Resolving Starlight: A Quantum Perspective

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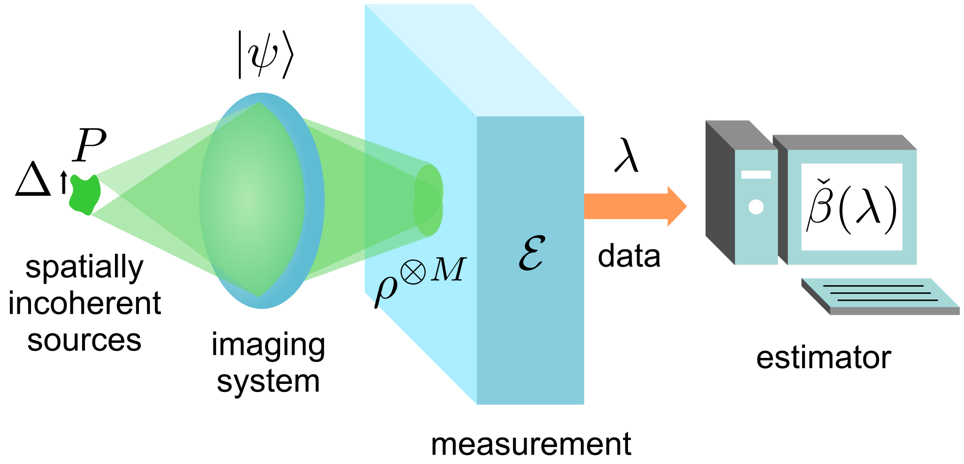
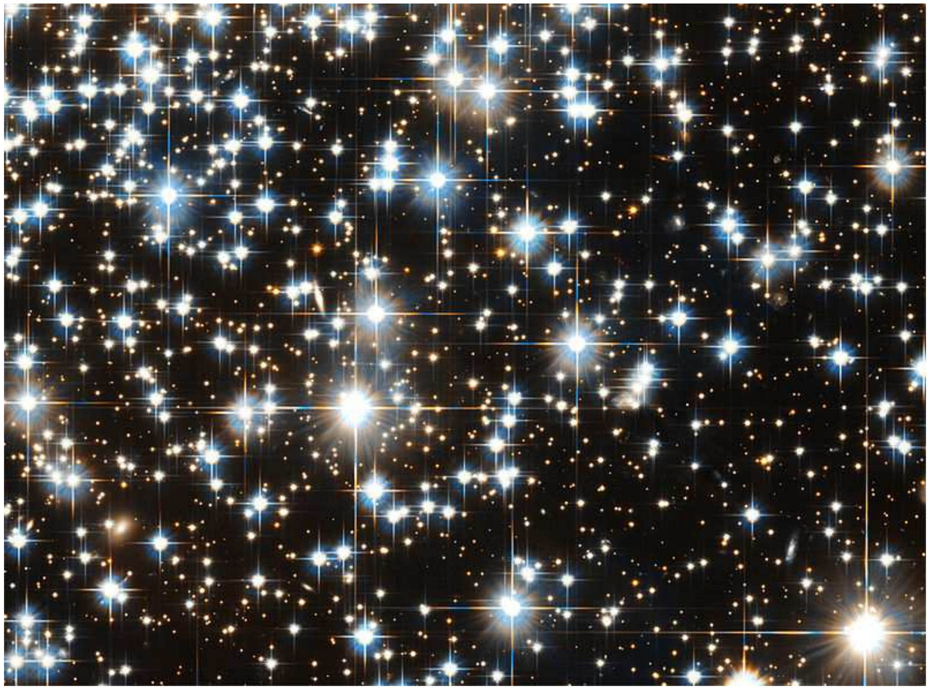
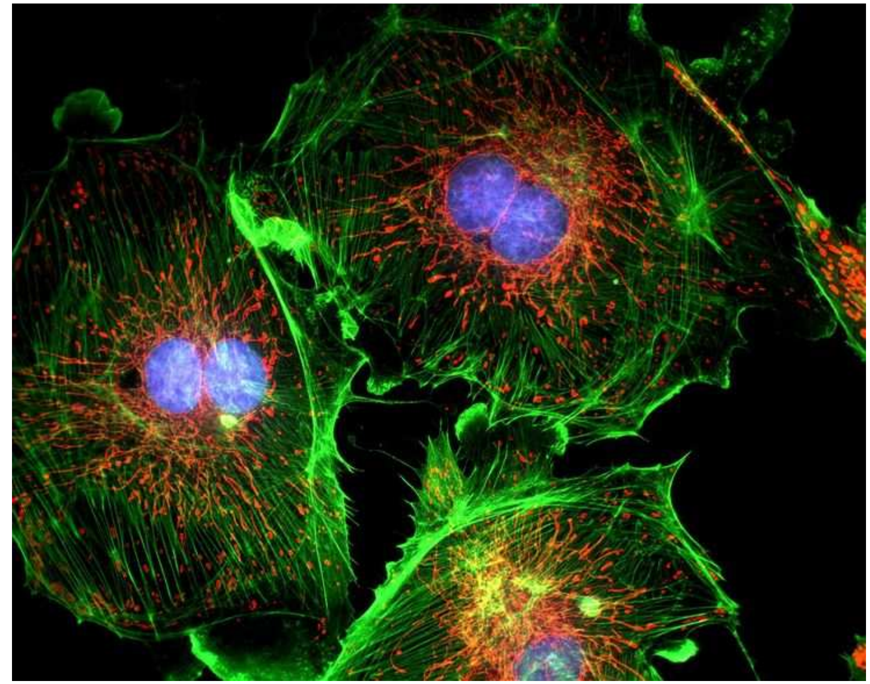


Fundamental Resolution of Incoherent Imaging



Observational Astronomy



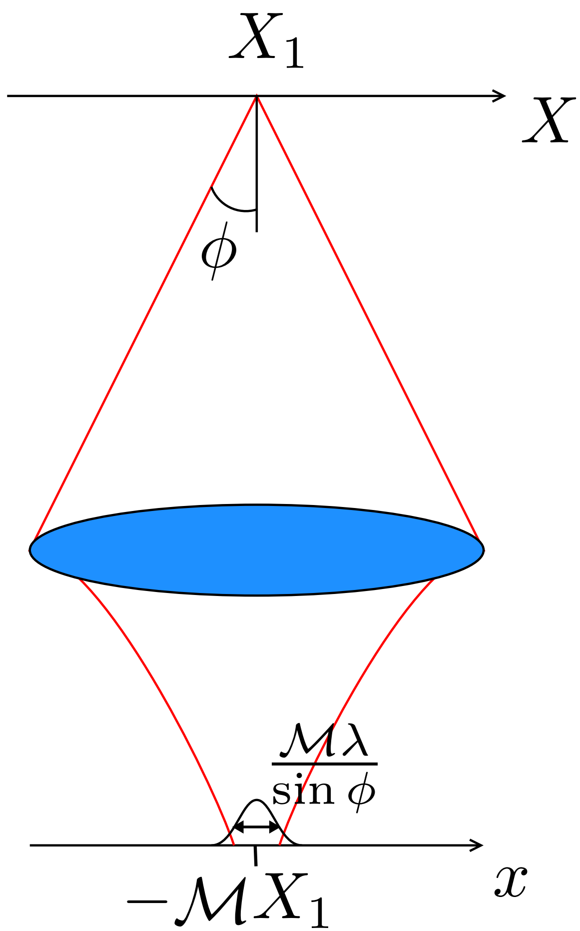
Fluorescence Microscopy

(images from the internet)

- ☐ Quantum estimation theory
- ☐ Quantum-inspired measurements

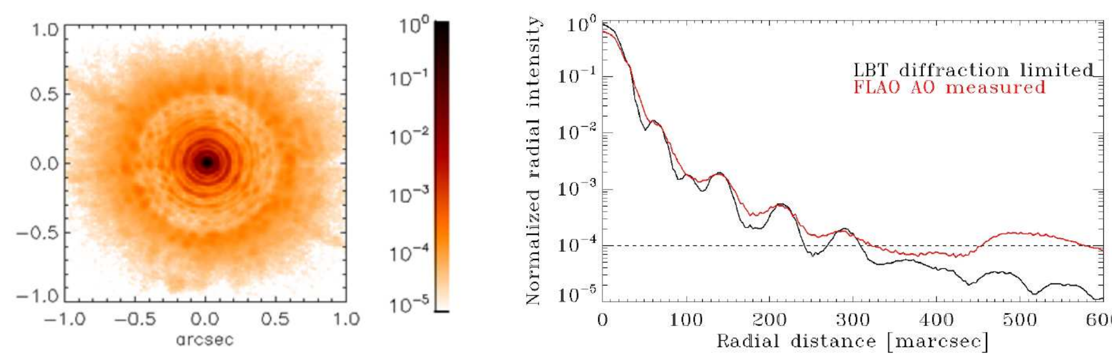
Part I: Introduction

Wave Nature of Light: Diffraction Limit



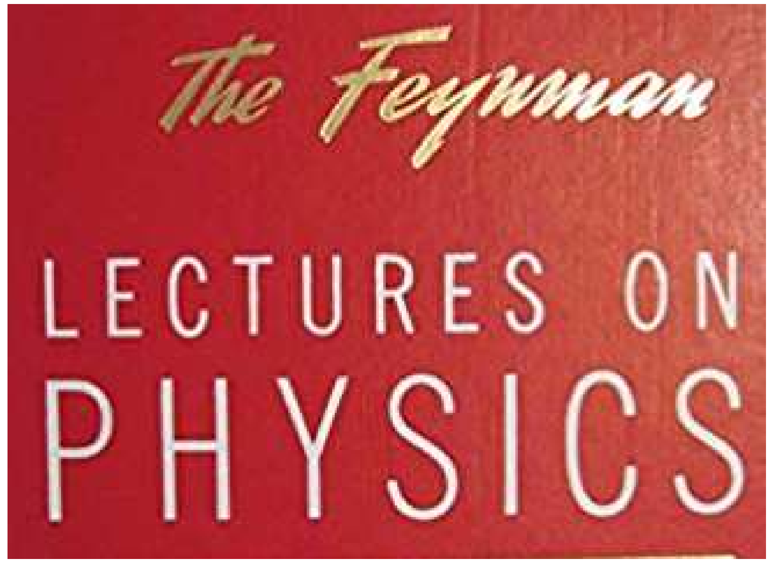
Diffraction-limited (λ/NA or λ/D)

1. Fluorescence microscopy
2. Space telescopes (Webb, \$10 billion)
3. Ground-based telescopes (corrected by **adaptive optics**):
   (a) Large Binocular Telescope (LBT) (Strehl ratio $> 80\%$, \$120 million)
   (b) Giant Magellan Telescope (GMT)
   (c) Thirty Meter Telescope (TMT)
   (d) European Extremely Large Telescope (E-ELT) (>\$1 billion each)



Esposito *et al.*, SPIE **8149**, 814902 (2011).

Diffraction Limit is a Rough Idea



"Rayleigh's criterion is a **rough idea**..." a better resolution can be achieved "if sufficiently careful **measurements of the exact intensity distribution** over the diffracted image spot can be made."

Chapter 30. *Diffraction*, Feynman Lectures on Physics, Vol. I

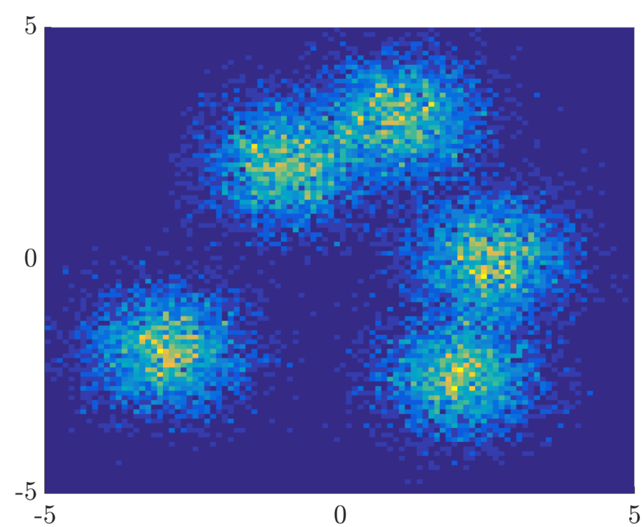
- Image processing/statistics: **superresolution**, **deconvolution**, **deblurring**, etc. Extremely sensitive to **SNR**
- **Slepian** (1954–); Bertero (1982–); de Villiers & Pike, *The Limits of Resolution*, etc.

Particle Nature of Light: Photon Shot Noise

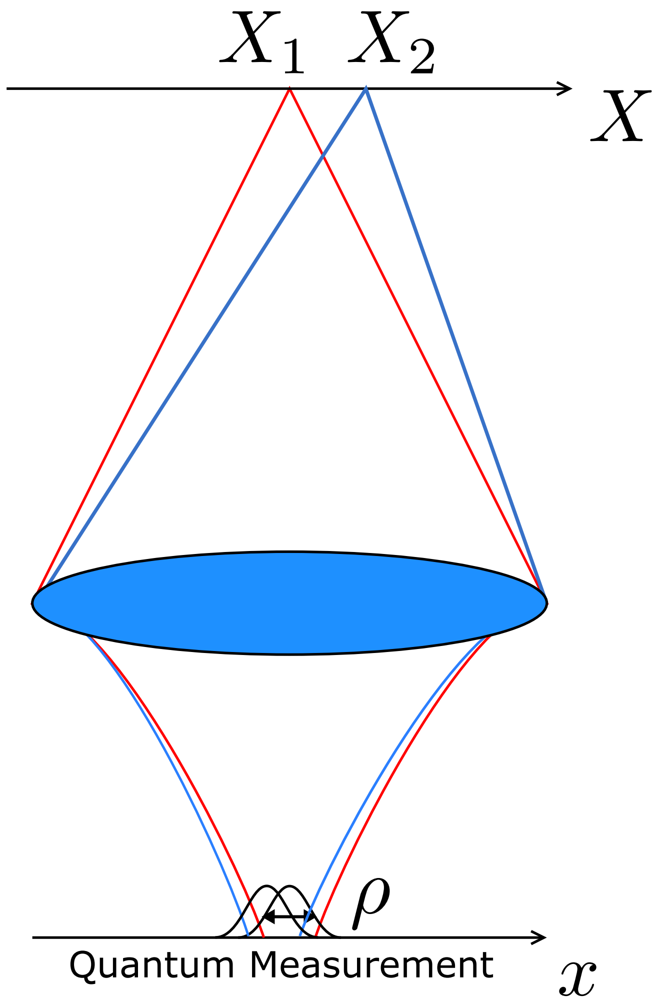
- Thermal sources (stars, etc.)
  - **Poisson**, bunching negligible at optical
  - Goodman, *Statistical Optics*; Zmuidzinas, JOSA A **20**, 218 (2003)

> considerations. If the count degeneracy parameter is much less than 1, it is highly probable that there will be either zero or one counts in each separate coherence interval of the incident classical wave. In such a case the classical intensity fluctuations have a negligible "bunching" effect on the photo-events, for (with high probability) the light is simply too weak to generate multiple events in a single coherence cell. If negligible bunching of the events takes place, the count statistics will be indistinguishable from those produced by stabilized single-mode laser radiation, for which no bunching occurs.

- Fluorophores (GFP, dye molecules, quantum dots, etc.)
  - **Poisson**, negligible anti-bunching
  - Pawley *ed.*, *Handbook of Biological Confocal Microscopy*; Ram, Ober, Ward, PNAS (2006)



Quantum Limit to Passive Imaging



- Quantum limit to **parameter estimation** with **ANY measurement:** direct imaging, linear, non-linear, any photonics, quantum computers, etc.
- Born's rule: $P(y|\theta) = \operatorname{tr} E(y)\rho(\theta)$
- **Helstrom** (1967), Nagaoka (1989), etc.: For any POVM, mean-square estimation error (MSE):

$$\text{MSE} \geq J^{-1} \boxed{\geq K^{-1},}$$

$$K_{jk}(\rho^{\otimes M}) = M \operatorname{Re} \operatorname{tr} S_j S_k \rho,$$

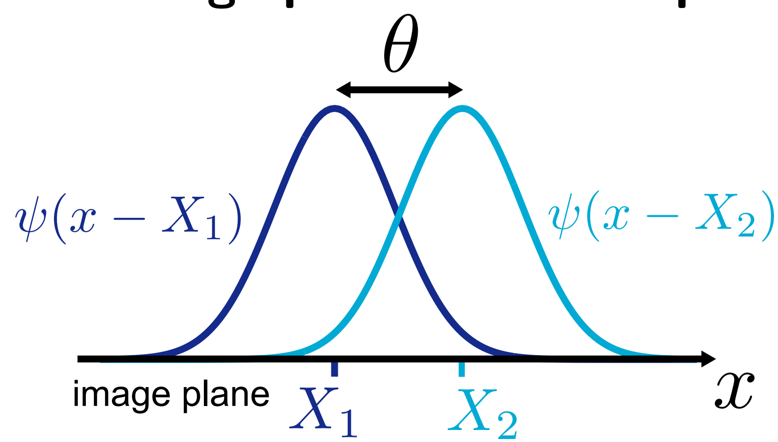
$$\frac{\partial \rho}{\partial \theta_j} = \frac{1}{2}\left(S_j \rho + \rho S_j\right).$$

- $K(\rho)$ is the quantum Fisher information, **the ultimate amount of information in the photons.**

Part II: Two Point Sources

Quantum Optics for Incoherent Imaging

- Thermal optical source: **average photon number per mode** $\epsilon \ll 1$



- Quantum state in M spectral modes $= \rho^{\otimes M}$, average photon number in all modes $N = M\epsilon$.

