); PRA (2009, 2010).

Proof-of-concept experiment

- Iwasawa, Makino, Yonezawa, Tsang, Davidovic, Huntington, Furusawa, "Quantum-Limited Mirror-Motion Estimation," PRL 111, 163602 (2013).



- Spin squeezing, magnetometry by smoothing: Bao, Mølmer, Xiao *et al.*, Nature (2020).

- “**Energetic quantum limit**” [Braginsky *et al.* (1992, 1999)]:
  - bound on **SNR**:

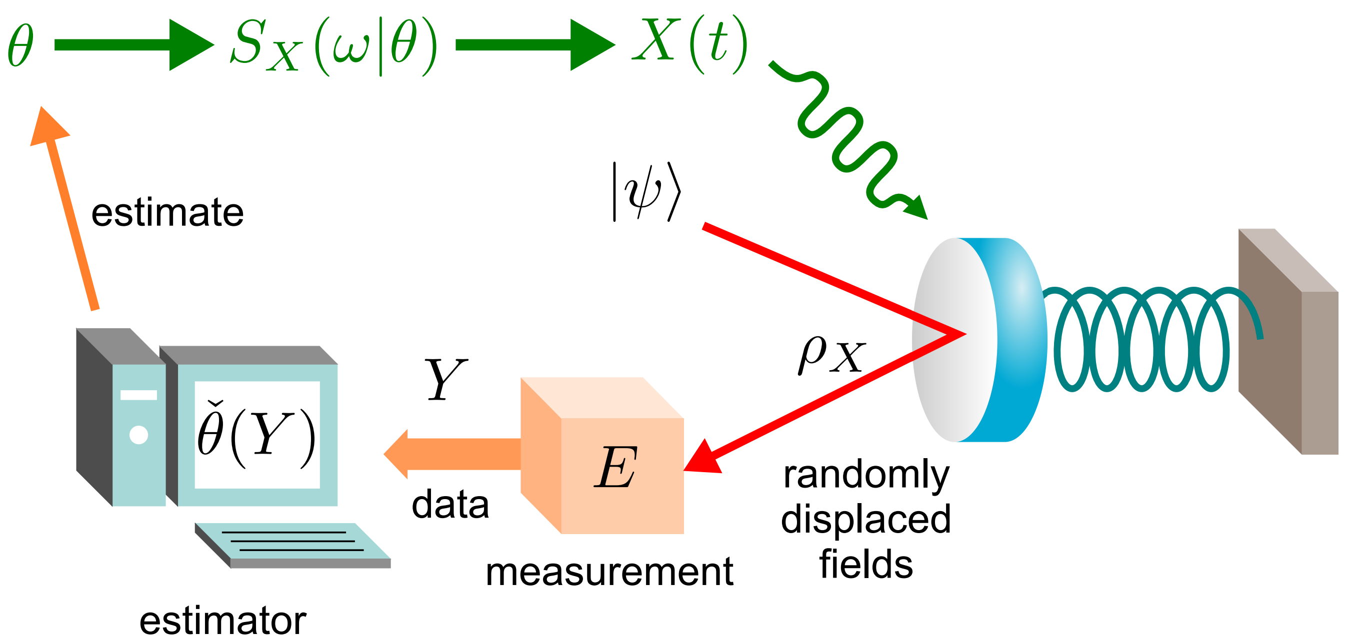
$$\mathrm{SNR} \le \int_0^T \int_0^T X(t)\Sigma_Q(t,t')X(t')dtdt' = \int_{-\infty}^{\infty} S_Q(\omega)|\tilde{X}(\omega)|^2 \frac{d\omega}{2\pi}, \quad (8)$$

$$\Sigma_Q(t,t') \equiv \mathrm{Re}\,\langle\psi|\,\Delta Q(t)\Delta Q(t')\,|\psi\rangle\,. \quad (9)$$

  - Since WWII: Error doesn’t depend only on SNR, also depends on
    1. **task** (estimation, detection, spectroscopy, etc.)
    2. **type of statistics** (Gaussian, Poisson, etc.)
    3. **data processing**

- Tighter quantum bounds with **optical loss**: Tsang, NJP (2013).
  - “Unfavorable purification” technique by Escher, Filho, Davidovich (2011).

