Classification of TQFTs

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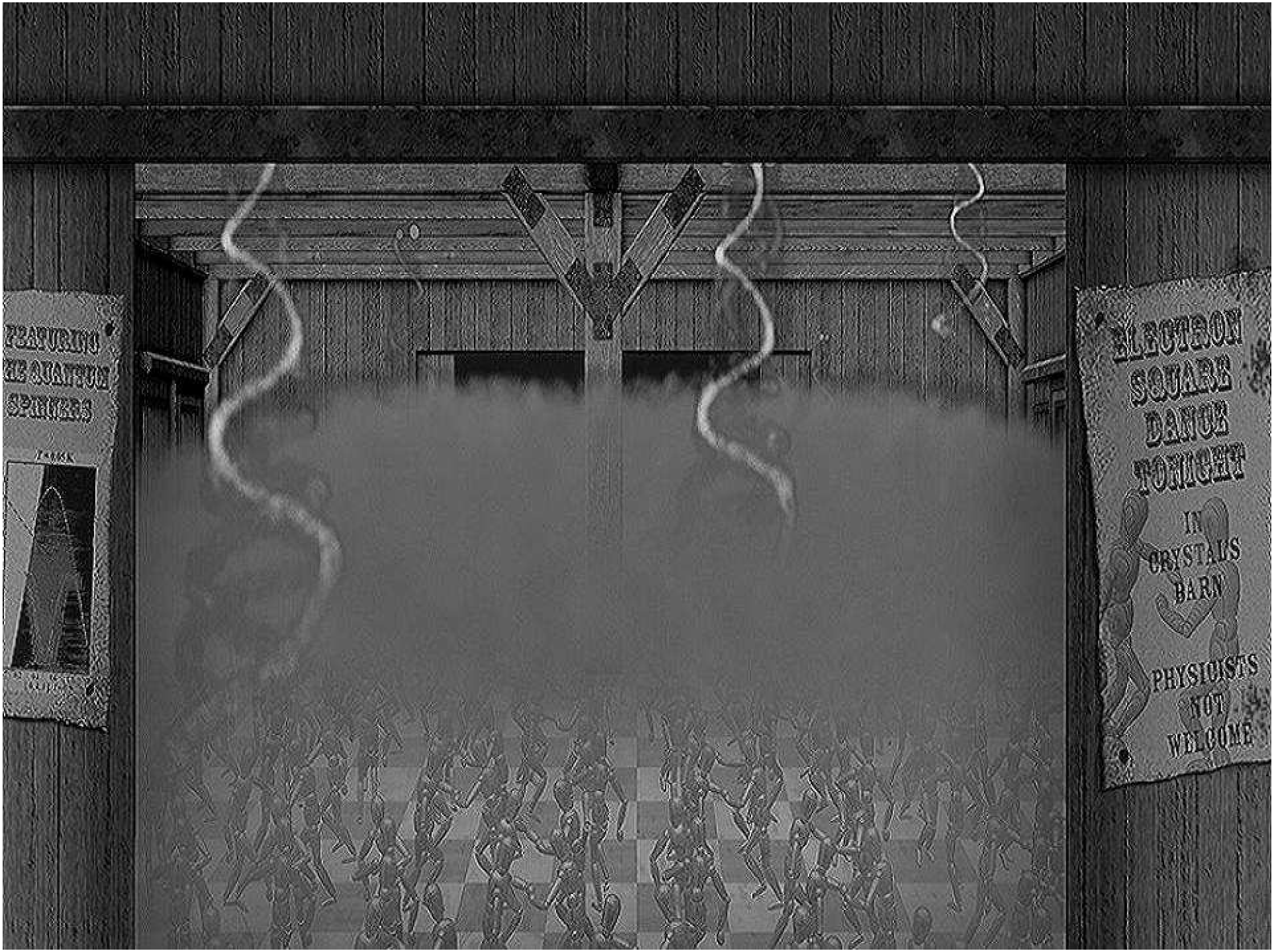
PRELIMNARY	**1**		**A=abelian** **N=non-abelian** **U=universal for anyonic QC** **#=number of UNITARY theories**	
	A 2 **$SU(2)_1$** Semion=Z_2	N 2 **$SO(3)_3$** Fib U2		
A 2 **Z_3** (ν=1/3)	N 8 **$SU(2)_2$** **Ising** (ν=5/2)	N 2 **$SO(3)_5$** U4		
A 4 **Z_4**	N 4 **$SU(2)_3$** (ν=12/5) U	N 2 **$SO(3)_7$** U6	N 4 **Fib×Fib** U	A 6 **$Z_2 \times Z_2$**


FEATURING
HE QUANTUM
SPINNERS
ELECTRON
SQUARE
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TONIGHT
IN
CRYSTALS
BARN
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WELCOME

How To Model and Classify the emerged topological orders?