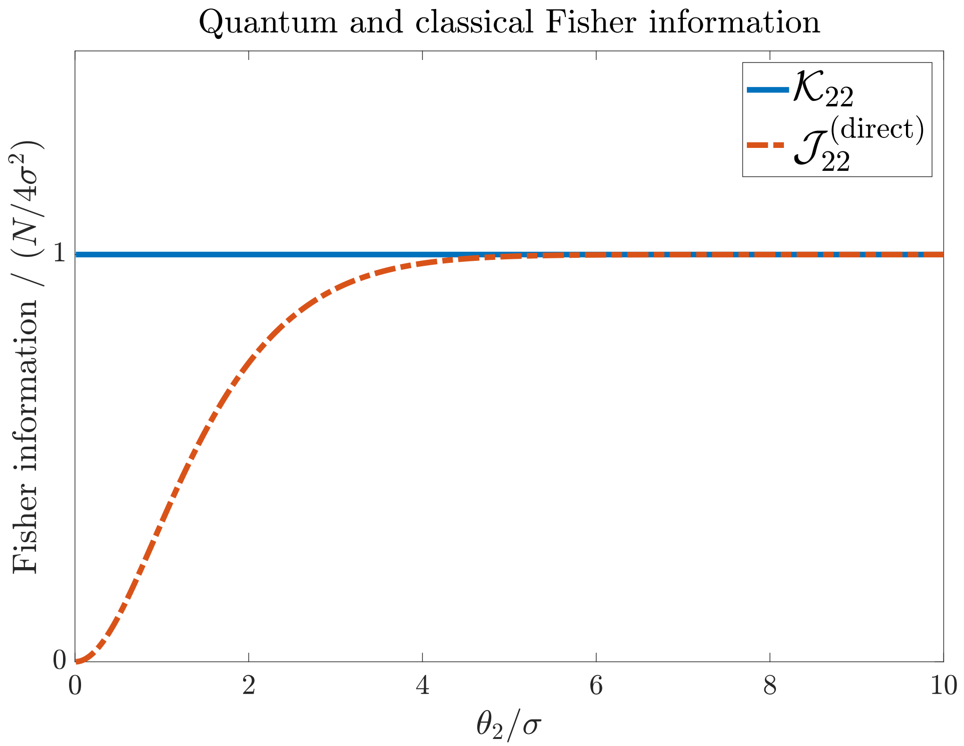
$$\boxed{\rho = (1-\epsilon)\left|\text{vac}\right\rangle\left\langle\text{vac}\right| + \frac{\epsilon}{2}\left(\left|\psi_1\right\rangle\left\langle\psi_1\right| + \left|\psi_2\right\rangle\left\langle\psi_2\right|\right) + O(\epsilon^2),}$$

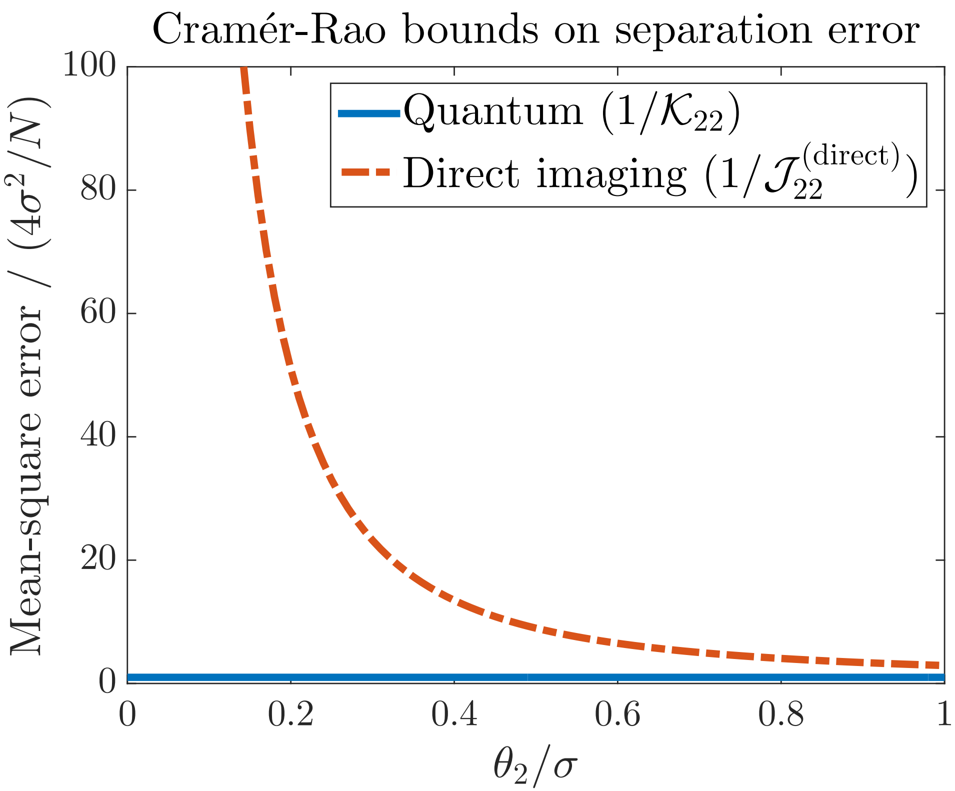
$$\boxed{\left|\psi_s\right\rangle \equiv \int_{-\infty}^{\infty} dx \psi(x - X_s)\left|x\right\rangle.}$$

- derived from Glauber representation.
- Consistent with **Poisson** counting statistics.
- see, e.g., **Tsang, PRL 107, 270402 (2011)**; Tsang, Nair, Lu, PRX (2016); Quantum (2021).

Plenty of Room at the Bottom

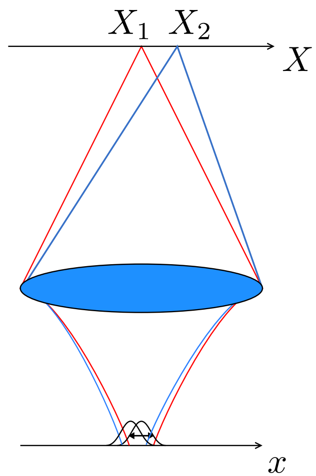
- Orange curve: **“direct imaging”** (image-plane intensity measurement). e.g., Ram-Ward-Ober, “Beyond Rayleigh’s criterion: A resolution measure with application to single-molecule microscopy,” PNAS (2006).

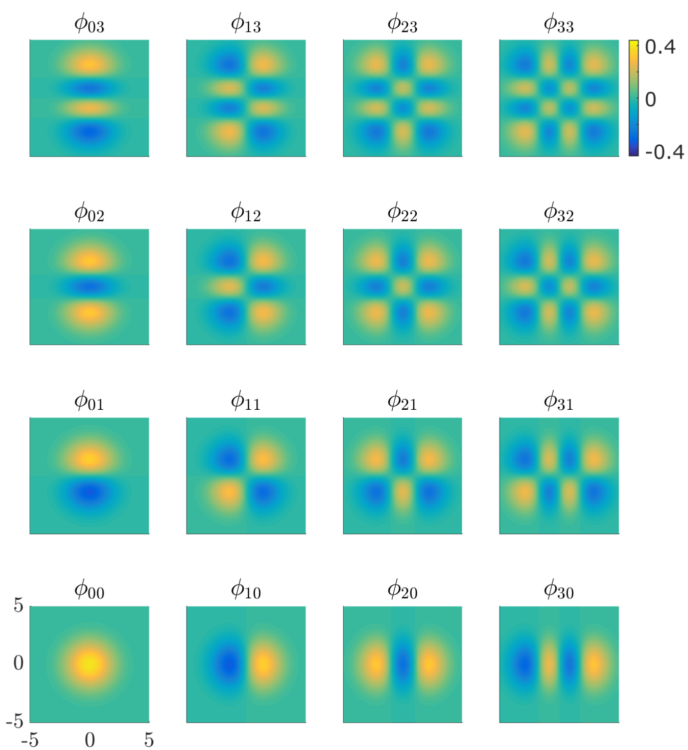




- Blue: **Quantum limit**. Tsang, Nair, Lu, PRX (2016).

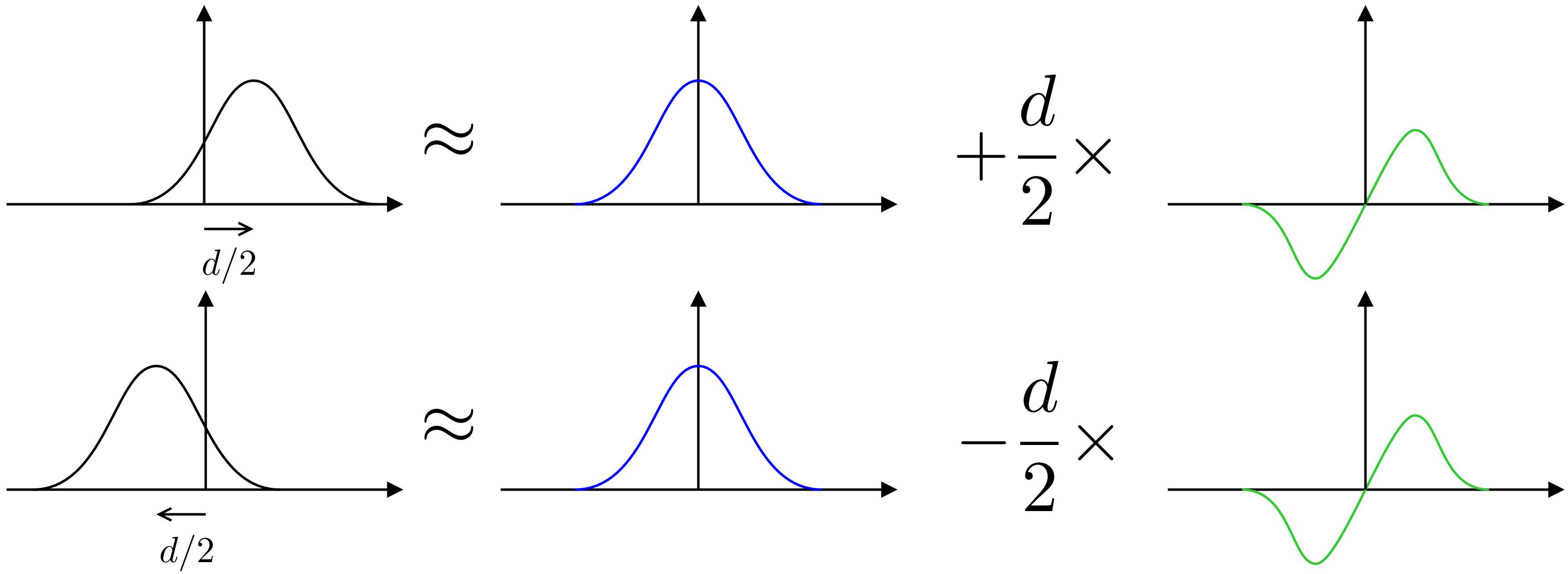
Optimal Measurement





- Sort the photons in **Hermite-Gaussian (TEM)** modes first, then do photon counting
- **Spatial-mode demultiplexing (SPADE)**
  - Tsang, Nair, Lu, PRX (2016); Rehacek *et al.*, OL **42**, 231 (2017).
- Many implementations (in optical comm., photonic circuits, interferometers, spatial light modulators, etc.)
- **Classical sources**
- **Linear optics/photon counting**
- **Killer applications** (astronomy, fluorescence microscopy, etc.)

Semiclassical Explanation



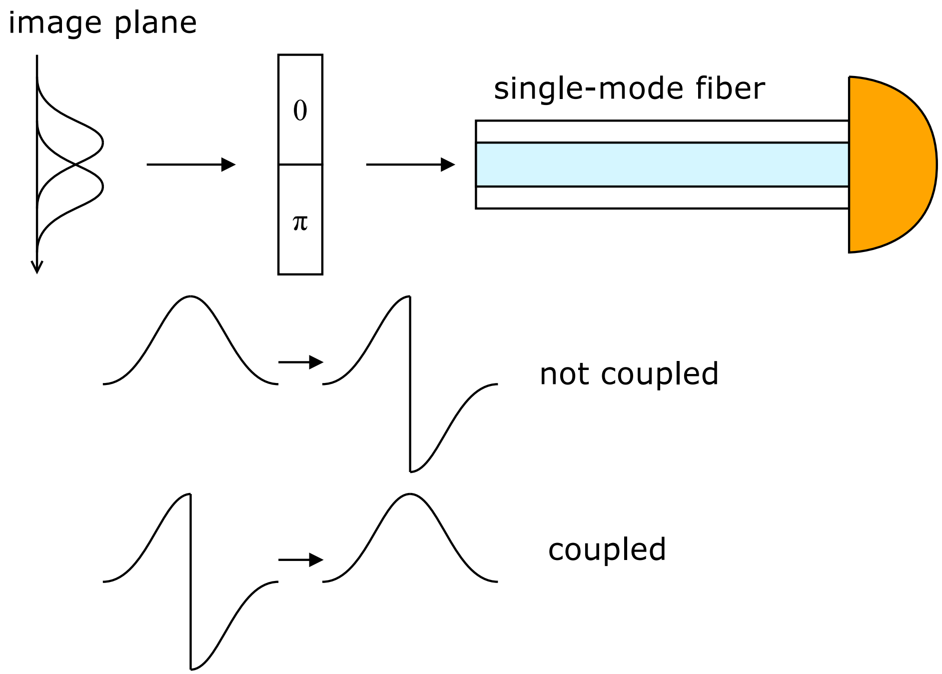
- **Incoherent sources**: energy in **1st-order mode** is

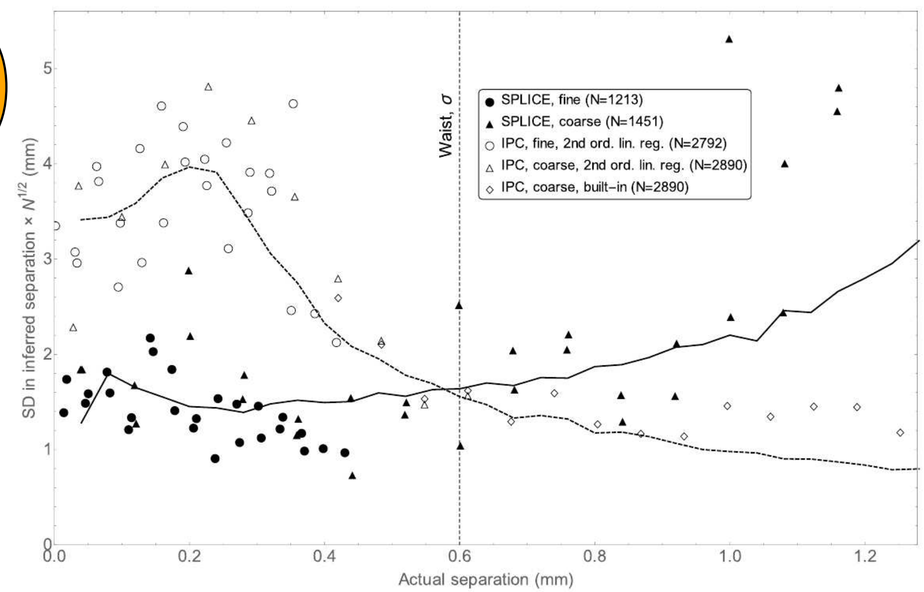
$$\propto \left(\frac{d}{2}\right)^2 + \left(-\frac{d}{2}\right)^2 = \frac{d^2}{2}.$$

- **0th-order mode** is just **background noise**; filtering it improves SNR.
- Incoherence (sources) + Coherence (diffraction) + Signal-dependent noise (Poisson)
- "Quantum-inspired" superresolution

Experimental Demonstrations

- **30+** experiments so far (`https://blog.nus.edu.sg/mankei/superresolution/`)
- Tham, Ferretti, Steinberg (Toronto), PRL **118**, 070801 (2017):

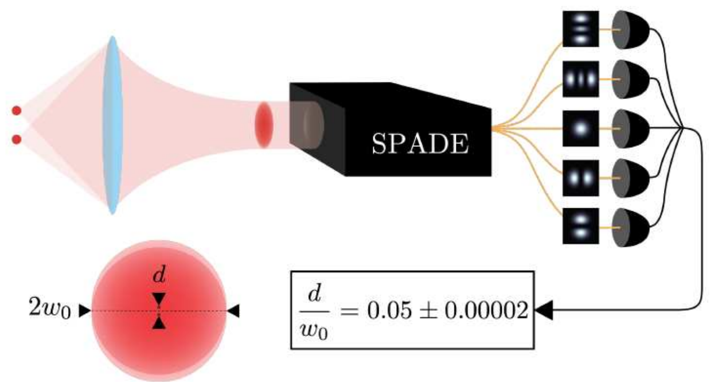




- MSE $\sim 5 \times$ QCRB
- $\sim 10\% \times$ MSE of experimental direct imaging

Recent Experiment

- Rouvière, Barral, Grateau, Karuseichyk, Sorelli, Walschaers, Treps (Lab. Kastler Brossel) arXiv:2306.11916 (2023):



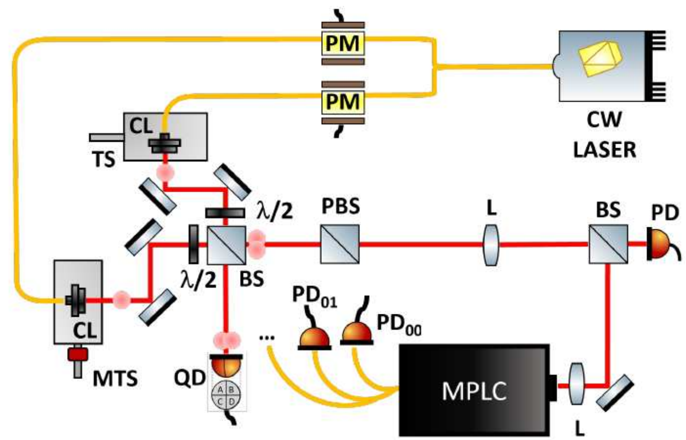
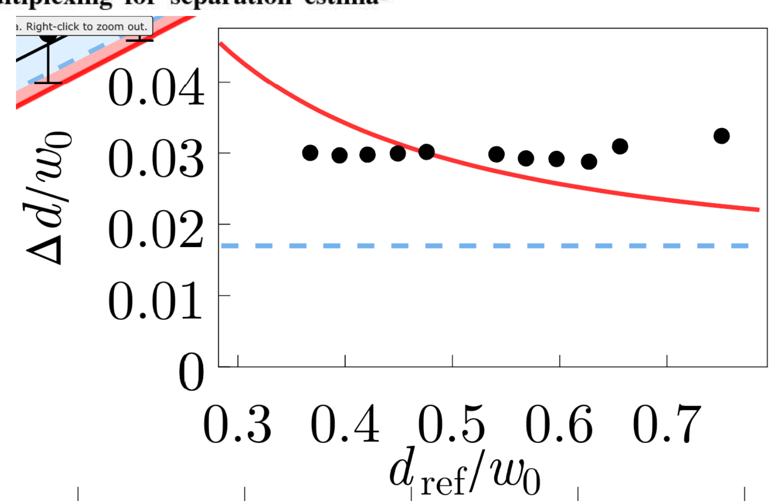