Noise spectroscopy



- Assume $X(t)$ is zero-mean stationary Gaussian random process. Examples:
  - Stochastic gravitational-wave background
  - Gravity-induced decoherence
  - Optical phase noise
  - Stochastic magnetic field

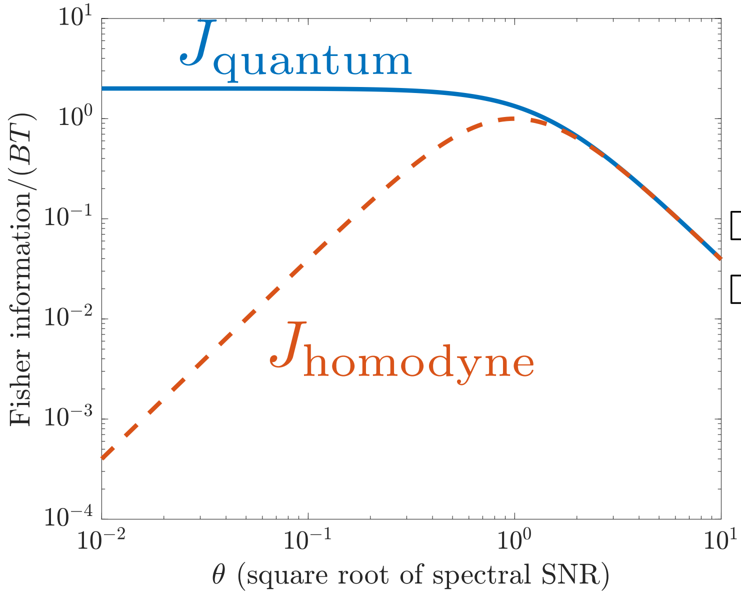
- estimate unknown parameter θ of PSD $S_X(\omega|\theta)$ of $X(t)$.
- θ is **static**; maximum-likelihood/Cramér-Rao bound/asymptotics make sense again [Whittle (1953); Shumway & Stoffer, *Time Series Analysis*]
- Focus on **weak optical phase modulation** (i.e., interferometer)

Quantum limit to noise spectroscopy

- CRB: $\Delta\theta^2 \geq 1/J$.
- Upper quantum bound on Fisher information J [Ng *et al.*, PRA (2016)]:

$$\boxed{J(\theta) \leq J_{\text{quantum}}(\theta) \equiv T \int_{-\infty}^{\infty} \frac{1}{2 + 1/[S_Q(\omega) S_X(\omega|\theta)]} \left[\frac{\partial}{\partial\theta} \ln S_X(\omega|\theta)\right]^2 \frac{d\omega}{2\pi}.} \tag{10}$$

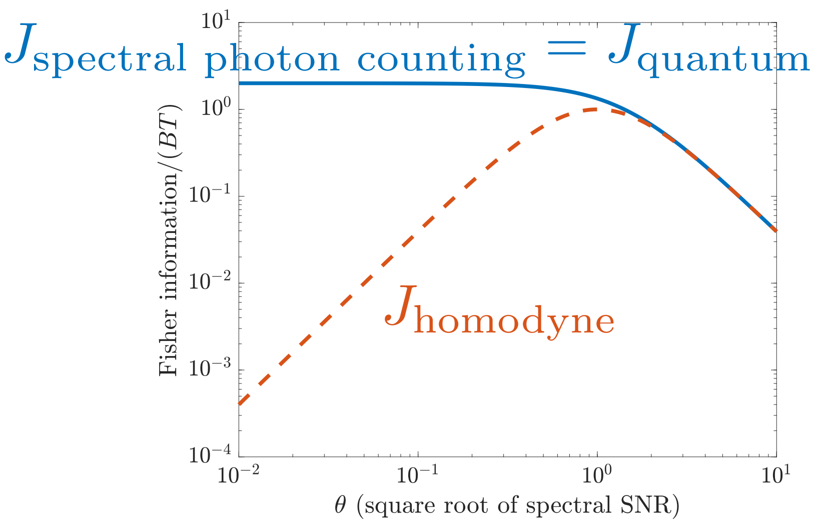
- derived using **extended convexity** of QFI [Alipour & Rezakhani, PRA (2015)].
- **Optical phase noise spectroscopy**, flat photon-flux PSD $S_Q(\omega)$, flat $S_X(\omega|\theta) \propto \theta^2$ with bandwidth B [Tsang, PRA (2023)]:

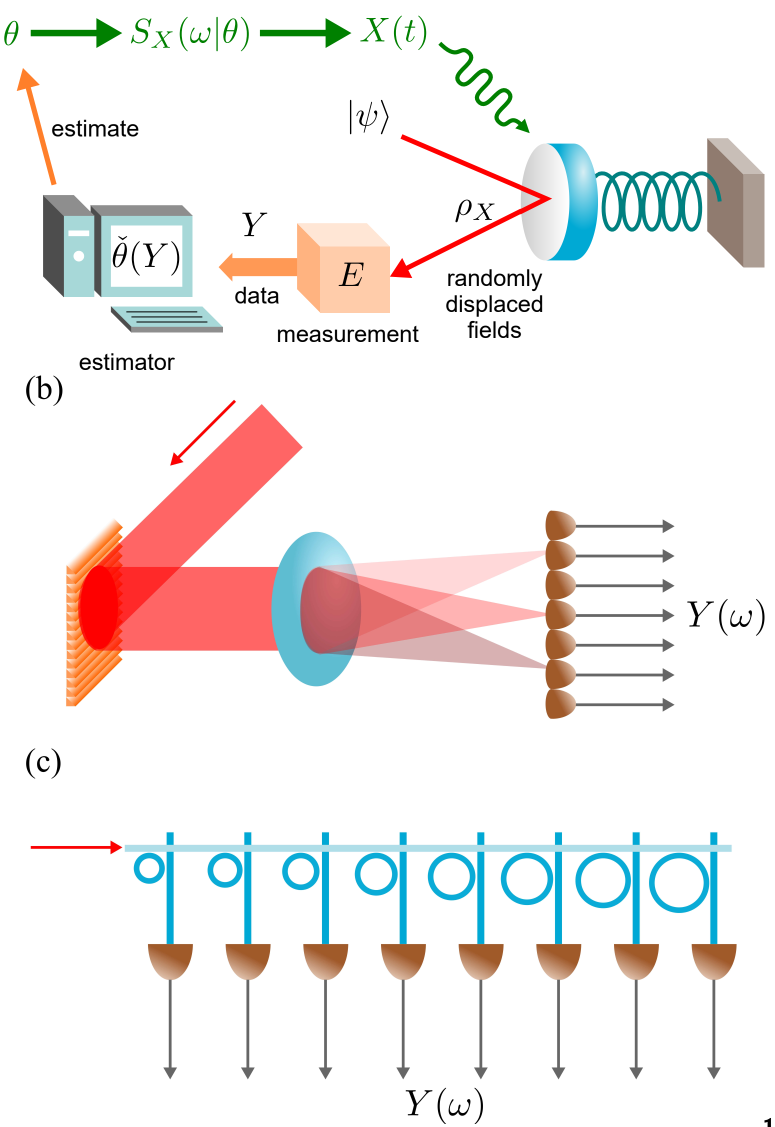


- **LOG-LOG PLOT**
- **Homodyne is much worse than quantum limit!**

Optimal measurement: spectral photon counting

- Assume **coherent-state input**.
- **Ng et al., PRA (2016):** Spectrometer (diffraction grating or resonator array) + photon counting: Fisher information = quantum bound.