Bulk: **TQFTs**

Boundary: **RCFTs (?)**

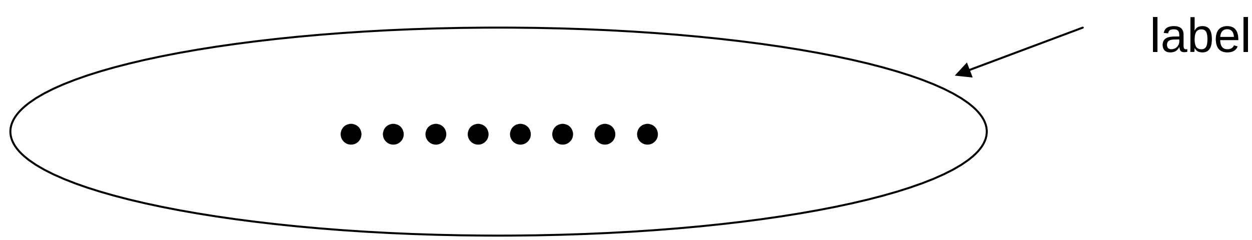
Supersymmetric version for electrons

Bulk ⟷ Boundary

(∂ Chern-Simons=WZW)

Topological Phases of Matter

Suppose a gapped quantum system on an oriented surface Σ has particle-like elementary excitations of types a_1, a_2, …, a_n localized at $z_1, z_2, \ldots, z_n$,
then the ground states of the system "outside" $z_1, z_2, \ldots, z_n$ form a Hilbert space $V(\Sigma, a_1, a_2, \ldots, a_n)$.



A diffeomorphism, e.g. braiding

$$f: \Sigma \backslash \{z_i\} \to \Sigma \backslash \{z_i\}$$

induces a unitary transformation

$$V(f): \; V(\Sigma, a_1, \ldots, a_n) \to V(\Sigma, a_1, \ldots, a_n)$$

If the collection $\{V(\Sigma, a_1, \ldots, a_n), V(f)\}$ form a TQFT, then the quantum system is topological.

Algebraic Data for a TQFT

A (unitary) ribbon tensor category:

a finite label set $\{X_0=1, X_1,\ldots,X_{n-1}\}$

(anyon type representatives, and $X \cong \oplus\, m_i\, X_i$)

with compatible

Fusion rule: $X_i \otimes X_j \cong \oplus\, h_{i,j}^{k}\; X_k$

Duality: $1 \to X \otimes X^*,\; X^* \otimes X \to 1$

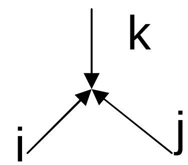
Braiding: $c_{X,Y}: X \otimes Y \to Y \otimes X$

Twist: $\theta_X: X \to X$

e.g. G=finite abelian group, $X_i=g$, $g \otimes h = g \cdot h$

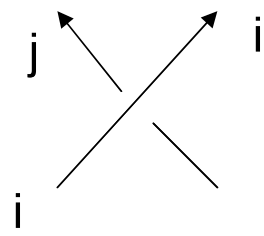
Graphical Calculus

- Fusion rule:

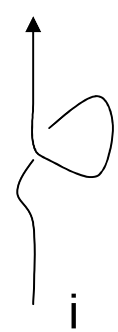


- Duality:

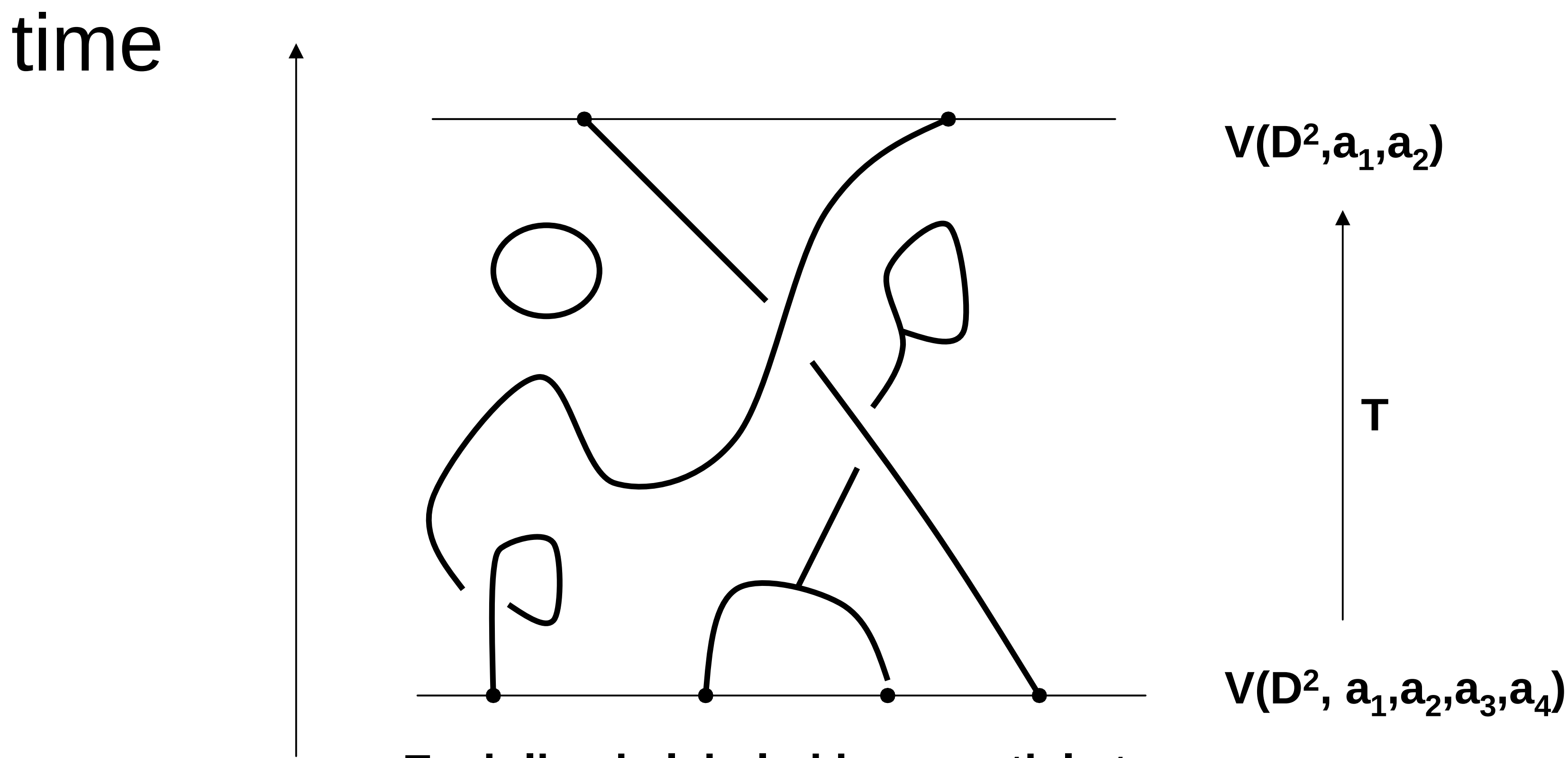
- Braiding:



- Twist:



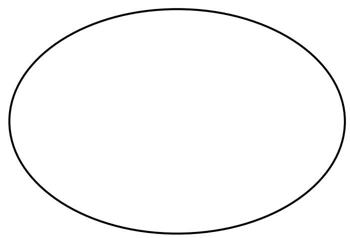
Invariant for Anyon Trajectory



Each line is labeled by a particle type
Move forward=particle, move backward=anti-partilce
Special trajectories are braids and links.

Quantum Dimensions

For each particle type X_i,

d_i=  **X_i**

Global dimension: $D^2=\sum_i d_i^2$

$d_i\, d_j=\sum_k h_{i,j}^k\, d_k$, **Let** $(Q_i)_{a,b}=h_{i,b}^a$

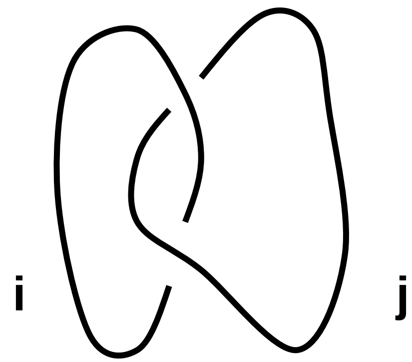
Then d_i is the Perron-Frobenius eigenvalue of Q_i.