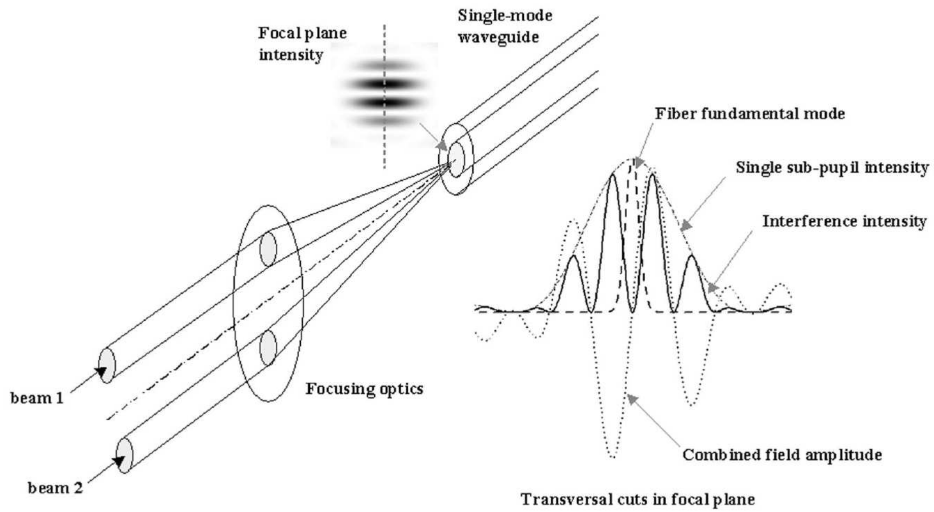
Figure 1. **Spatial-mode demultiplexing for separation estima-**

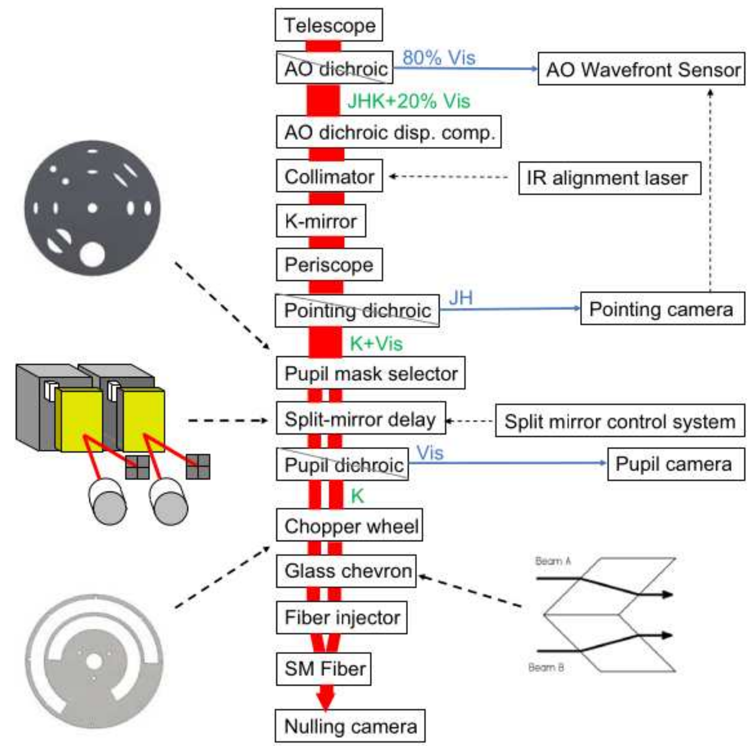


- MSE $< 4 \times$ QCRB
- beats ideal direct-imaging CRB

Nulling Interferometry

- **Palomar Fiber Nuller**: Haguenauer and Serabyn AO (2006); Serabyn *et al.*, MNRAS (2019)

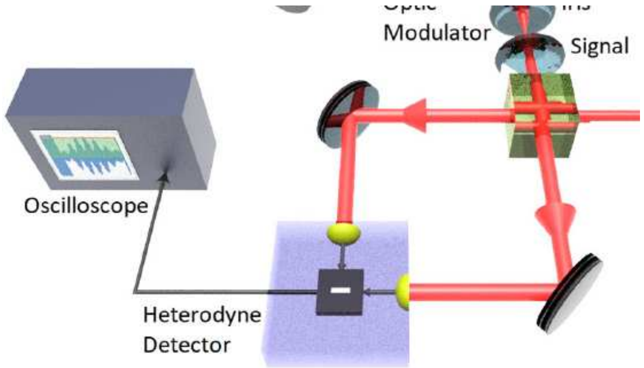




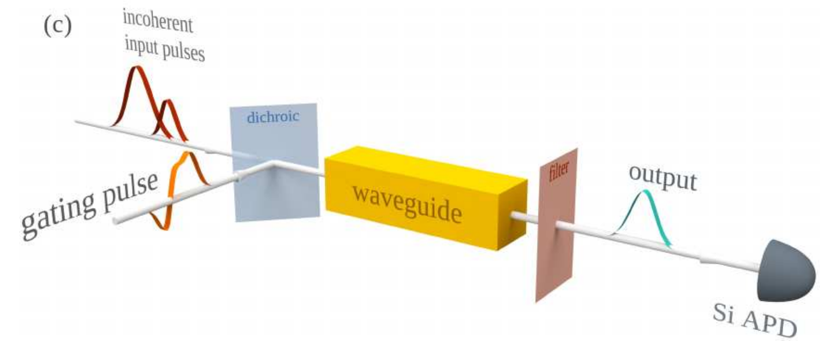
- Our work:
  - Quantum limits $\sim$ laws of thermodynamics
  - Spatial-mode demultiplexing with the **optimal** modes $\sim$ Carnot engine

Other Implementations

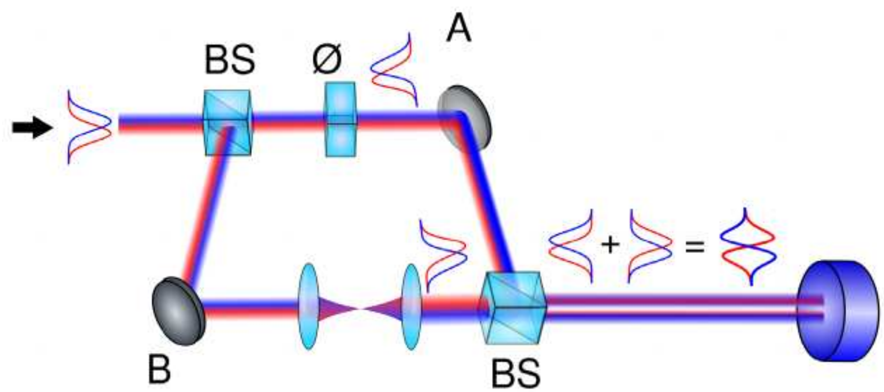
Yang *et al.* (Calgary), Optica (2016); Pushkina *et al.* (Oxford), PRL (2021)



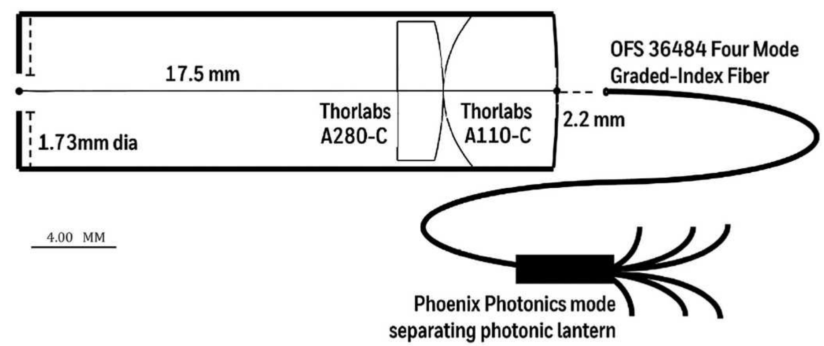
Ansari *et al.* (Paderborn), PRXQ (2020)



Tang *et al.* (CQT), OE (2016), etc.

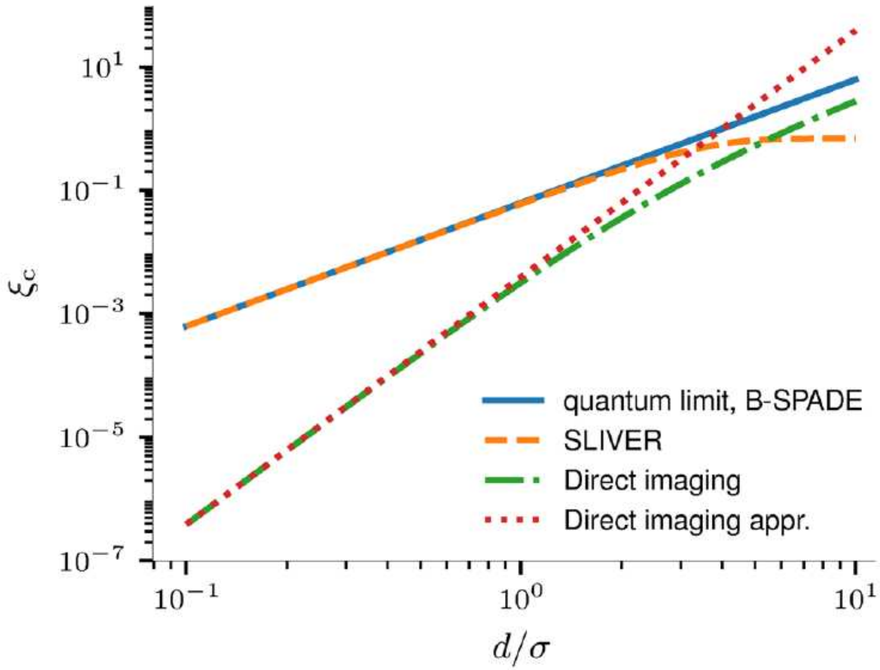


Salit *et al.* (Honeywell), AO (2020).



Other Statistical Problems

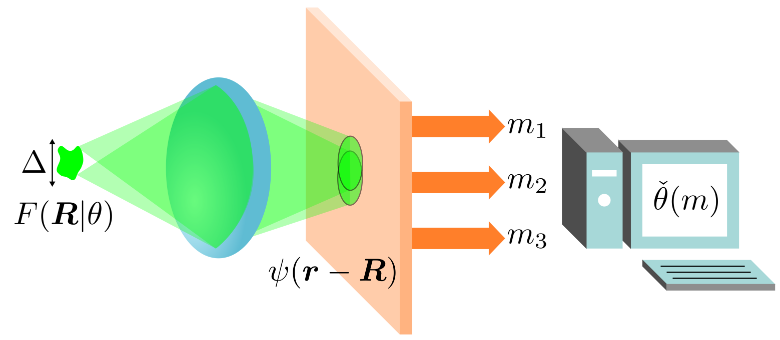
- Quantum hypothesis testing
  - Lu, Krovi, Nair, Guha, Shapiro, NPJQI (2018): one vs two, Chernoff exponents.



  - Zixin Huang, Cosmo Lupo, & collaborators: Neyman-Pearson, relative entropy
  - Multi-hypothesis testing: Michael Grace, Saikat Guha's group, Amit Ashok's group (Arizona), etc.

Part III: Extended Objects

Arbitrary Source Distribution



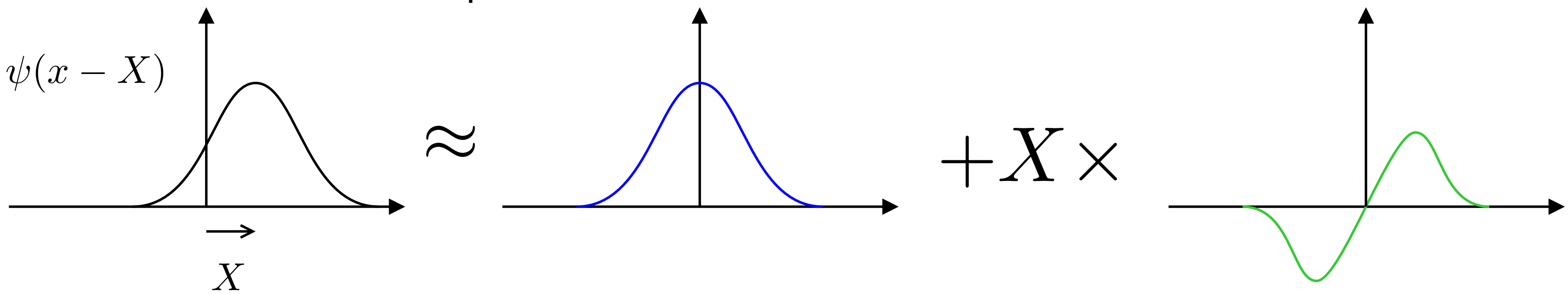
- Source distribution $= F(X)$. One-photon density operator:

$$\tau_1 = \int e^{-i\hat{k}X} |\psi\rangle \langle\psi| e^{i\hat{k}X} F(X) dX.$$

- Arbitrary F:
  - **UNKNOWN/INFINITE number of incoherent point sources**
  - **Infinite-dimensional parameter space!! (semiparametric)**
  - Multiple parameters/multiple point sources: Bisketzi *et al.*, NJP (2019); Bonsma-Fisher *et al.*, NJP (2019); Dutton *et al.*, PRA (2019); Bao *et al.*, OL (2021); Liang *et al.*, PRA (2021); Bearne *et al.*, OE (2021); Pushkina *et al.*, PRL (2021); Fiderer *et al.*, PRXQ (2021); Matlin and Zipp *et al.*, SR (2022); Lee *et al.*, arXiv (2022); Krovi, arXiv (2022); Liang *et al.*, OE (2023); Prasad, PRA (2020, 2023); Lee *et al.*, IEEE JSTSP (2023); Frank *et al.*, Optica (2023); etc.

SPADE for Even Moment Estimation

- Wavefunction from each point source:



- For one point source,

$$\text{Energy in first-order mode} \propto X^2.$$

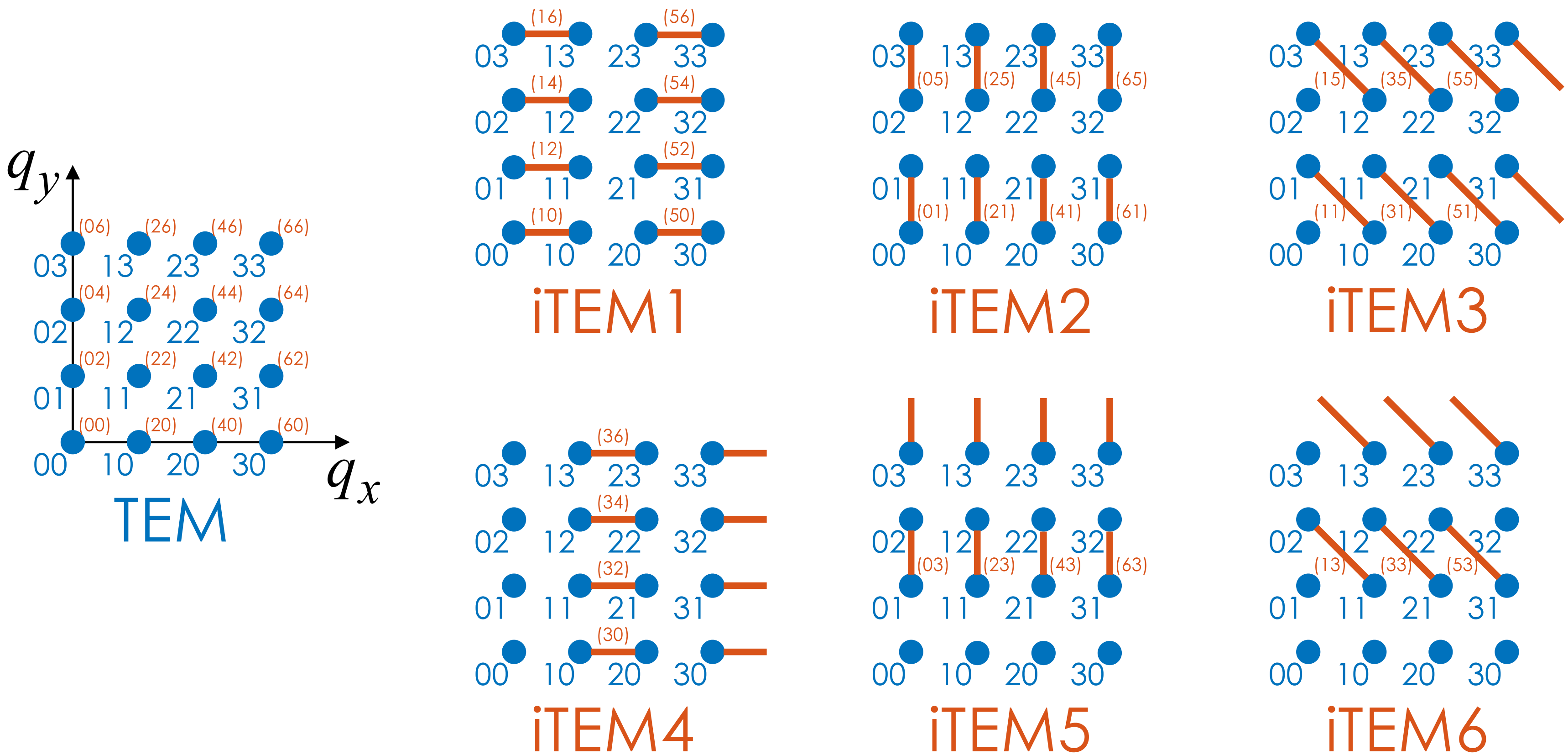
- A distribution of **incoherent** sources:

$$\text{Total energy in first-order mode} \propto \int X^2 F(X) dX.$$

- 2nd mode gives 4th moment, 3rd mode gives 6th moment, etc.
- unknown parameter $F(X)$ (in ∞-dim. function space), parameter of interest $\beta(F)$ is a functional (scalar).