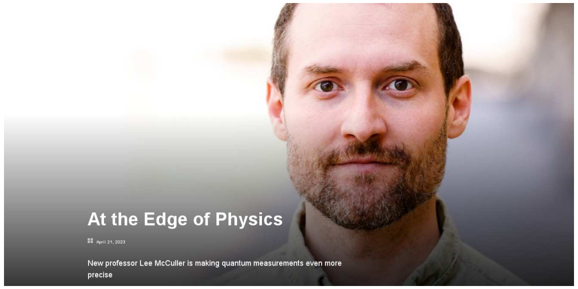


- Cannot be explained by "energetic quantum limit."
- Joint estimation of phase and phase diffusion: Vidrighin, Datta *et al.*, NC (2014).
  - Single mode
  - POVMs, homodyne, "displaced Sagnac polarization interferometer"

Caltech/LIGO proposed experiment

- Lee McCuller (Caltech/LIGO), "Single-Photon Signal Sideband Detection for High-Power Michelson Interferometers," arXiv:2211.04016 (2022).

> **VII. OUTLOOK**
>
> This work establishes and analyzes photon counting as a promising methodology for using Michelson interferometers to search for new physics. The analysis focuses on incoherent stochastic-noise-like signals, yet utilizes an orthonormal Fourier-like temporal basis to decompose down the observables and statistics for both kinds of searches. The focus on incoherent signals avoids known quantum Fisher information limits in signal detection. This key

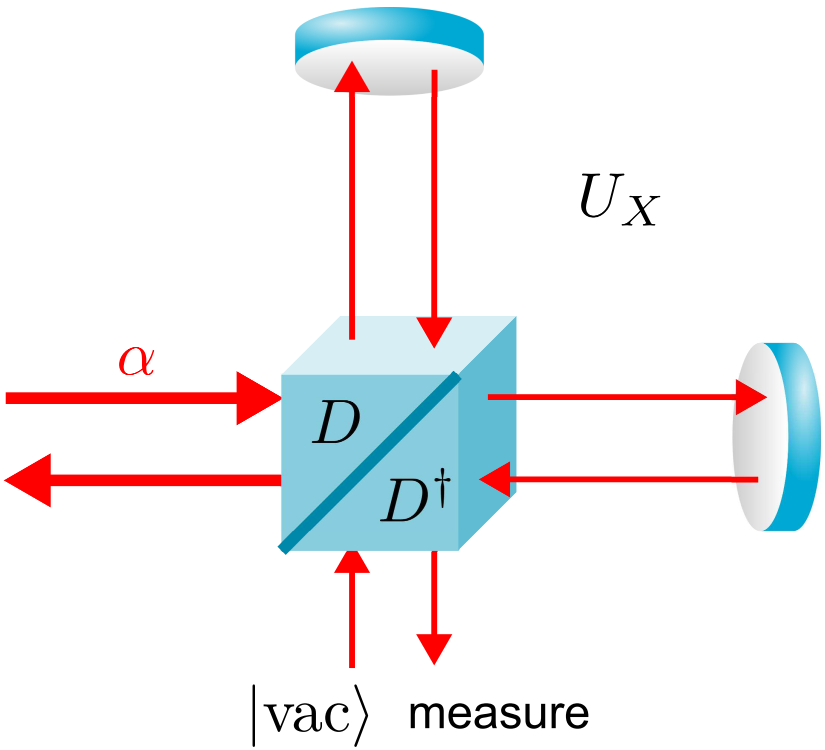


- To detect signatures of quantum gravity (GQuEST):

1. https://www.caltech.edu/about/news/at-the-edge-of-physics
2. https://magazine.caltech.edu/post/quantum-gravity

Heuristic, ad-hoc explanation

- Consider dark port of Michelson (nulled mean field)
- Average output photon number per mode $\boxed{\epsilon \ll 1}$. (Number of temporal modes $M = BT$)
  - With random phase, **vacuum state most of the time.**
- **Photon counting**:
  - when there is no photon, variance $= 0$
  - $\sim$ Poisson, variance $= \Theta(\epsilon)$
- Homodyne/heterodyne/linear amplifiers:
  - **vacuum fluctuations all the time**.
  - $\sim$ additive Gaussian, variance $= \Theta(1)$
- Different from waveform estimation!
  - estimating X versus $\theta = \sqrt{\mathbb{E}(X^2)}$.