If $d_i<2$, then $d_i=2\cos(\pi/r)$ for some $r>2$.

(Kronecker)

Modular S-matrix

$s_{ij}=1/D$

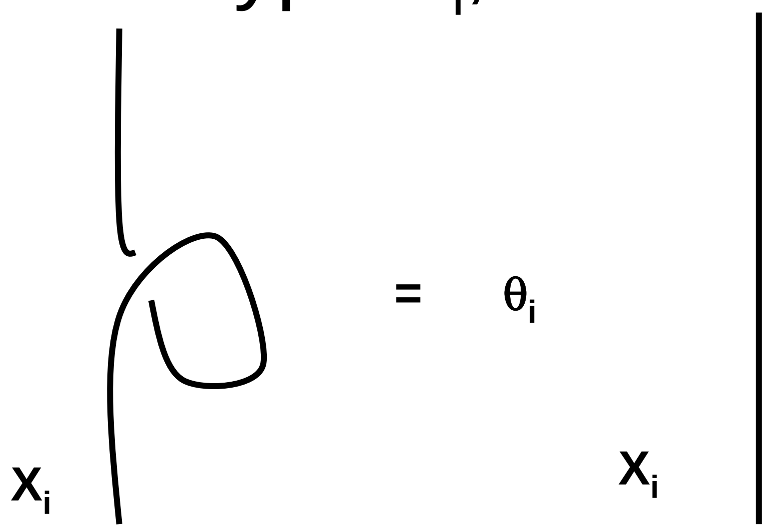


$S=(s_{ij})$ is the modular S-matrix.

If it is non-singular, then the RTG is called a MTG $\longrightarrow$ TQFT (Turaev)

Twists

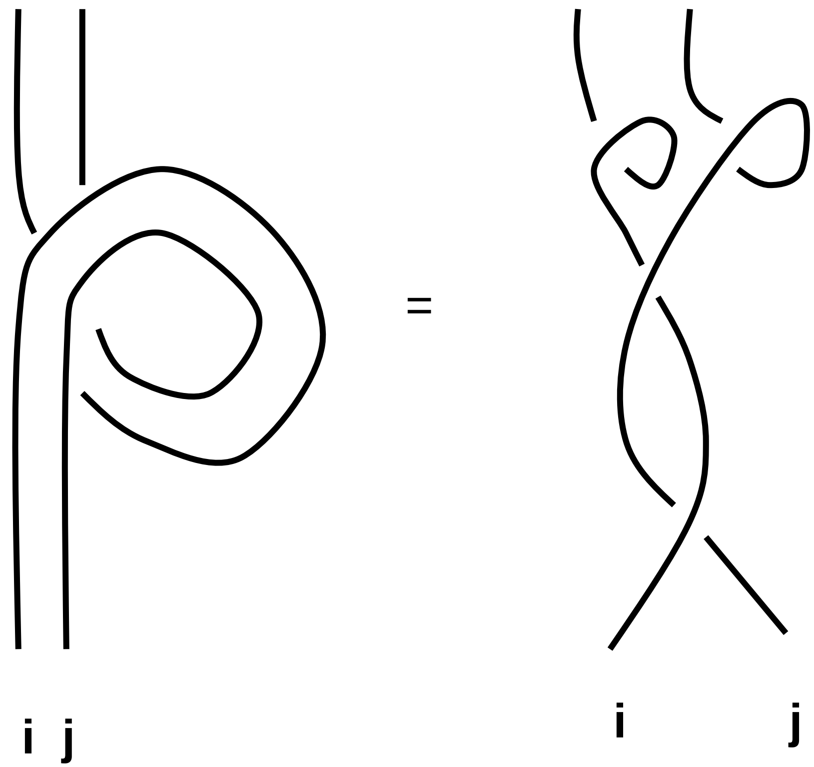
Given a particle type X_i,



θ_i is a root of unity (Vafa)