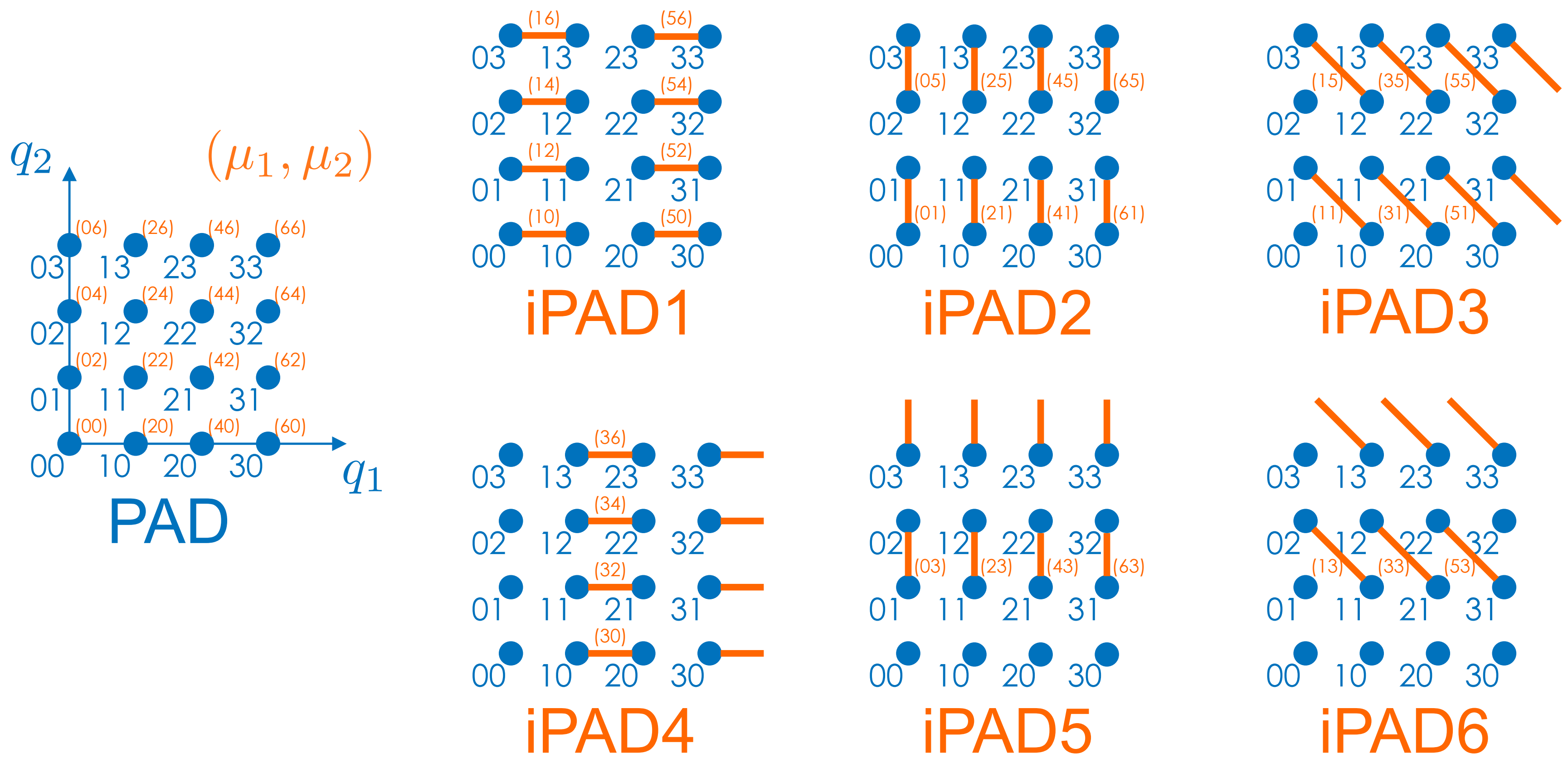
Generalized SPADE for Moment Estimation in 2D Imaging

- Gaussian PSF [Tsang, NJP (2017)]:
  - For moments with even μ_1 & even μ_2: TEM basis
    - See also Yang *et al.*, Optica (2016)
  - For other moments: interference of pairs of TEM modes



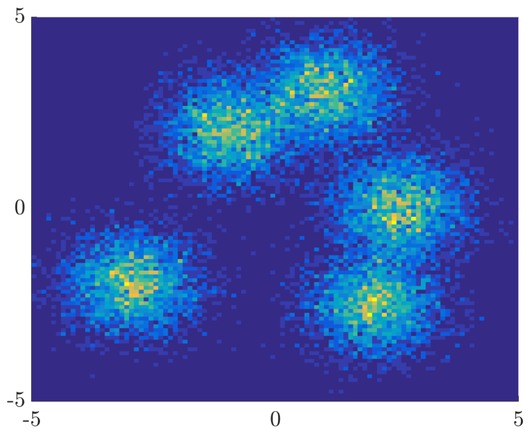
More General PSFs

- Any centrosymmetric and separable PSF [Tsang, PRA 97, 023830 (2018)]:
  - For moments with even μ_1, even μ_2: **"PSF-adapted" (PAD) basis** (Rehacek *et al.* OL (2017), generalizes TEM)
  - For other moments: interference of pairs of PAD modes

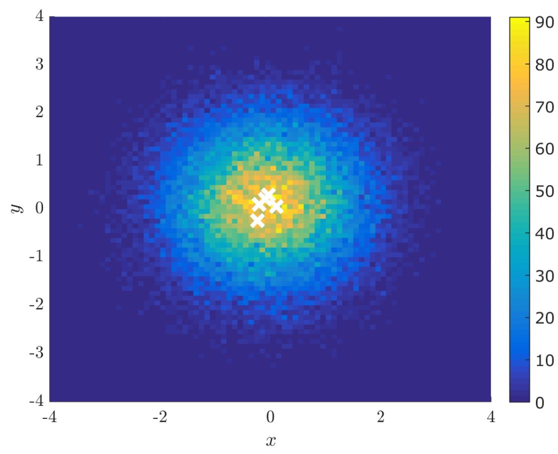


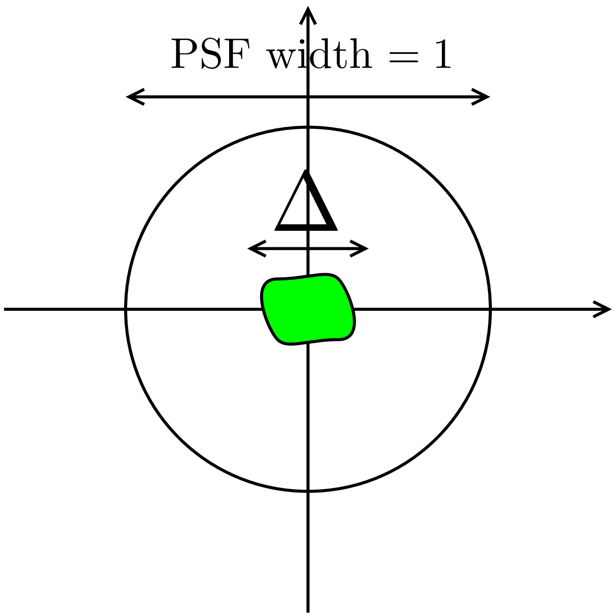
Defining the Subdiffraction Regime

Sparse (**Good Regime**, PALM, STED, compressed sensing, etc.)



Subdiffraction (Worst-Case Regime)





object width (relative to origin) $\equiv \Delta \ll 1.$

Define the parameter of interest as an object moment:

$$\beta = \int [X^\mu + o(\Delta^\mu)] F(X) dX, \quad \mu = \text{integer}$$

Arbitrary F, scalar parameter of interest: **Semiparametric**

Performance of Generalized SPADE

- Tsang, NJP (2017); PRA 97, 023830 (2018):

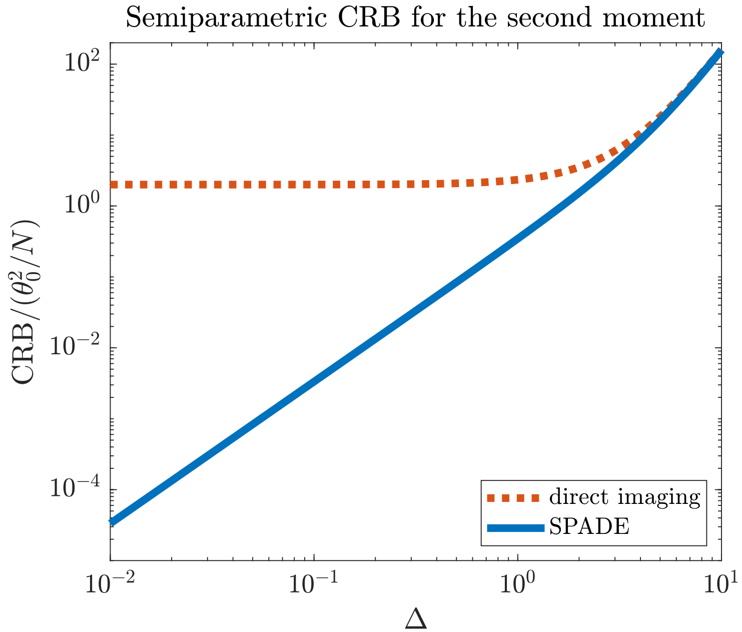
$$\boxed{\mathrm{MSE}^{(\mathrm{SPADE})} = \frac{\Theta\left(\Delta^{2\lfloor \mu/2 \rfloor}\right)}{N}.}$$

- **Good news**: compare with direct imaging (Gaussian PSF) [Tsang, PRR (2019)]:

$$\boxed{\mathrm{CRB}^{(\mathrm{direct})} = \frac{\Theta(1)}{N}.}$$

big enhancement over direct imaging when

– $\Delta \ll 1$ (subdiffraction)
– $\mu = \mu_X + \mu_Y \geq 2$



- **Bad news**: $\beta^2 = O(\Delta^{2\mu})$, SNR:

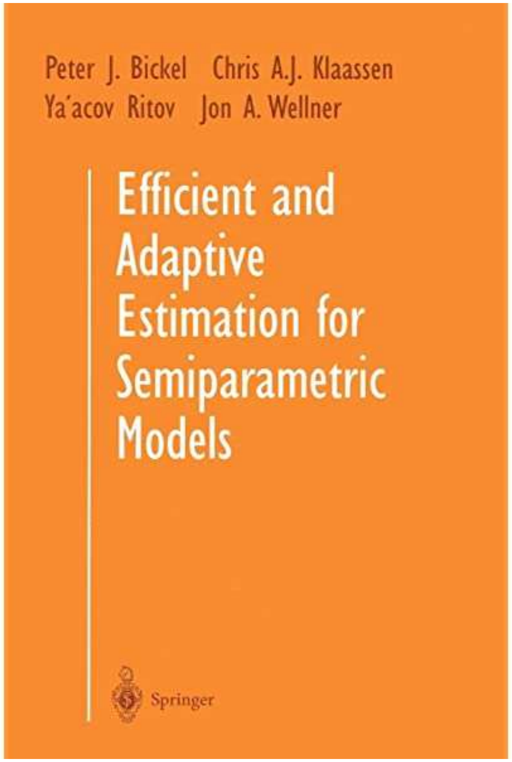
$$\boxed{\frac{\beta^2}{\mathrm{MSE}^{(\mathrm{SPADE})}} = N O(\Delta^{2\lceil \mu/2 \rceil}).}$$

Need many photons, especially for large μ.

- **Quantum limit?**

Classical Semiparametric Estimation

- CRB$^{\text{(direct)}}$ is not for the faint-hearted!
- multiparameter CRB: $(\partial\beta)^\top J^{-1}\partial\beta$, J is **infinite-dimensional**
- Bickel *et al.*, *Efficient and Adaptive Estimation for Semiparametric Models.*



- Reformulate CRB using **abstract Hilbert-space theory**
- Gives **exact** CRB$^{\text{(direct)}}$ for Gaussian PSF [Tsang, PRR (2019)]

Quantum Semiparametric Estimation

PHYSICAL REVIEW X **10,** 031023 (2020)

Quantum Semiparametric Estimation

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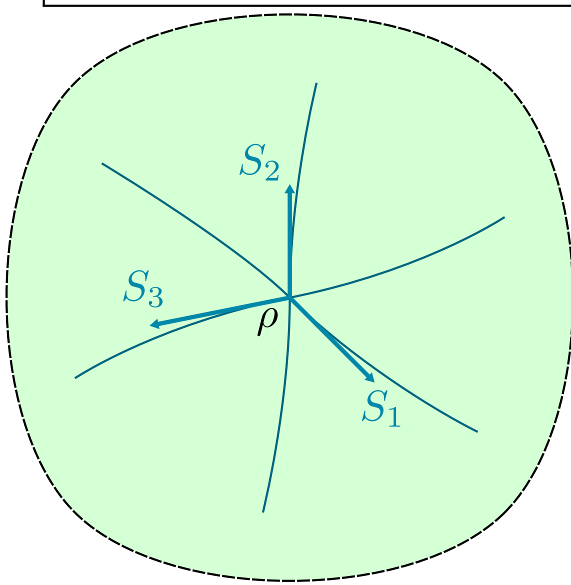
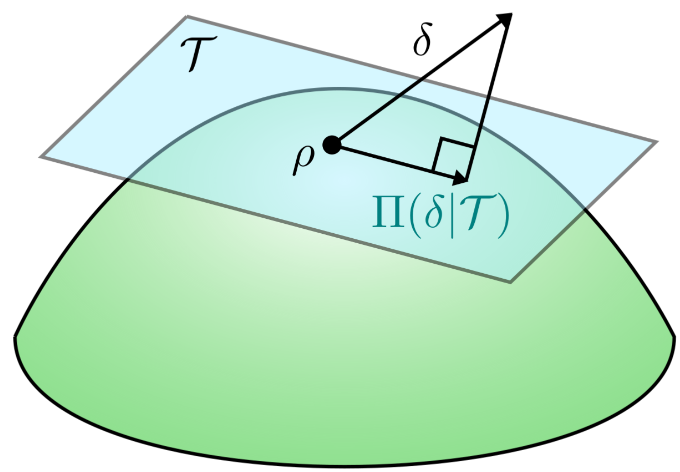




- Tsang, Albarelli, Datta, PRX (2020).

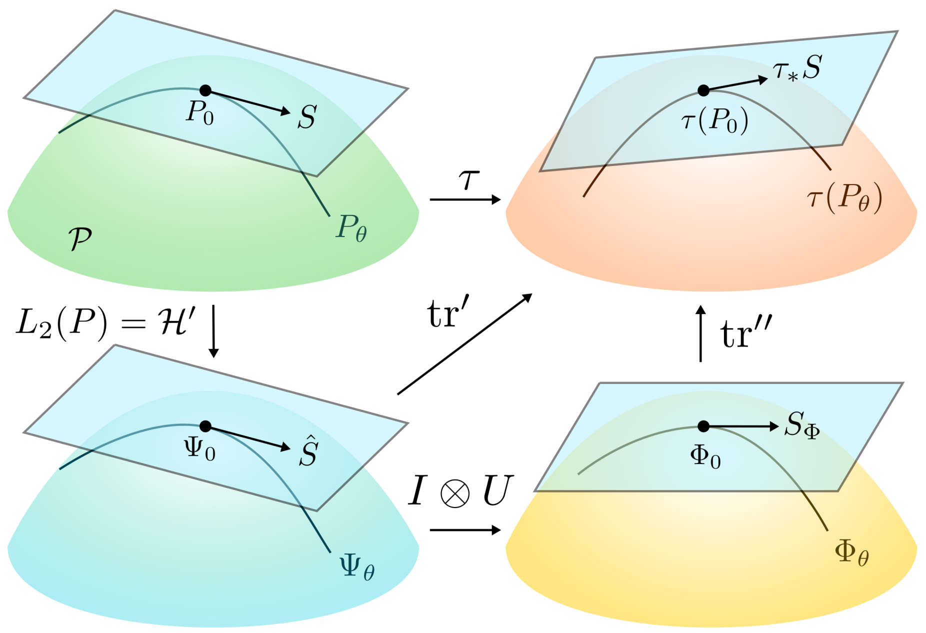
Quantum Limit on Semiparametric Imaging

- One-photon density operator: $\rho_1(F) = \int e^{-i\hat{k}X} |\psi\rangle \langle\psi| e^{i\hat{k}X} F(X) dX$, $F \in \{\text{any probability density}\}$, $\beta(F) = \int [X^\mu + o(\Delta^\mu)] F(X) dX$.
- **Rigorous quantum bound** [Tsang PRA 104, 052411 (2021), purification method (Escher, Filho, Davidovich, NP 2011) + other tricks]:

$$\boxed{\text{QCRB} = \frac{\Omega(\Delta^{2\lfloor \mu/2 \rfloor})}{N},} \tag{1}$$

compare with

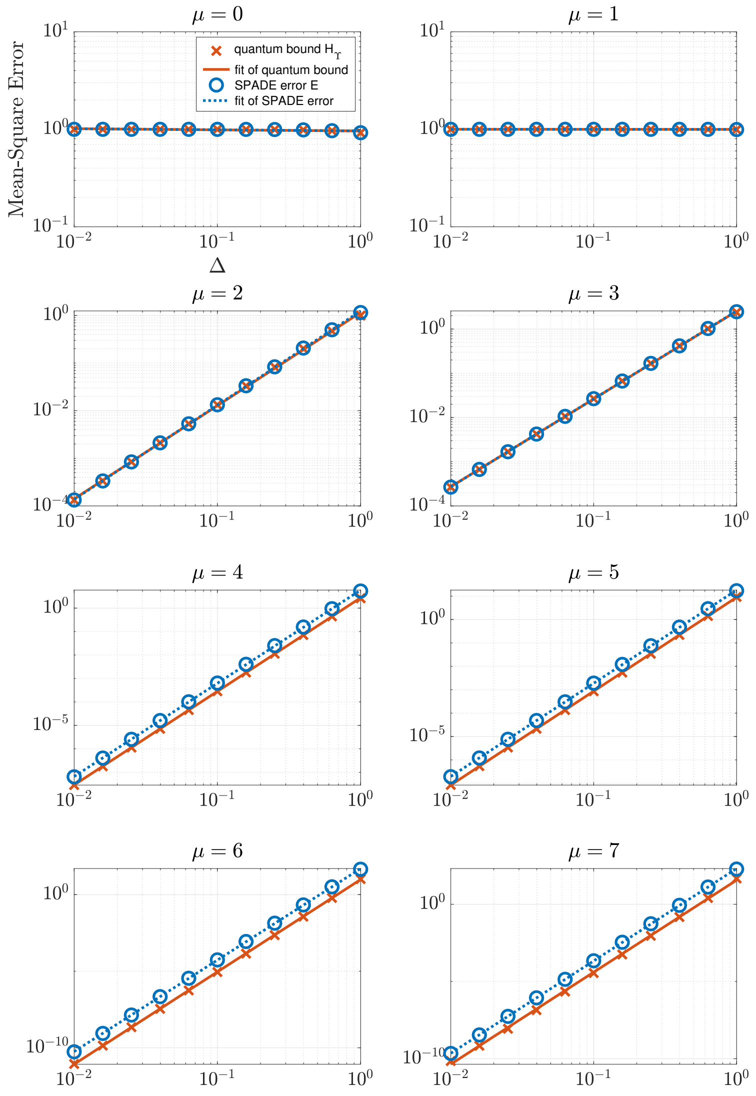
$$\text{MSE}^{(\text{SPADE})} = \frac{\Theta(\Delta^{2\lfloor \mu/2 \rfloor})}{N}. \tag{2}$$



- See also Tsang, PRA 99, 012305 (2019); Zhou & Jiang (Yale/Chicago), PRA (2019).