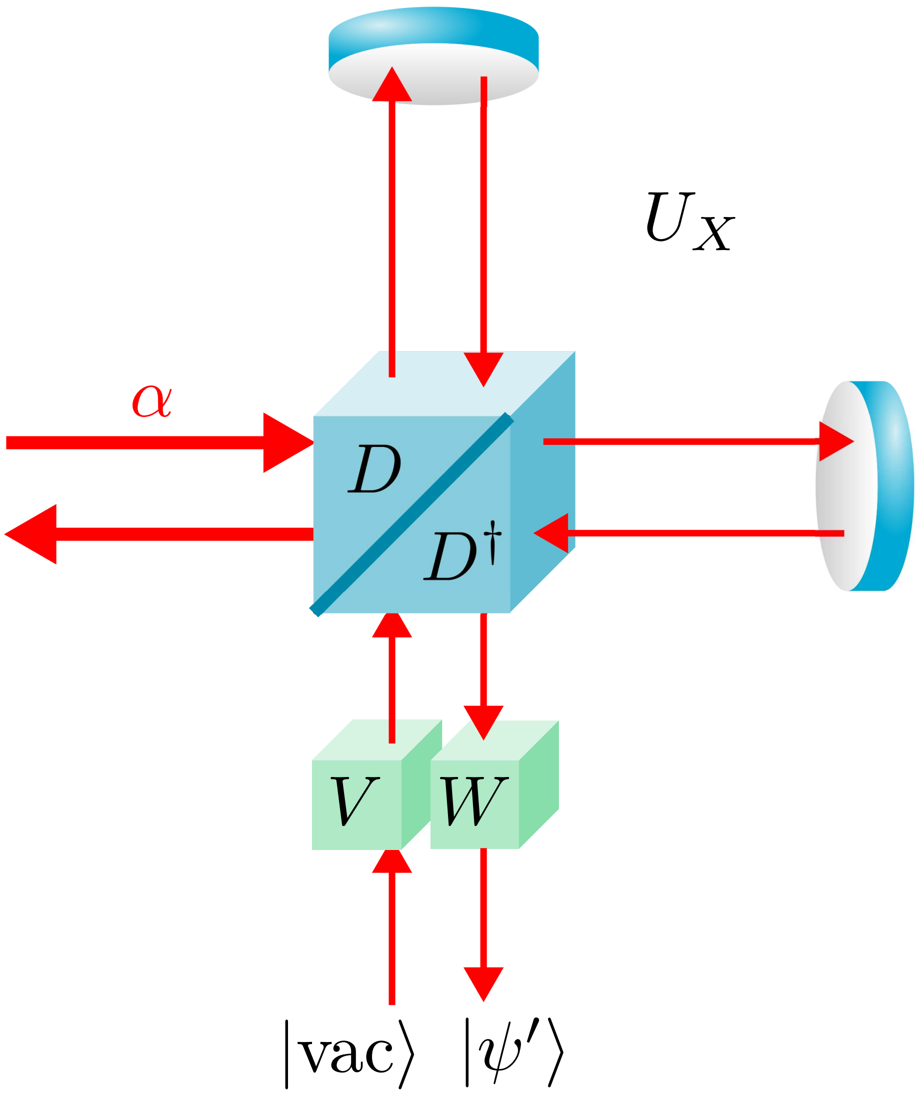


Superiority of photon counting for random signals

- **Thermal radiation** (**random fields**):
  - photon counting $\gg$ linear detectors (heterodyne, homodyne, linear amplifiers) when $\epsilon \ll 1$
  - Siegman (1966), Kingston (1978), Prasad (1994), Townes, etc.
- **Axion dark matter search**:
  - dark matter = random field, may convert to **thermal microwave**
  - **random displacements in both microwave quadratures**
  - Photon counting $\gg$ linear detectors when $\epsilon \ll 1$ [Lamoreaux *et al.*, PRD (2003); Dixit *et al.* PRL (2021)]
- **Phase noise spectroscopy**:
  - **coherent input**, random displacement of **optical phase quadrature**.
- **Incoherent imaging**:
  - Random fields + randomly displaced **photons**
  - Photon counting $\gg$ linear detectors when $\epsilon \ll 1$ [Yang *et al.*, PRA (2017)]
  - Hermite-Gaussian mode basis (SPADE) $\gg$ position basis for sub-Rayleigh sources [Tsang *et al.* (2016–)]

Optimal measurement for squeezed input

- **Unsqueeze + spectral photon counting**
  - Tsang, "Quantum noise spectroscopy as an incoherent imaging problem," PRA (2023).