Suspender Formulas

Identity



$$D\, s_{ij} = \theta_i^{-1}\, \theta_j^{-1} \sum_k h_{ij}^k\, \theta_k\, d_k$$

Verlinde Formulas

1. **Verlinde Conj:**

$$\mathbf{S\,Q_i\,S^{-1}=D_i, \quad (D_i)_{ab}=\delta_{ab}s_{ia}/s_{0a}}$$

2. Given a surface $\Sigma_{g,n}$ of genus=g, and n boundaries labeled by $a_1,\ldots,a_n$

   Then dim $V(\Sigma_{g,n})=\sum_x (\prod_i s_{a_i\,x})s_{0x}^{\chi(\Sigma)}$, where $\chi(\Sigma)=2-2g-n$

Central Charge

Let $p_{\pm}=\sum_i \theta_i^{\pm} d_i^2$

Then $p_+ p_- = D^2$

$p_+/D = e^{2\pi i/8\ c}$

where c is the central charge of a boundary CFT mod 8

Classification

- **Low rank TQFTs, rank=# of particle types:**
  rank=2,3,4 (Belinschi, Stong, Rowell, W.)
- **Abelian TQFTs**: $X_i \otimes X_j = X_{\mathbf{k}}$
  fusion rules=abelian group,
  braid reps=1-dimensional
  (Ludwig, Read, W.)
- **General case**:
  ? **Fix rank, there are only finitely many TQFTs**
  (True if fusion rules are fixed---Ocneanu rigidity)

Jones-Witten SU(2)-Theory

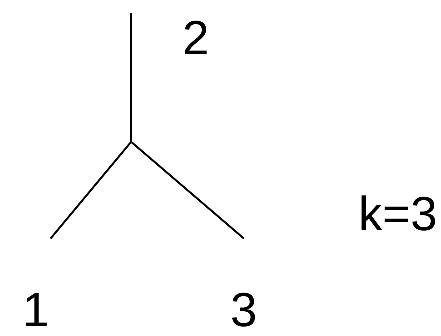
Pictorial formulation of the SU(2)-Chern-Simons theory by Kauffman (different for some levels).

Fix r$\geq$ 3, and k=r-2, called the level

The particle types are L={0,1,…,r-2} and each is its own dual, a*=a

Fusion rules:

$a \otimes b = \oplus\, c$ such that



1) **a+b+c even**
2) **a+b$\geq$ c, b+c$\geq$ a, c+a$\geq$ b**
3) **a+b+c$\leq$ 2(r-2)**

Restricted to even labels, called SO(3) at level=k

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Rank=2 TQFTs

- **Particle types: 1, X, $d=d_X$, $\theta=\theta_X$**
- **Fusion rules: $X^2=1+mX$**

$d^2=1+md$

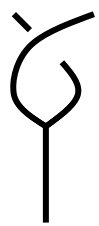
$(1+\theta\, d^2)(1+\theta^{-1}d^2)=1+d^2$

$\rightarrow \theta+\theta^{-1}=1-d^2=-md$

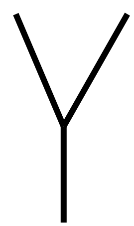
But $d\geq 1$, so m=0,1

Fibonacci TQFT

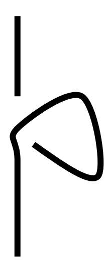
- Particle types: $\{1,\tau\}$
- Quantum dimensions: $\{1,\tau\}$
- Fusion rules: $\tau^2=1+\tau$
- Braidings:



$= e^{3\pi i/5}$



- Twists:



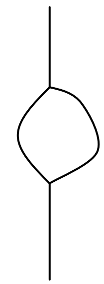
$= e^{4\pi i/5}$

Modular S-matrix:

$$S = 1/D \begin{bmatrix} 1 & \tau \\ \tau & 1 \end{bmatrix} \qquad S_\tau = (e^{3\pi i/10})$$

F-matrix:

$$F = \begin{bmatrix} \tau^{-1} & \tau^{-1/2} \\ \tau^{-1/2} & -\tau^{-1} \end{bmatrix}$$