Numerics [Tan & Tsang, arXiv:2308.04317 (2023)]

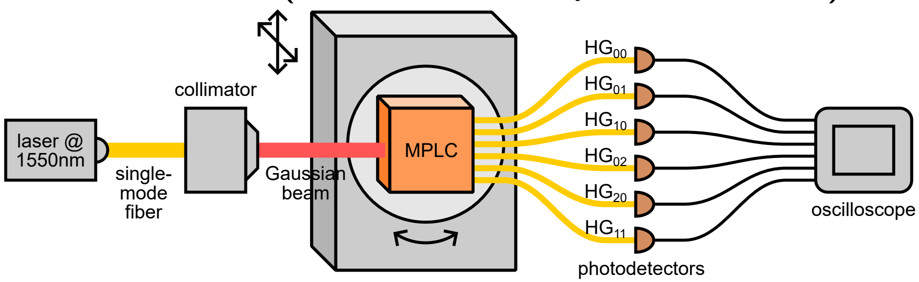
- ☐ Compare SPADE with QCRB of finite-dimensional "parametric submodel"
- ☐ **Same Δ scaling as theory**.
- ☐ Substantial gaps between SPADE and QCRB for $\mu \geq 4$



Part IV: Our Experiment

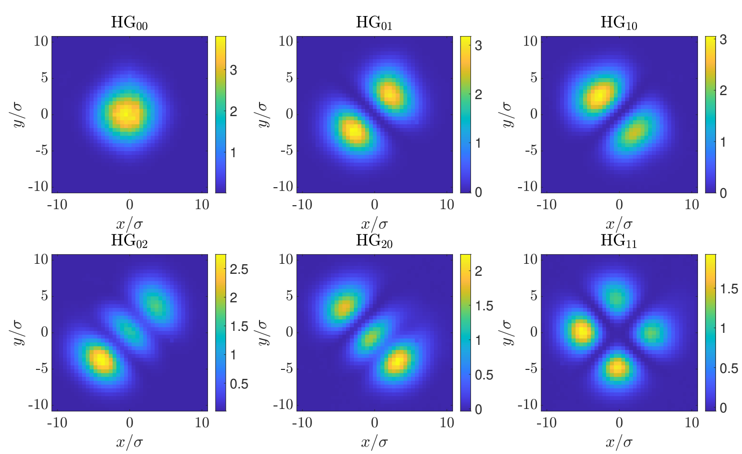
Experiment [Tan, Qi, ..., Tsang, Optica 10, 1189 (2023)]

- with laser (nowhere near quantum limit):



- Gaussian beam $\sim$ point-spread function with one point source, demultiplexer from Cailabs (MPLC).
- **MPLC output Y_{nm} of mode ϕ_{nm} vs beam displacement $\boldsymbol{R}$ (point-source location)**:

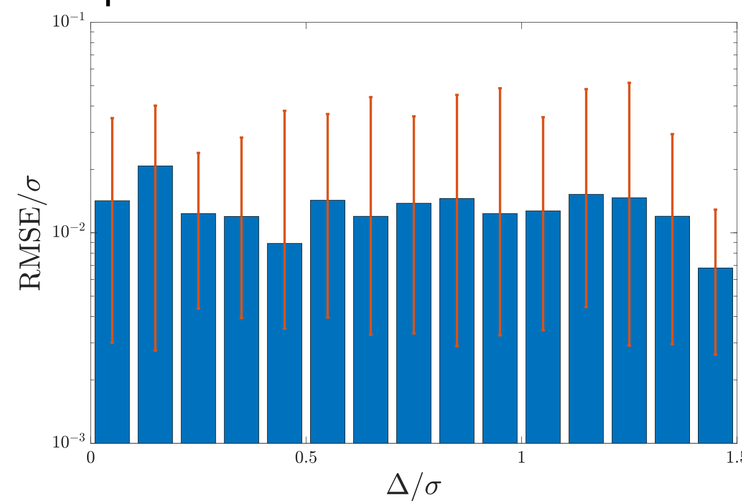
$$\bar{Y}_{nm}(\boldsymbol{R}) = \left| \langle \phi_{nm} | \underbrace{e^{-i\hat{\boldsymbol{k}}\cdot\boldsymbol{R}} |\psi\rangle}_{\text{displaced Gauss beam}} \right|^2.$$

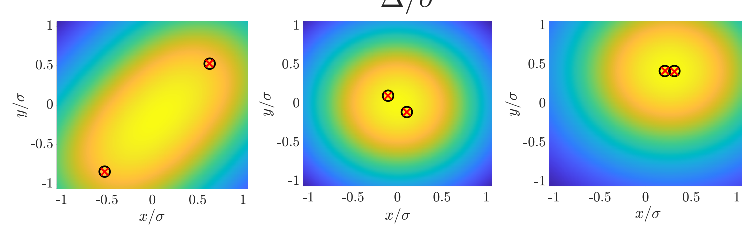


- For each mode ϕ_{nm}, sum data points with S displacements $\{\boldsymbol{R}_1, \ldots, \boldsymbol{R}_S\}$:

$$Z_{nm}(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_S) \equiv \sum_{s=1}^{S} Y_{nm}(\boldsymbol{R}_s)$$

$\sim$ total output of mode ϕ_{nm} from distribution of S incoherent point sources.
- Two point sources:



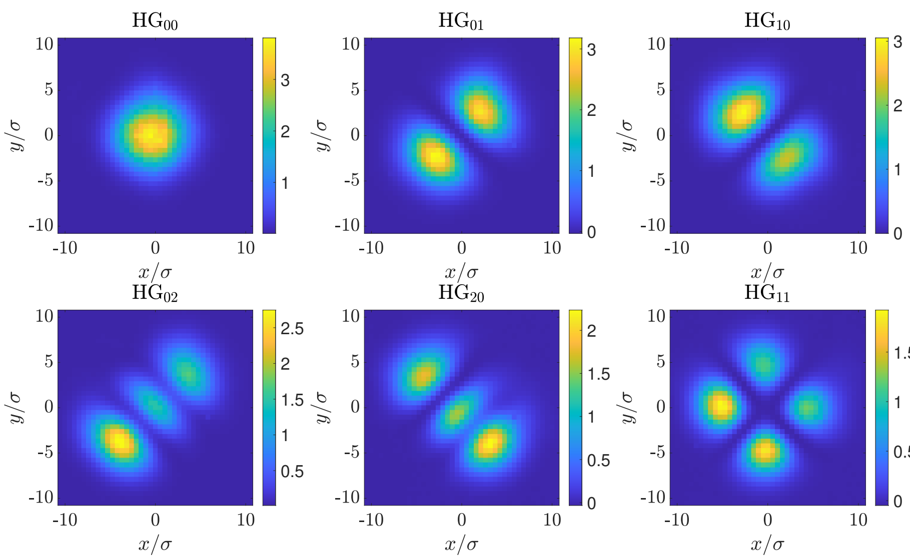


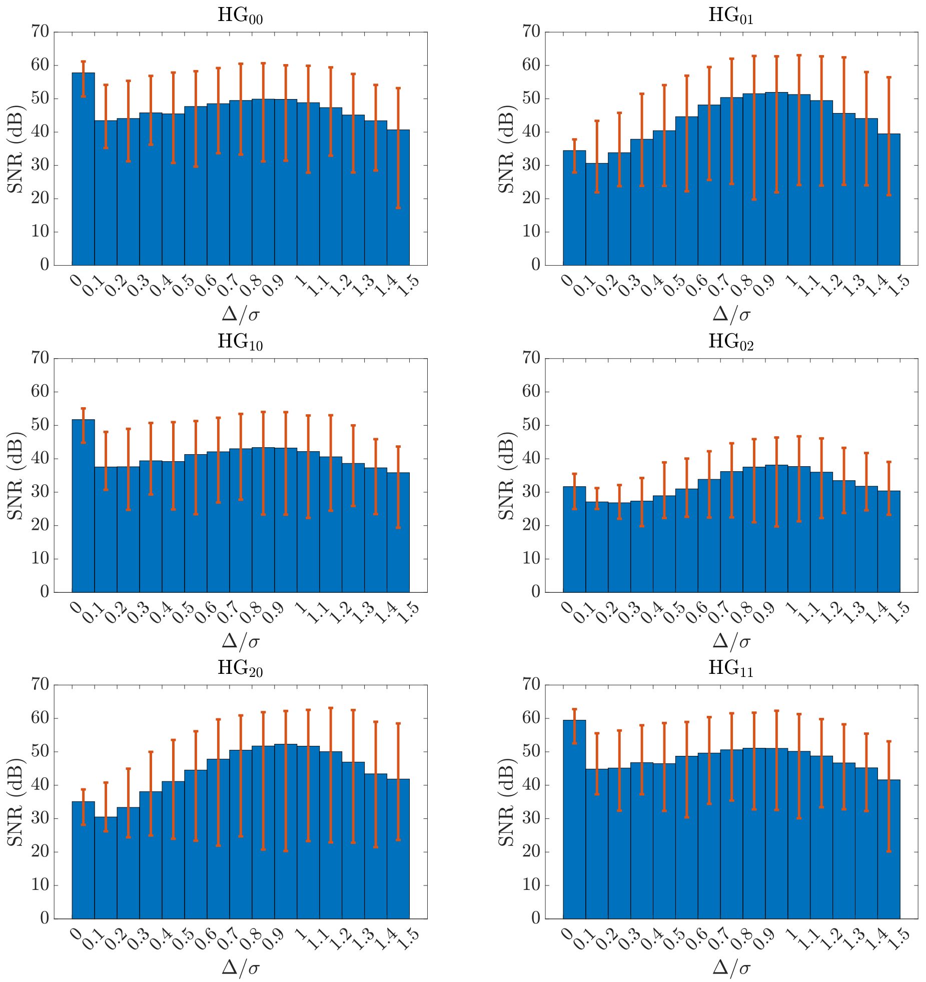
Semiparametrics with Multiple Sources

- Define parameter of interest:

$$\beta_{nm} = \int \bar{Y}_{nm}(\boldsymbol{R}) F(\boldsymbol{R}) d^2\boldsymbol{R},$$

$$\bar{Y}_{nm}(\boldsymbol{R}) = \left| \langle \phi_{nm} | \, e^{-i\hat{\boldsymbol{k}}\cdot\boldsymbol{R}} \, |\psi\rangle \right|^2 .$$



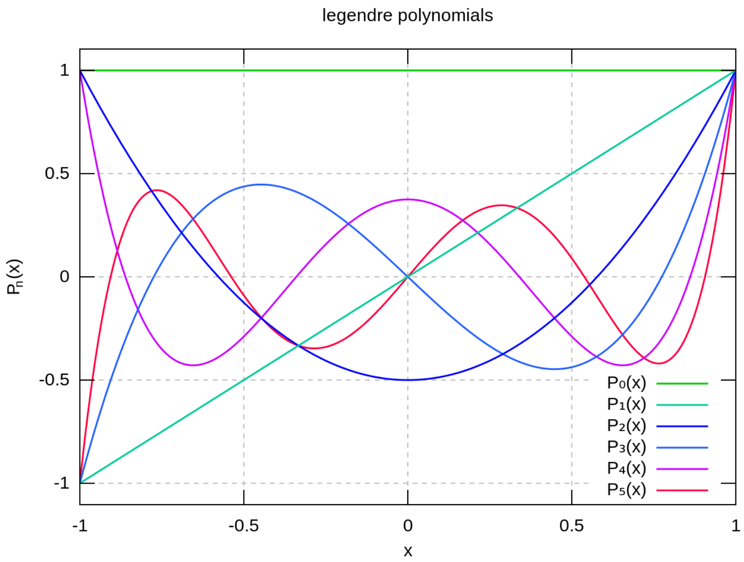


Superoscillation

□

$$\alpha_\mu = a_0\beta_0 + a_1\beta_1 + \cdots + a_\mu\beta_\mu = \int \underbrace{\left(a_0 + a_1 X + \cdots + a_\mu X^\mu\right)}_{\text{polynomial}} F(X)dX \quad (3)$$

□ The polynomial can be an **orthogonal polynomial** (oscillatory).



(Legendre polynomials, from wikipedia)

□ α_μ is like a **Fourier coefficient**.

□ Resembles **superoscillation** and **Slepian** theory

□ Tsang, IEEE JSTSP (2023).

Generalizations

- □ So far: scalar parameter of interest $\beta(\theta) \in \mathbb{R}$, Helstrom's QCRB.
- □ Multiple parameters of interest ($\beta(\theta)$ is multidimensional):
  - Holevo's QCRB:

$$\text{Helstrom} \leq \text{Holevo} \leq 2 \times \text{Helstrom}. \tag{4}$$

    [Carollo *et al.* JSM (2019); Erratum (2020); Tsang, Albarelli, Datta, PRX (2020)]
  - Asymptotically achievable by **entangling measurement** on $\rho^{\otimes N}$.
- □ Multiparameter + measurement restrictions: only linear optics, only separable, etc.?
  - Nagaoka-Hayashi bound?
- □ Remove unbiased condition: Bayesian/minimax?
  - Tsang, JMO (2018).

Part IV: Quantum Noise Spectroscopy

Random Displacement Models

- Incoherent imaging: randomly displaced **photons**

$$\rho_\theta = \int d^2\boldsymbol{R} F_\theta(\boldsymbol{R}) e^{-i\hat{\boldsymbol{k}}\cdot\boldsymbol{R}} \left|\psi\right\rangle \left\langle\psi\right| e^{i\hat{\boldsymbol{k}}\cdot\boldsymbol{R}}.$$

- Noise spectroscopy: randomly displaced **mechanical objects or spin ensembles or optical fields**:

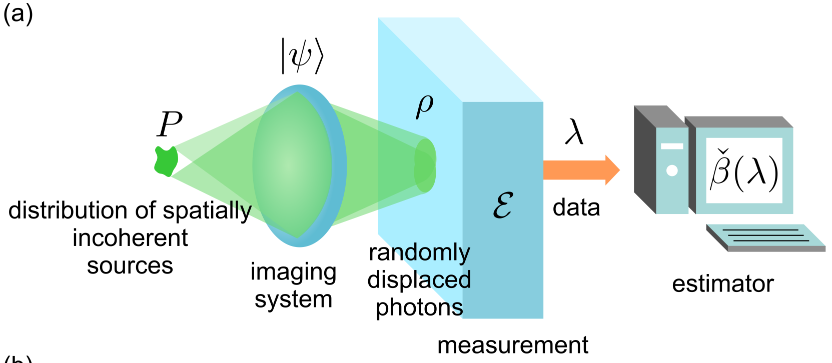
$$\rho_\theta = \int dP_\theta(X) U_X \left|\psi\right\rangle \left\langle\psi\right| U_X^\dagger,$$

$$U_X = \mathcal{T} \exp\left[-i\int_0^T \hat{k}(t)X(t)\right].$$

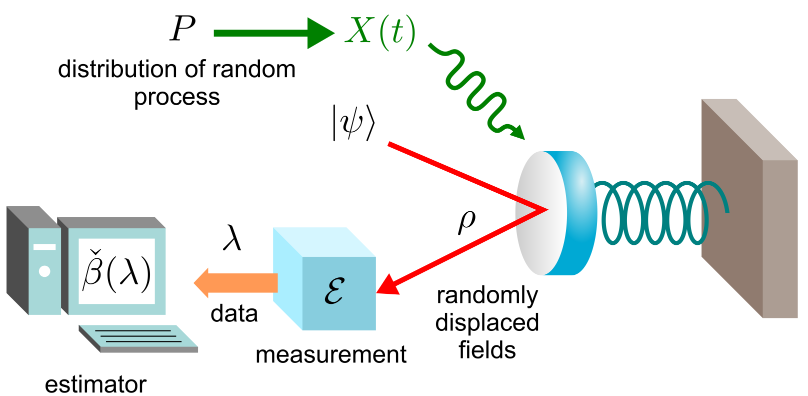
e.g., stochastic g-wave background, stochastic magnetic fields, stochastic optical phase modulation

- Search for **axion dark matter** with microwave cavities (Lehnart): Gorecki, Riccardi, Maccone, PRL (2023); Shi & Zhuang, NPJQI (2023).