- Similar ideas by Gorecki *et al.*, PRL (2023); Shi & Zhuang, NPJQI (2023).
- Time reversal in quantum metrology [discussion by Davidovich]



Deterministic waveform detection

- **Binary hypothesis testing**:
  - Error probabilities: miss, false-alarm, **average error probability** $\approx C \exp(-\Gamma)$.
  - Γ is **error exponent**, $\propto$ time T, SNR, photon number, etc.
  - Classical: Neyman-Pearson, Bayesian, likelihood-ratio test, Chernoff, etc. [Van Trees' books]
- Assume
  1. Null hypothesis: $X(t) = 0$
  2. Alt. hypothesis: **known** $X(t)$.
- Upper quantum bound on error exponent via fidelity $F \equiv (\operatorname{tr}\sqrt{\sqrt{\rho_1}\rho_0\sqrt{\rho_1}})^2$ [Nair & Tsang, PRA (2012)]:

$$\boxed{\Gamma \le \Gamma_{\mathsf{quantum}} = -\ln F = \int_0^T \int_0^T X(t)\Sigma_Q(t,t')X(t')dtdt',} \tag{11}$$

$$\Sigma_Q(t,t') \equiv \operatorname{Re} \langle\psi| \Delta Q(t)\Delta Q(t') |\psi\rangle . \tag{12}$$

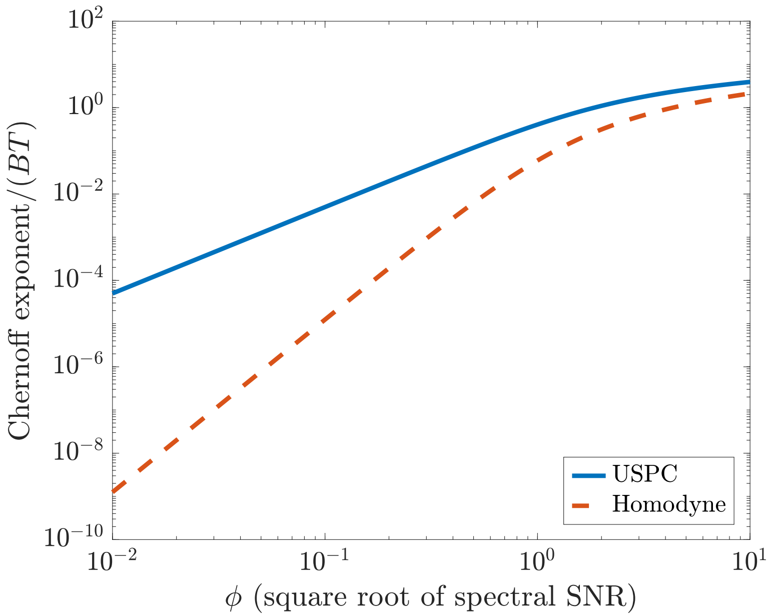
- $\Gamma_{\mathsf{homodyne}} = \Gamma_{\mathsf{quantum}}/2$ (suboptimal but pretty good)
- Coherent-state input: $\Gamma_{\mathsf{nulling\ +\ photon\ counting}} = \Gamma_{\mathsf{quantum}}$ (Kennedy receiver)