(a)



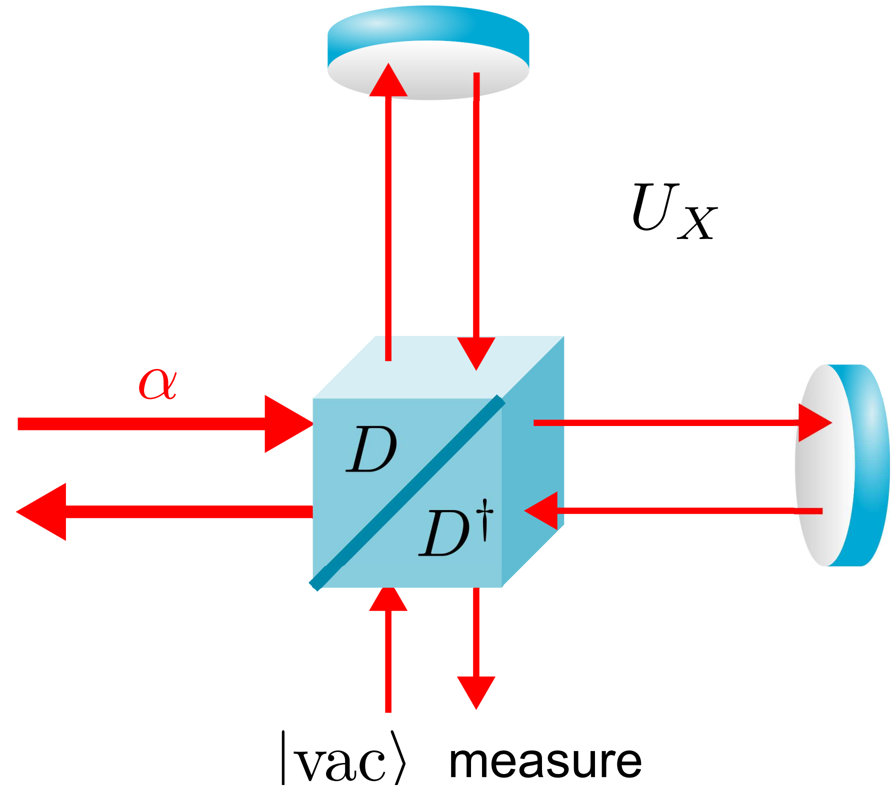
(b)

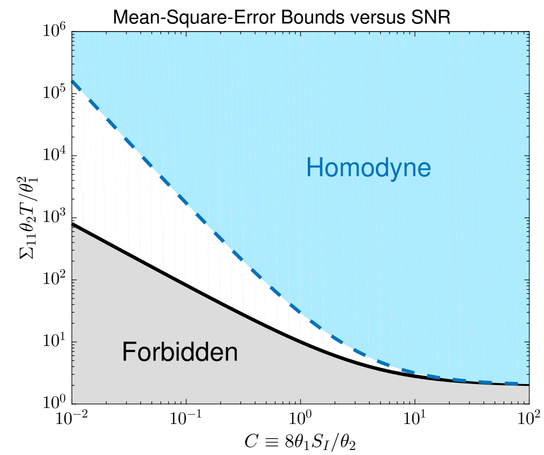


- For continuous/sequential measurements (e.g., LIGO), use **principle of deferred measurements** to model as one final-time measurement [Tsang, Wiseman, Caves, PRL (2011)].

Example: Michelson Interferometer

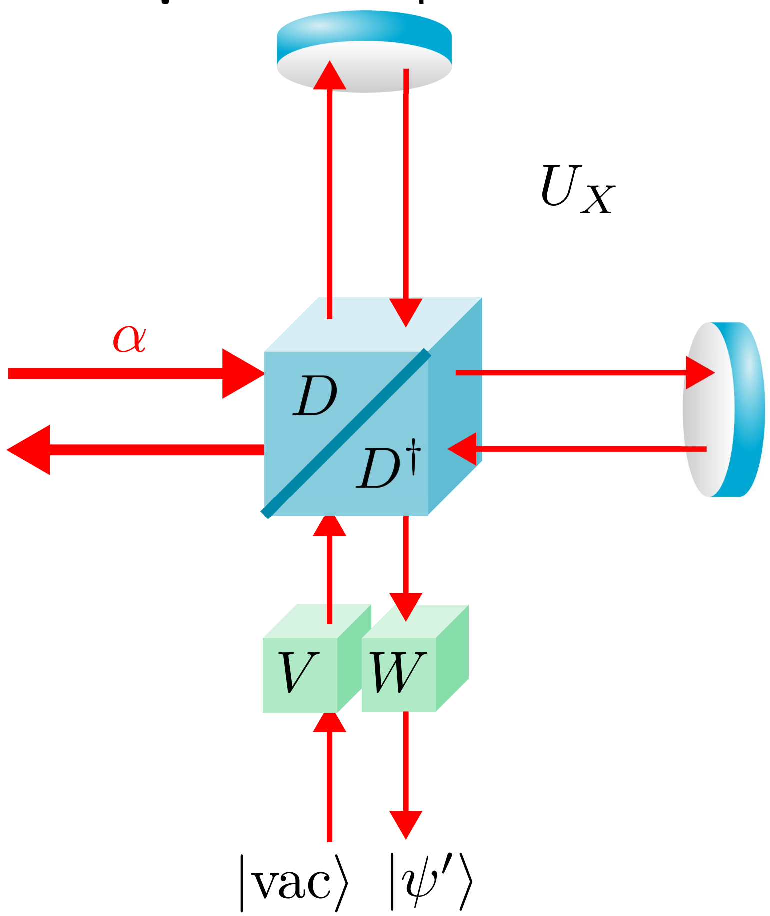
- **Example**: weak stochastic phase modulation $X(t)$ on Michelson interferometer.
  - Assume X is Gaussian stationary process, power spectral density $S_\theta(\omega)$ depends on θ.
  - When dark-port input is vacuum, **Spectrometer + Photon Counting** can
    - saturate QCRB,
    - beat homodyne by **a lot**
  - Ng *et al.*, "Spectrum analysis with quantum dynamical systems," PRA (2016).





Squeezed Vacuum Input

- Optimal measurement: **Unsqueeze** + Spectrometer + Photon Counting



- Tsang, "Quantum noise spectroscopy as an incoherent imaging problem," PRA 107, 012611 (2023).
- See also Gorecki, Riccardi, Maccone, PRL (2023); Shi & Zhuang, NPJQI (2023).

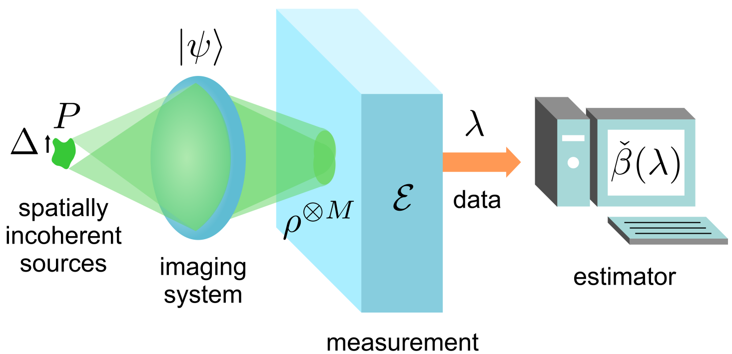
Experimental Difficulties

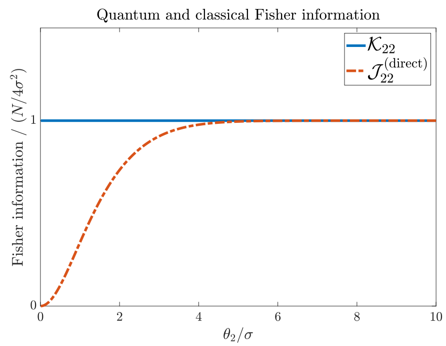
- Caves, PRD (1980):

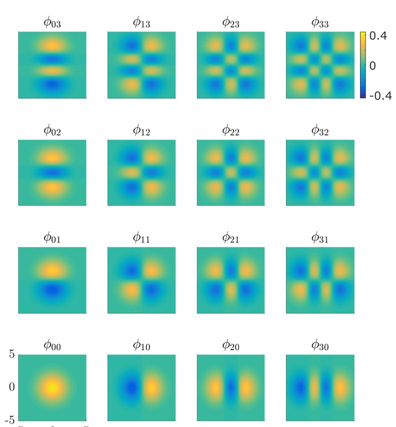
IV. CONCLUSION

The squeezed-state technique outlined in this paper will not be easy to implement. A refuge from criticism that the technique is difficult can be found by retreating behind the position that the entire task of detecting gravitational radiation is exceedingly difficult. Difficult or not, the squeezed-state technique might turn out at some stage to be the only way to improve the sensitivity of interferometers designed to detect gravitational waves. As interferometers are made longer, their strain sensitivity will eventually be limited by the photon-counting error for the case of a storage time approximately equal to the desired measurement time. Further improvements in sensitivity would then await an increase in laser power or implementation of the squeezed-state technique. Experimenters might then be forced to learn how to very gently squeeze the vacuum before it can contaminate the light in their interferometers.

Conclusion

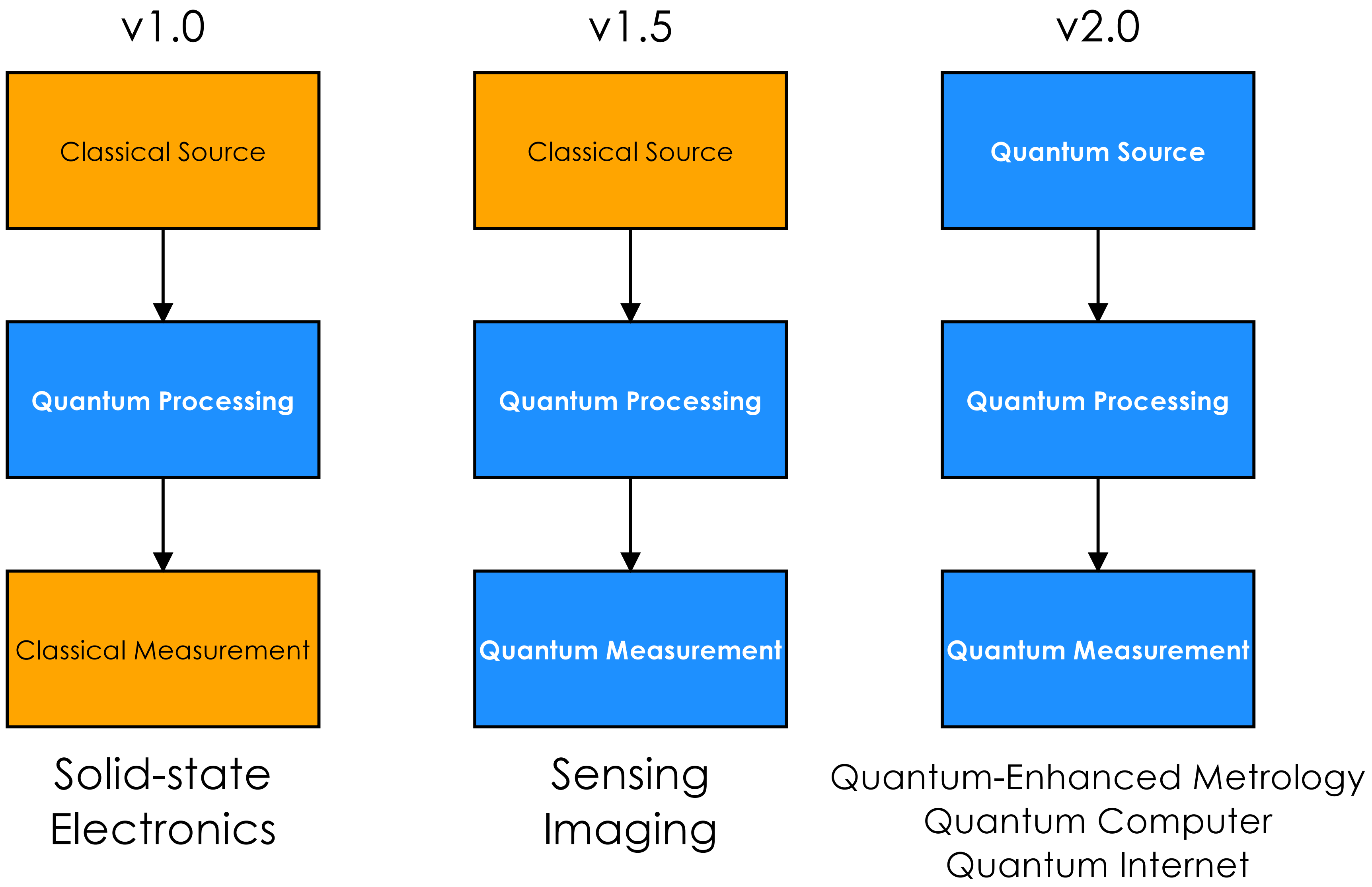






- **Quantum limits to imaging**
  - analogy: laws of thermodynamics
- **Optimal measurement**: SPADE
  - analogy: Carnot engine
  - substantial improvement over direct imaging
- **Random displacement models**: Michelson, magnetometry, LIGO, dark matter search, etc.
- National Research Foundation Singapore
  - Quantum Engineering Programme (QEP-P7): experiments.
- mankei@nus.edu.sg
- https://blog.nus.edu.sg/mankei/

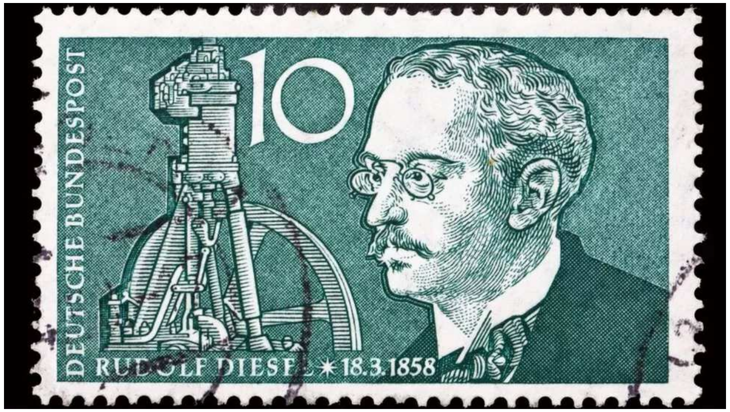
Quantum Technology 1.5



From Carnot to Diesel

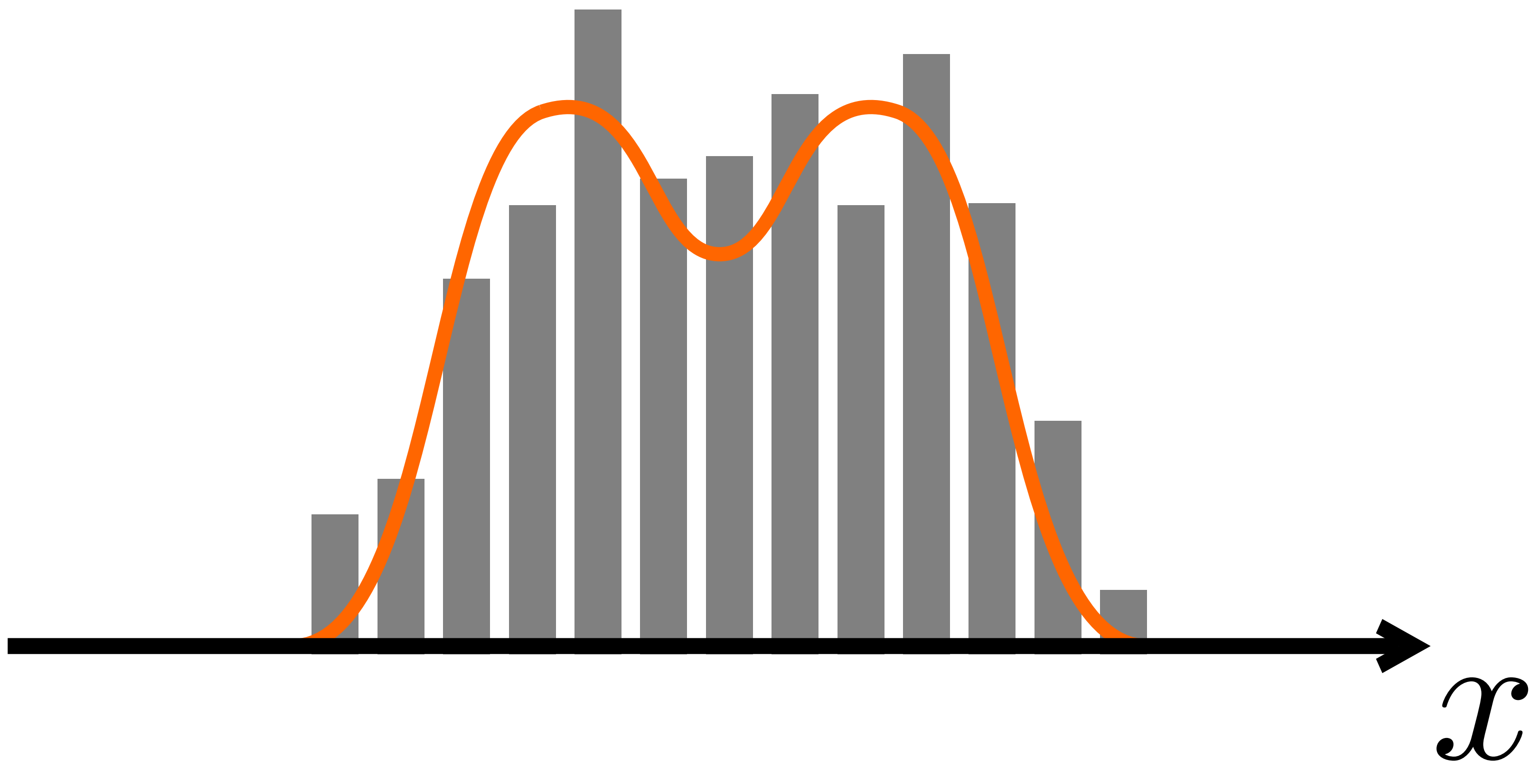
- 1824: Carnot published "*Reflections on the Motive Power of Fire*"
- 1832: asylum, died of cholera, aged 36
- 1850s: inspired Clausius, Kelvin to formulate 2nd law
- 1873–1879: As student, Diesel learned about Carnot engine, huge theoretical improvement over steam engines. Inspired to implement.
- 1892–1893: first experimental proposal (wrong)
- 1897: first successful demo
- 1898: millionaire
- 1913: disappeared (suicide or murdered)

(wikipedia)