Stochastic waveform detection

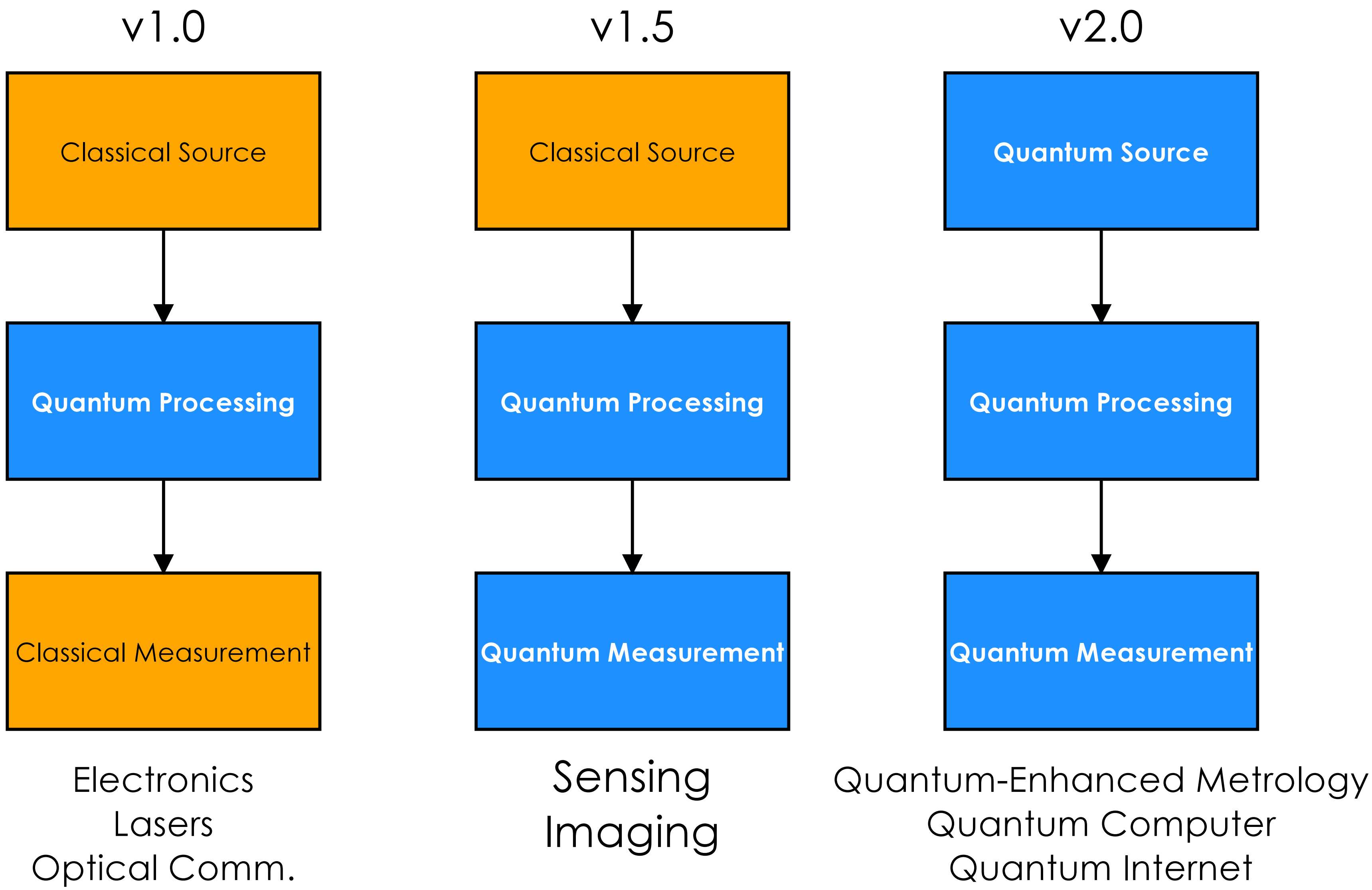
- Assume
  1. Null hypothesis: $X(t) = 0$
  2. Alt. hypothesis: **stochastic** $X(t)$ (stationary Gaussian).
- Upper quantum bound on error exponent [Tsang & Nair, PRA (2012)]:

$$\boxed{\Gamma \le \Gamma_{\text{quantum}} = -\ln F = \frac{T}{2}\int_{-\infty}^{\infty} \ln\left[1 + 2S_Q(\omega)S_X(\omega)\right]\frac{d\omega}{2\pi}.} \tag{13}$$

- **LOG-LOG PLOT**
- Homodyne is **terrible**
- **Unsqueeze + spectral photon counting** is **optimal** [Tsang, PRA (2023)].



Quantum Technology 1.5



Experimental Difficulties

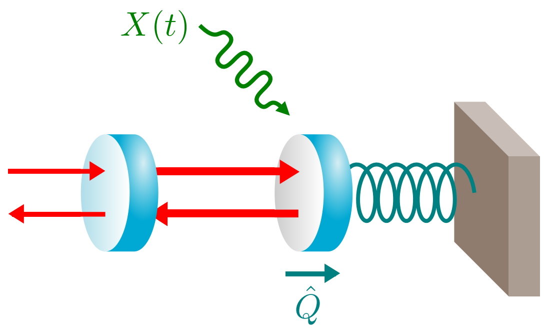
- Caves, PRD (1980):

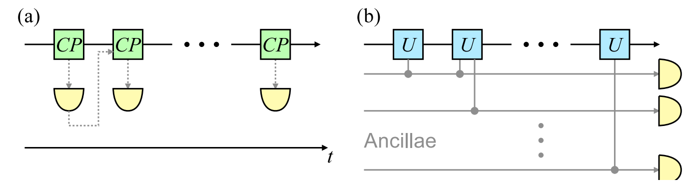
IV. CONCLUSION

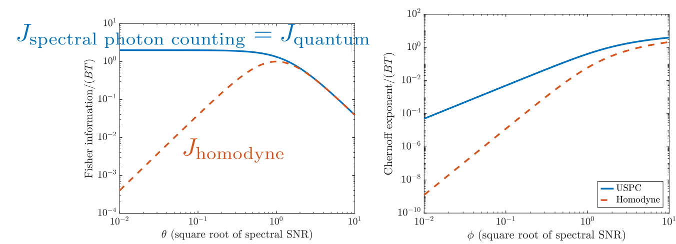
The squeezed-state technique outlined in this paper will not be easy to implement. A refuge from criticism that the technique is difficult can be found by retreating behind the position that the entire task of detecting gravitational radiation is exceedingly difficult. Difficult or not, the squeezed-state technique might turn out at some stage to be the only way to improve the sensitivity of interferometers designed to detect gravitational waves. As interferometers are made longer, their strain sensitivity will eventually be limited by the photon-counting error for the case of a storage time approximately equal to the desired measurement time. Further improvements in sensitivity would then await an increase in laser power or implementation of the squeezed-state technique. Experimenters might then be forced to learn how to very gently squeeze the vacuum before it can contaminate the light in their interferometers.

Conclusion

- Quantum limits to waveform sensing ~ laws of thermodynamics
- **Optimal measurements, data processing** ~ Carnot engine
- Optimal measurement depends on the **task**.
  - **Photon counting** far superior for **noise spectroscopy/detection**.
- National Research Foundation, Singapore
  - Fellowship (NRF-NRFF-2011-07, 2011–2016)
  - Quantum Engineering Programme (QEP-P7, 2019–2024)
- Ministry of Education, Singapore (R-263-000-C06-112, 2016–2019)
- mankei@nus.edu.sg, https://blog.nus.edu.sg/mankei/







LOG-LOG PLOTS