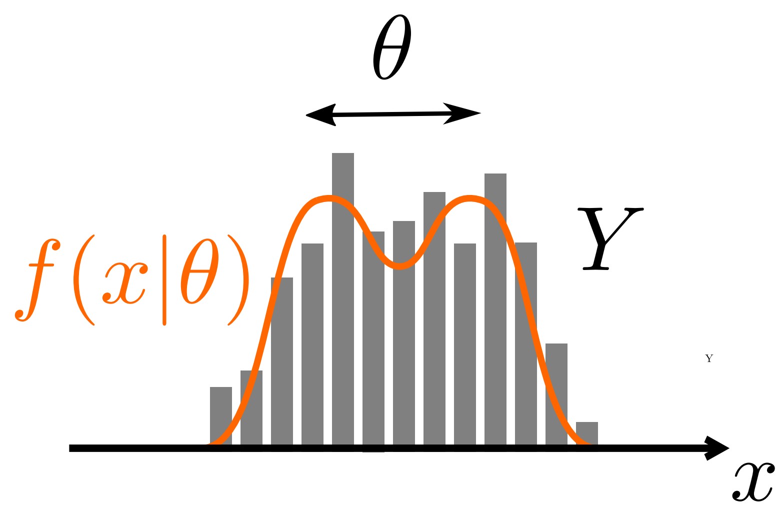
(bbc.com)

Photon Shot Noise



- **Poisson** at optical frequencies
- Bunching (thermal)/anti-bunching (fluor.) negligible in practice
- Random arrival of "photons"

Parameter Estimation

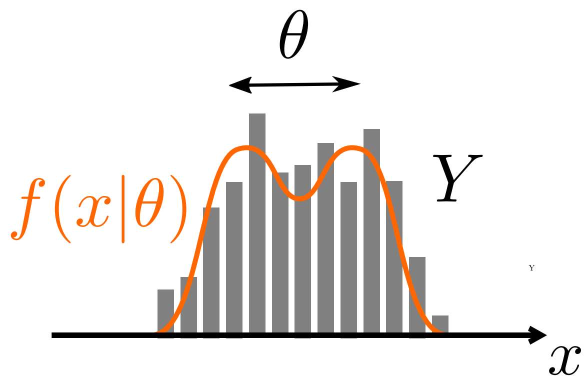


- **Estimator** $\check{\theta}(Y)$: guess θ from noisy data Y
- **Mean-square error**: $\mathsf{MSE} = \mathbb{E}\left[\check{\theta}(Y) - \theta\right]^2$.
- Cramér-Rao bound (unbiased estimators):

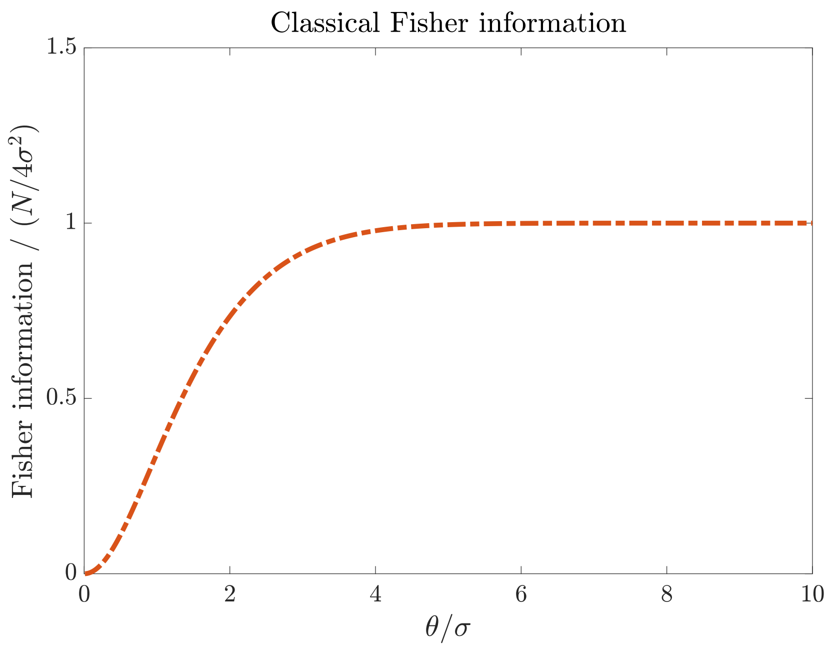
$$\mathsf{MSE}(\theta) \geq J(\theta)^{-1}, \qquad J(\theta) = N\int_{-\infty}^{\infty} dx f(x|\theta)\left[\frac{\partial}{\partial\theta}\ln f(x|\theta)\right]^2.$$

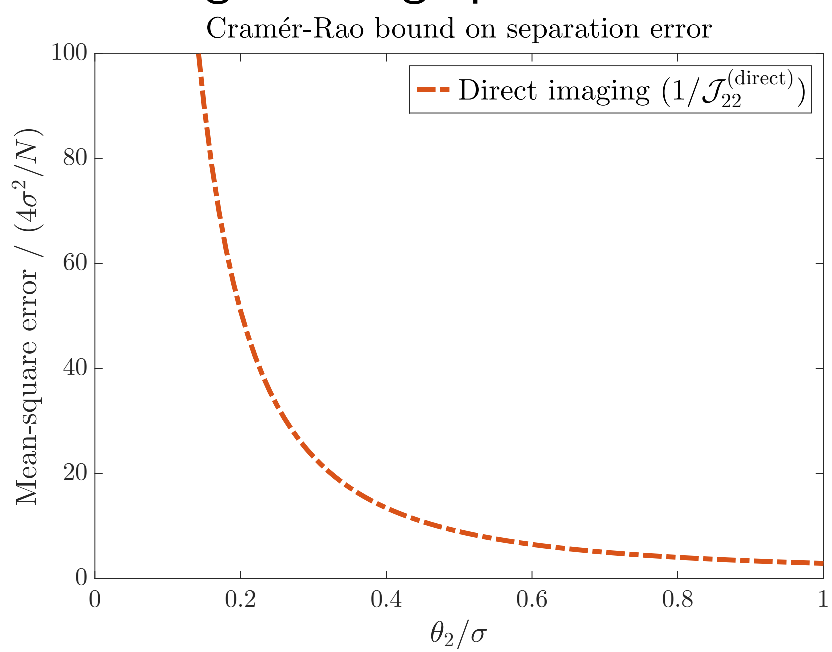
- J: Fisher information
- $\mathsf{MSE} \to J(\theta)^{-1}$ via maximum-likelihood.

Estimating Two-Point Separation



- ☐ Conventional **direct imaging** (photon counting on image plane, Poisson noise):

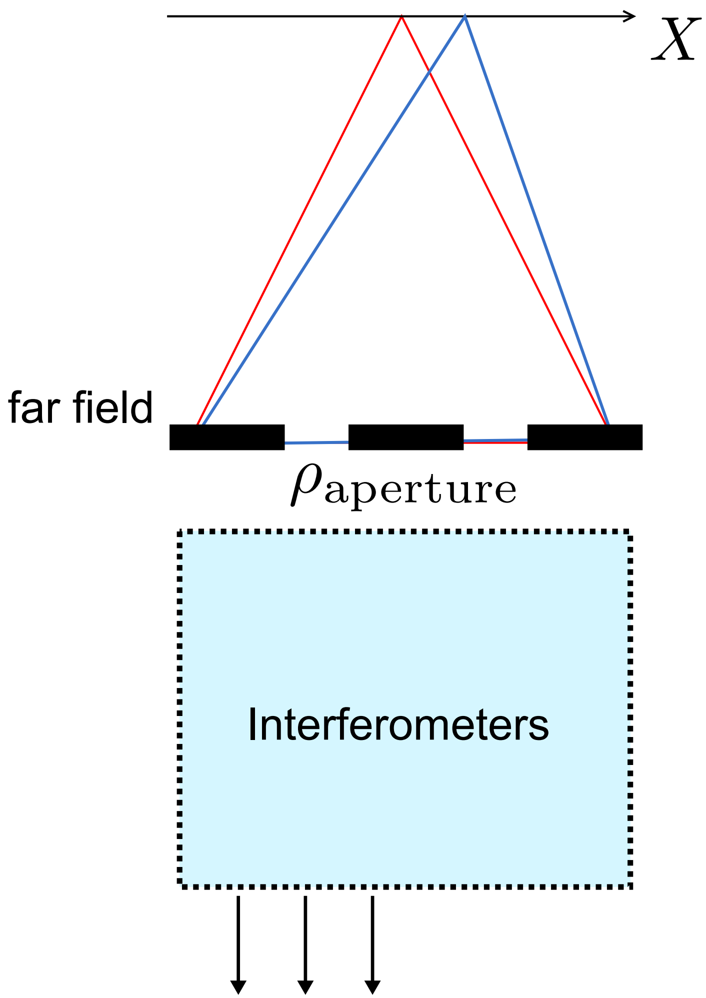


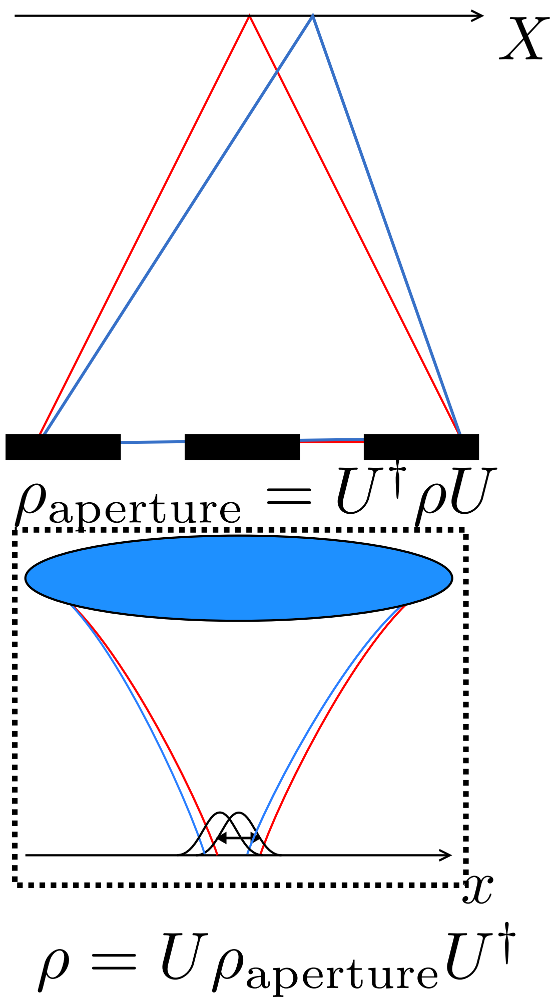


$$J(0) = 0,\ J(\infty) = \frac{N}{4\sigma^2},\ \sigma = \frac{\lambda}{\text{NA}} \qquad \frac{1}{J(0)} = \infty,\ \frac{1}{J(\infty)} = \frac{4\sigma^2}{N}$$

- ☐ **“Rayleigh’s curse”**
- ☐ See, e.g., Ram, Ward, Ober, PNAS **103**, 4457 (2006).

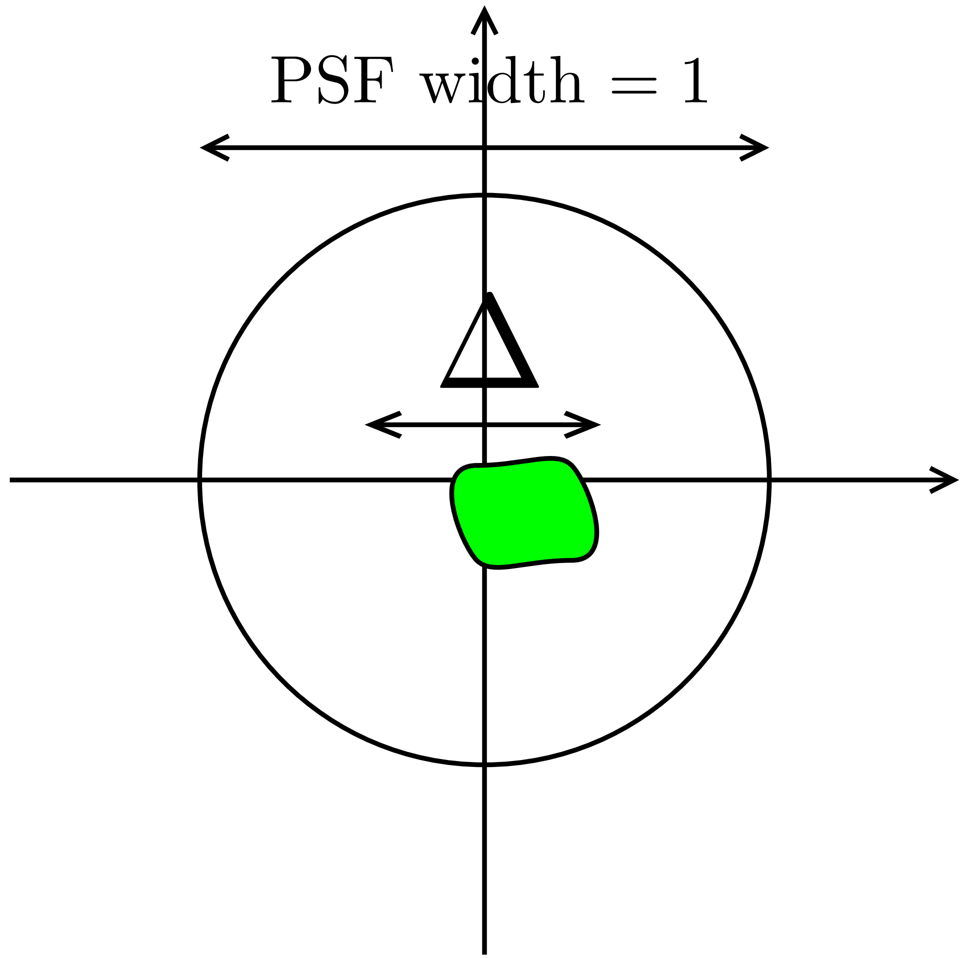
Stellar Interferometry





- Same principles apply to stellar interferometry if we **back-propagate**.
- Use the aperture function as the optical transfer function $(\psi(x) \to \Psi(k))$

Misalignment

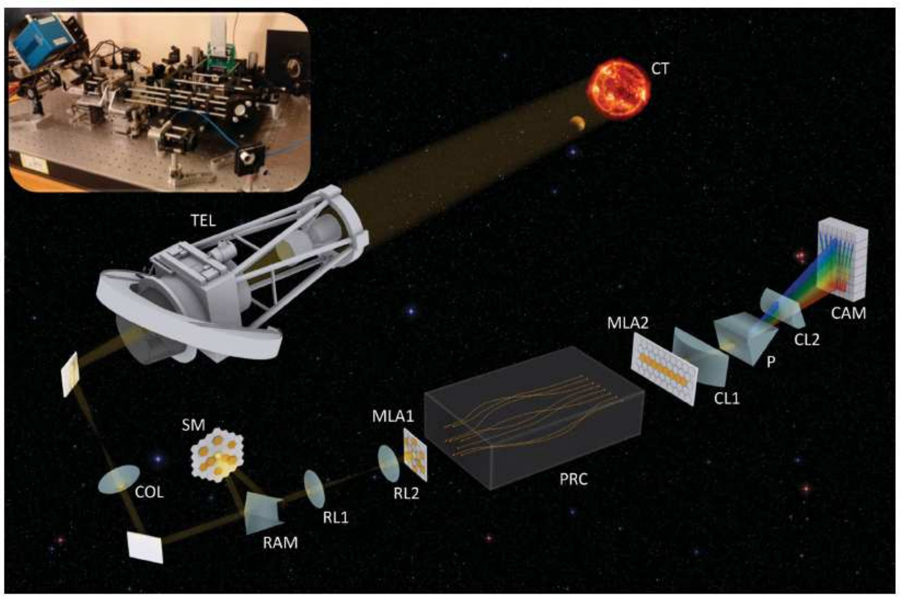


$$\Delta = \textbf{centroid displacement} + \textbf{object size} \ll 1$$

Adaptive measurements: Grace *et al.* (Arizona), JOSAA **37**, 1288 (2020).

Stellar Interferometry

- **Astrophotonics**: photonic circuits for stellar interferometry

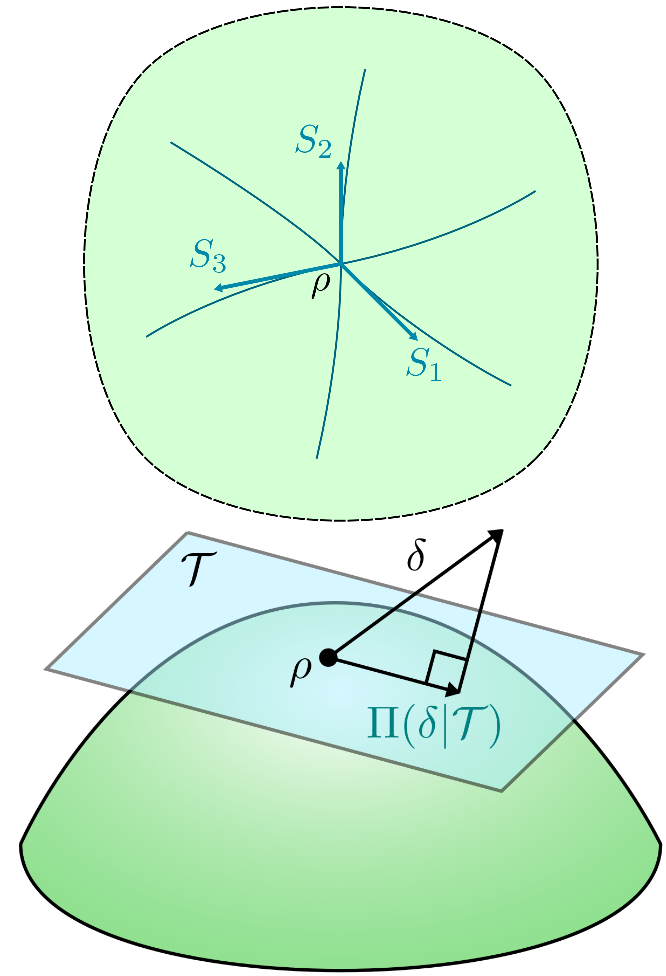


"Dragonfly," Jovanovic *et al.*, Mon. Not. R. Astron. Soc. **427**, 806 (2012)

- **Conventional wisdom**: robust against atmospheric turbulence, **can't compete with direct imaging at diffraction limit**; see, e.g.,
  - Goodman, *Statistical Optics*
  - Zmuidzinas, JOSA A (2003): *It is important to remember that the imperfect beam patterns of sparse-aperture interferometers extract a sensitivity penalty as compared with filled-aperture telescopes, even after accounting for the differences in collecting areas.*
- **Our work**:
  - **Advantage with diffraction + photon shot noise**
  - **Optimal modes adapted to PSF/aperture**
  - **Quantum limit**

Detour: Quantum Semiparametric Estimation

- Let ρ be the true density operator.
- **Hilbert space of observables** $L_2(\rho)$ with $\langle a, b\rangle \equiv \operatorname{tr}(a \circ b)\rho$ (Holevo), $a \circ b \equiv (ab + ba)/2$.
- Zero-mean subspace: $\mathcal{Z} \equiv \{h \in L_2(\rho) : \langle h, I\rangle = \operatorname{tr}\rho h = 0\} \subset L_2(\rho)$.
- Each 1D **parametric submodel** with scalar θ_j gives a **SLD score operator** S_j that obeys $\partial\rho/\partial\theta_j = S_j \circ \rho$.
- **Tangent space**: $\mathcal{T} \equiv \overline{\operatorname{span}}\{S_j\} \subseteq \mathcal{Z}$.
- Define parameter of interest $\beta(\theta)$, error operator $\delta \equiv \int \check{\beta}(x)E(dx) - \beta(\theta)$ (E = POVM).
- δ is an "influence operator," defined by $\partial_j\beta = \langle S_j, \delta\rangle$. δ acts like a gradient of β.

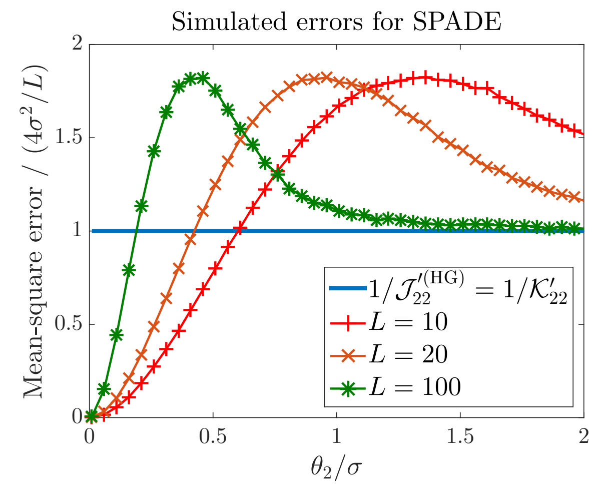


$$\boxed{\text{MSE} \geq \|\delta\|^2 \geq \text{QCRB} = \|\Pi(\delta|\mathcal{T})\|^2.} \quad (5)$$

- Tsang, Albarelli, Datta, PRX (2020); see also Fujiwara's thesis [Chap. 18 in Hayashi (2005)].

- If $\{S_j\}$ is a finite and linearly independent set, QCRB $= (\partial\beta)^\top K^{-1}\partial\beta$, $K_{jk} \equiv \langle S_j, S_k\rangle$.

Detour: Bayesian Cramér-Rao Bounds



- CRB works for **unbiased** estimators/in a **local asymptotic** sense only. For finite sample size, CRB may be violated.
- Solution: Bayesian/minimax bounds
- Assume $\theta \in \Theta \subseteq \mathbb{R}^p$, a prior density $\pi(\theta)$, $\beta : \Theta \to \mathbb{R}$, n i.i.d. samples. Schutzenberger/Van Trees/**Gill-Levit**:

$$\langle \mathsf{MSE} \rangle \geq \mathsf{BCRB} \equiv \frac{\langle v^\top \partial \beta \rangle^2}{n\langle v^\top J v\rangle + \langle \mathsf{P} \rangle}, \quad \langle \cdots \rangle \equiv \int \cdots \pi(\theta) d^p\theta, \quad \mathsf{P} \equiv \left[\frac{1}{\pi}\partial^\top (\pi v)\right]^2.$$

  (J =Fisher information matrix)
- Problems:

  1. **Reparametrization** of θ leads to a different bound.
  2. **Family of bounds**: How to choose the vector v?

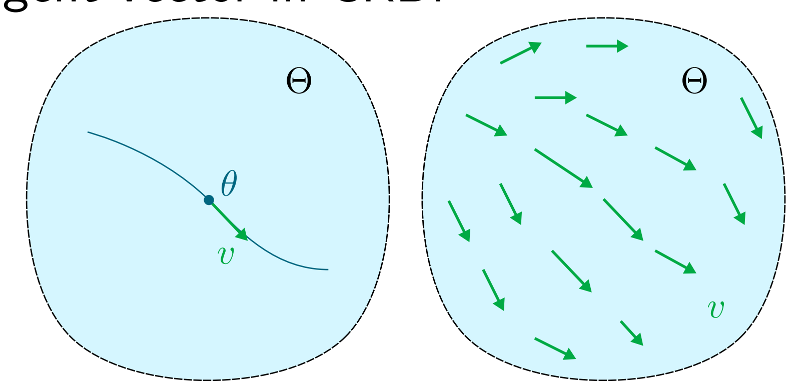
Invariant Gill-Levit Bound

- Treat Θ as a **manifold**. Assume a Riemannian metric g_{ab}. Define $\epsilon \equiv \sqrt{\det g} d^p\theta$, $\rho \equiv \pi/\sqrt{\det g}$. Define covariant derivative ∇_a.
- **Invariant Gill-Levit Bound** [Tsang PRA (2020); see also Jupp JMVA (2010)]:

$$\mathsf{BCRB} = \frac{\langle v^a \nabla_a \beta \rangle^2}{n\langle v^a J_{ab} v^b \rangle + \langle \mathsf{P} \rangle}, \quad \langle \cdots \rangle \equiv \int \cdots \rho\epsilon, \quad \mathsf{P} = \left[\frac{1}{\rho}\nabla_a(\rho v^a)\right]^2. \quad (6)$$

(differential geometry notations)

- Treat v as a **vector field**. Transform v^a in a contravariant way upon reparametrization.
- v generalizes the tangent vector in CRB:



Optimal Gill-Levit Bound

- There exists a **least favorable** v that maximizes the bound. It obeys the differential equation

$$\boxed{nJ_{ab}v^b - \nabla_a\left[\frac{1}{\rho}\nabla_b\left(\rho v^b\right)\right] = \nabla_a\beta.} \tag{7}$$

- Tsang, PRA (2020)
- Example 1: assume $\beta(\theta) = u^\top\theta$, $g = I$ (flat manifold), J independent of θ, prior $\rho(\theta) = \pi(\theta)$ is Gaussian with covariance matrix Σ. Then

$$v = \left(nJ + \Sigma^{-1}\right)^{-1}u, \qquad \text{BCRB} = u^\top\left(nJ + \Sigma^{-1}\right)^{-1}u, \tag{8}$$

coincides with Schutzenberger/Van Trees. Tight for linear Gaussian models.
- Example 2: $n \to \infty$

$$nJv \approx \nabla\beta, \qquad v \approx \frac{1}{n}J^{-1}\nabla\beta, \qquad \text{BCRB} \approx \frac{1}{n}\left\langle(\nabla\beta)^\top J^{-1}\nabla\beta\right\rangle. \tag{9}$$

coincides with the least favorable tangent vector for CRB, consistent with local asymptotic minimax theorem.

Minimax Bound

- Minimax:

$$\sup_{\theta\in\Theta} \mathsf{MSE}(\theta) \geq \langle \mathsf{MSE} \rangle \text{ for any } \pi \text{ with support } \subseteq \Theta. \tag{10}$$

  Choose unfavorable prior to tighten bound.
- Assume $\beta = \theta$ (scalar), $u = 1$. Choose constant $v = 1$, $g = 1$ (Van Trees version).
- Trick: let $\pi(\theta) = [\psi(\theta)]^2$.

$$\mathsf{BCRB} = \frac{1}{n\langle \mathsf{J} \rangle + \langle \mathsf{P} \rangle}, \qquad n\langle \mathsf{J} \rangle + \langle \mathsf{P} \rangle = \int \left[nJ\psi^2 + \psi(-4\partial^2)\psi \right] d\theta. \tag{11}$$

  Think of $\psi(\theta)$ as a **wavefunction**, $nJ(\theta)$ as **potential energy**, $-4\partial^2$ as **kinetic energy**. Choose $\pi(\theta)$ to maximize the bound $\rightarrow$ choose $\psi(\theta)$ to minimize total energy. Best bound $=$ $1/$**ground-state energy**.
- Tsang, JMO (2018); PRA (2020)
- Example 1: If $\min_\theta J(\theta) > 0$, then ground-state energy $= n \min_\theta J(\theta) + o(n)$, BCRB $= \Theta(n^{-1})$ (parametric rate).
- Example 2: If $J(\theta) \leq c\theta^m$, then ground-state energy $= O(n^{2/(m+2)})$, BCRB $= \Omega(n^{-2/(m+2)})$ (subparametric rate).
- Example 3: $J(\theta) \leq c\theta^2$ (quadratic potential), then ground-state energy $= O(n^{1/2})$, BCRB $= \Omega(n^{-1/2})$.

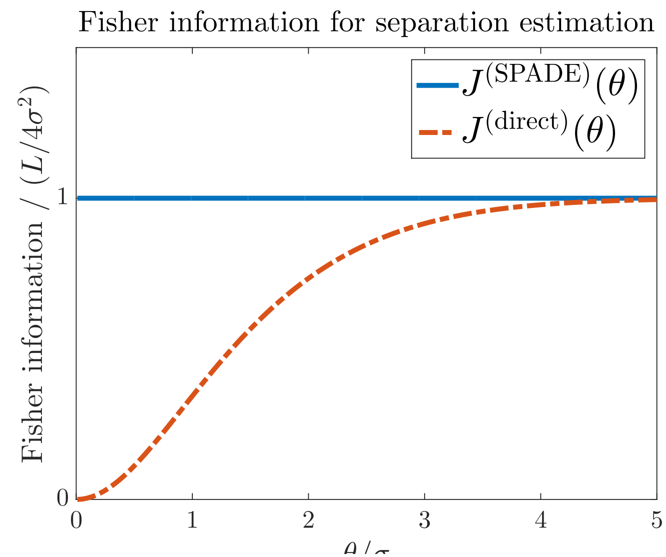
Imaging Example

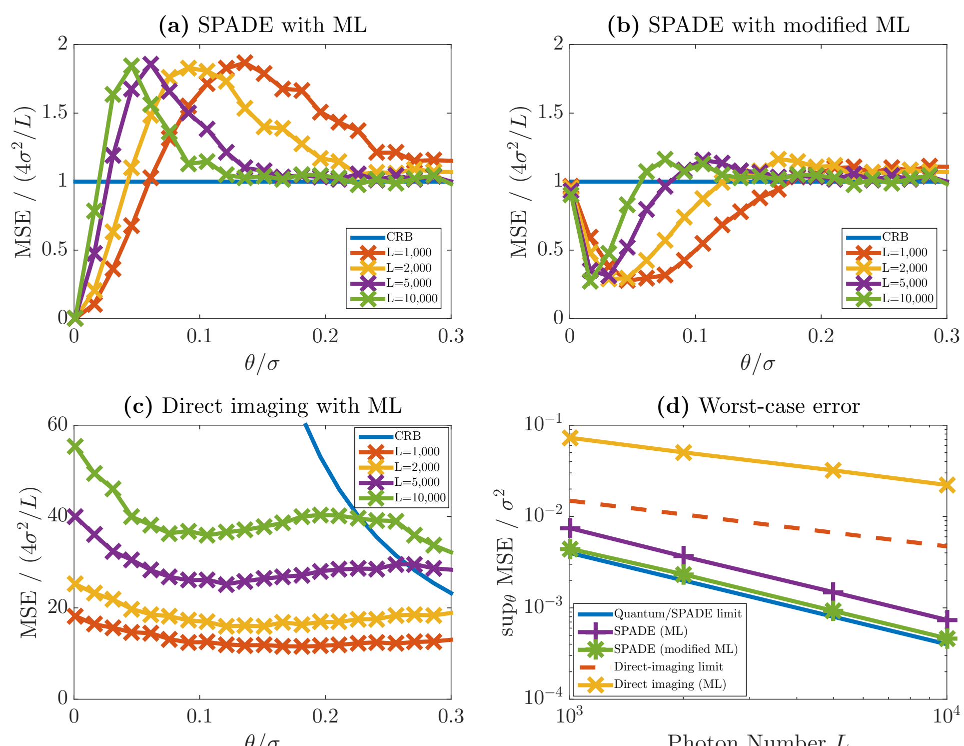
- Two-point separation estimation, direct imaging with Gaussian PSF:

$$J \leq c\theta^2,$$

$$\sup_\theta \mathsf{MSE}(\theta) = \Omega(n^{-1/2}).$$

- SPADE: $\sup_\theta \mathsf{MSE}(\theta) = \Omega(n^{-1})$.
- Tsang, JMO (2018)





Semiparametric Examples

- If parameter space = {any density operator} (nonparametric), any $h \in \mathcal{Z}$ can be a score because I can construct a parametric submodel from any h, e.g.,

$$\sigma(\theta) = \frac{[I + \tanh(\theta h/2)]\rho[I + \tanh(\theta h/2)]}{\mathrm{tr}(\text{numerator})}. \tag{12}$$

  Hence $\boxed{\mathcal{T} = \mathcal{Z},}$ tangent space is **full-dimensional**.
- Example 1: $\beta = \mathrm{tr}\, Y\rho$. $\partial_j \beta = \mathrm{tr}\, Y \partial_j \rho = \mathrm{tr}\, Y(S_j \circ \rho) = \langle Y, S_j \rangle$.
- $\Pi(Y|\mathcal{Z}) = Y - \langle Y, I\rangle = Y - \mathrm{tr}\, \rho Y = \delta$ is an influence operator.
- Efficient influence: $\Pi(\delta|\mathcal{T}) = \Pi(\delta|\mathcal{Z}) = Y - \mathrm{tr}\, \rho Y$.

$$\boxed{\text{QCRB} = \|Y - \mathrm{tr}\, \rho Y\|^2 = \mathrm{tr}\, \rho\, (Y - \mathrm{tr}\, \rho Y)^2\,.} \tag{13}$$

  Measurement of Y is efficient. (See also Watanabe *et al.*, PRL (2010))
- Example 2: $\beta = \mathrm{tr}\, \rho^2$. $\partial_j \beta = 2\, \mathrm{tr}\, \rho \partial_j \rho = 2\, \mathrm{tr}\, \rho(S_j \circ \rho) = \langle S_j, 2\rho \rangle$.
- $\Pi(2\rho|\mathcal{Z}) = 2\rho - \langle 2\rho, I\rangle$ is an influence operator.
- Efficient influence: $\Pi(\delta|\mathcal{T}) = \Pi(\delta|\mathcal{Z}) = 2\rho - \langle 2\rho, I\rangle$.

$$\boxed{\text{QCRB} = 4\, \mathrm{tr}\, \rho \left(\rho - \mathrm{tr}\, \rho^2\right)^2\,.} \tag{14}$$

- **Open question**: optimal measurement?

Quantum Limit on Semiparametric Imaging

- One-photon density operator: $\rho_1(F) = \int e^{-i\hat{k}X} |\psi\rangle \langle\psi| e^{i\hat{k}X} F(X) dX$, $F \in \{\text{any probability density}\}$, $\beta(F) = \int [X^\mu + o(\Delta^\mu)] F(X) dX$.
- Exact QCRB is still intractable! ($\mathcal{T}$ is a nontrivial subspace of $\mathcal{Z}$)
- Find a looser bound using an unfavorable 1D submodel $\{\rho_1(F_\theta) : \theta \in \mathbb{R}\}$ [Lemma 2, Tsang, PRA (2021)]:

$$\text{QCRB} = \sup_{\text{submodel} \in \{\text{1D submodels}\}} \text{QCRB}^{(\text{submodel})}.$$

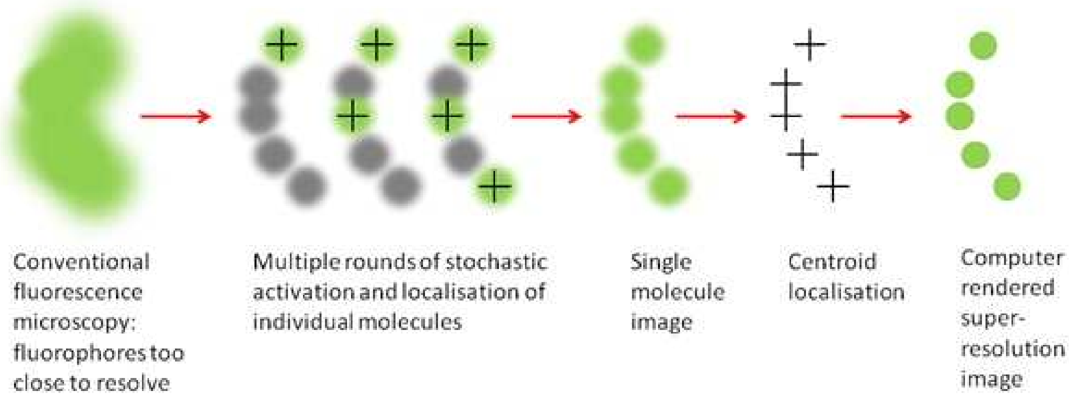
  Classical: Stein (1956).
- Still intractable! (ρ_1 is mixed and ∞-dim) Find lower bound on $\text{QCRB}^{(\text{submodel})}$ in terms of a unfavorable purification of ρ_1.
- **Quantum bound** [Tsang, PRA (2021)]:

$$\boxed{\text{QCRB}^{(\text{purified 1D submodel})} = \frac{\Omega(\Delta^{2\lfloor \mu/2 \rfloor})}{N},} \quad \text{compare with } \text{MSE}^{(\text{SPADE})} = \frac{\Theta(\Delta^{2\lfloor \mu/2 \rfloor})}{N}.$$

- See also Tsang, PRA (2019); Zhou & Jiang (Yale/Chicago), PRA (2019).

Superresolution Microscopy

- PALM, STORM, STED, etc.: make **sparse subsets** of fluorophores emit



https://cam.facilities.northwestern.edu/588-2/single-molecule-localization-microscopy/

- **avoid violating Rayleigh**
- Need **controllable** fluorophores
- **slow**, **phototoxicity**
- doesn't work for stars, passive imaging