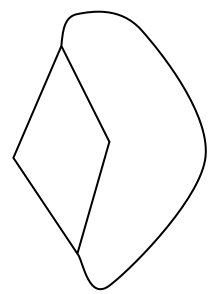


$= \tau^{1/2}$

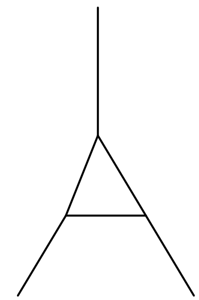
Theta symbol:

$$\theta = \tau^{3/2}$$

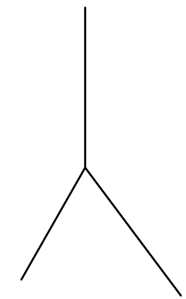
Tetrahedron symbol:



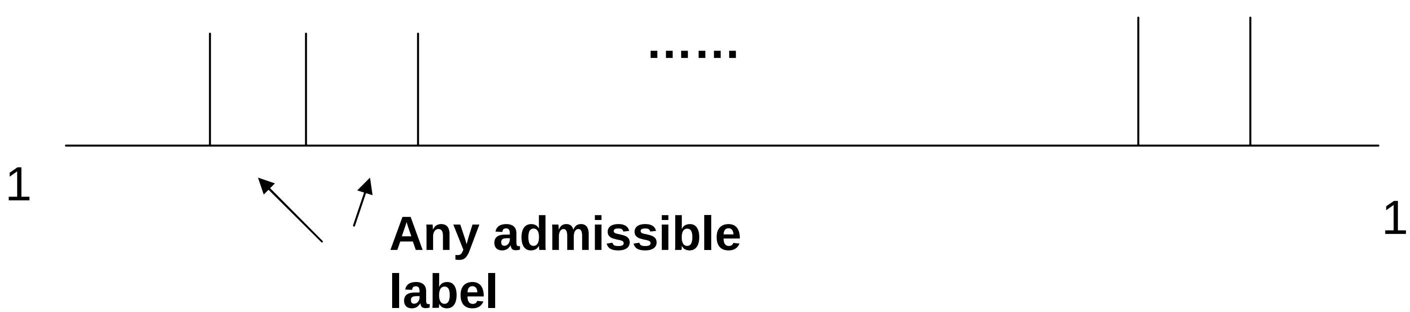
$= -\tau$



$= -\tau^{-1/2}$

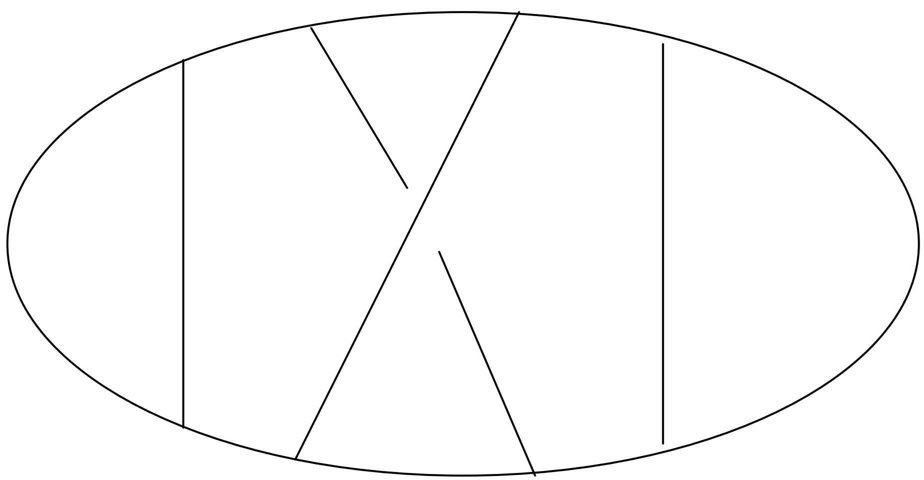


Conformal block basis:



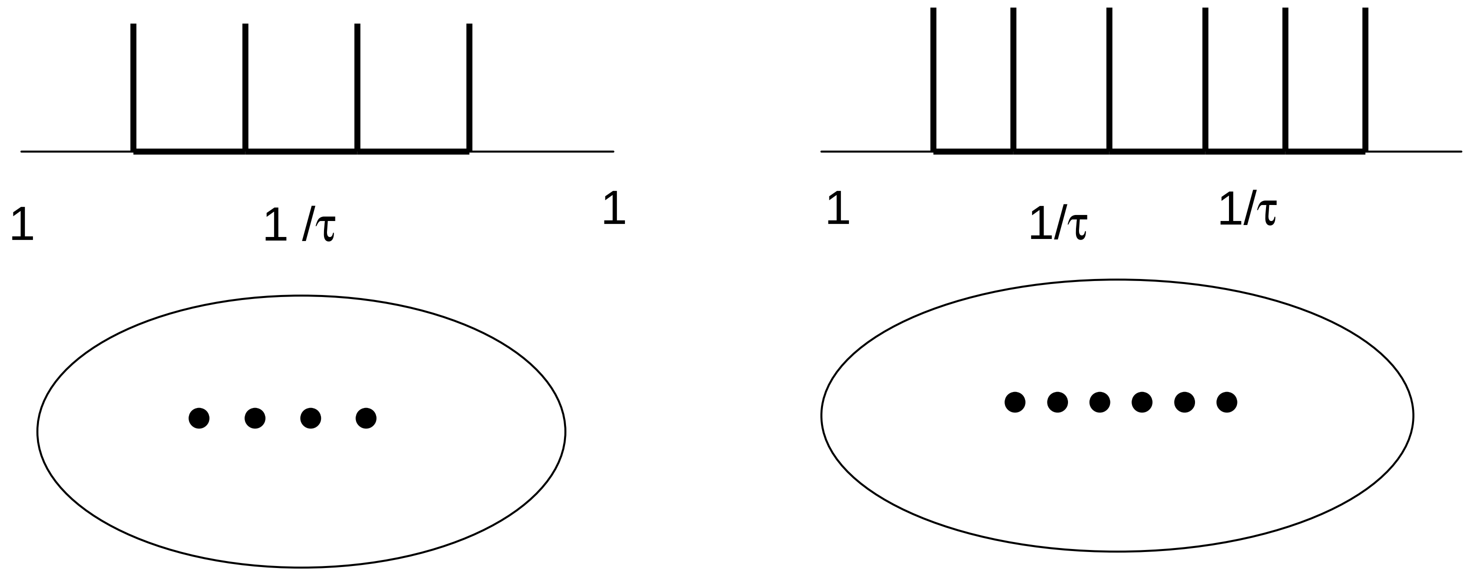
Number Fields

- Invariants of trivalent graphs are rational functions of $\tau^{\pm 1/2}$ and $\xi_{20}=e^{2\pi i/20}$



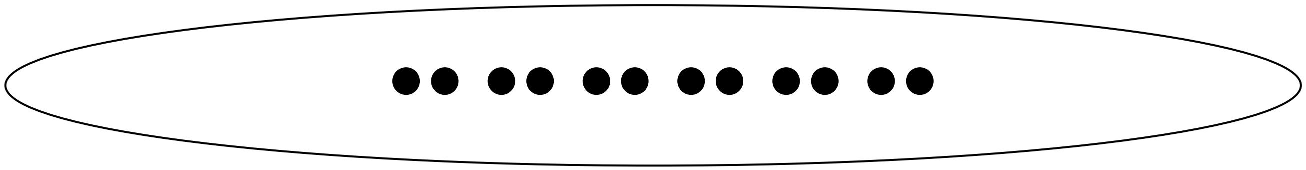
Fibonacci Quantum Computer

4 τ's in a disk is C^2---qubit. 6 τ's C^5---2 qubits+NC.
For 1-qubit gates, ρ: $B_4 \to U(2)$
For 2-qubits gates, ρ: $B_6 \to U(4) \subset U(5)$



For n qubits, consider the 2n+2 punctured disk D_{2n+2} and

$$\rho: B_{2n+2} \to U(F_{2n})$$



Given a quantum circuit on n qubits:

$$U_L: (C^2)^{\otimes n} \to (C^2)^{\otimes n}$$

Ideally to find a braid $b \in B_{2n+2}$ so that the following diagram commutes:

$$\begin{array}{ccc} (C^2)^{\otimes n} & \to & V(D_{2n+2}) \\ U_L \downarrow & & \downarrow \rho(b) \\ (C^2)^{\otimes n} & \to & V(D_{2n+2}) \end{array}$$

Non-Realization of F_N

- **Fibonacci TQFT can be constructed over $K=Q(\tau^{1/2}, \xi_{20})$.**

Since there are only finitely many roots of unity in K, the Fourier transforms $F_N=(\omega^{ij})$, $\omega=e^{2\pi i/N}$ for large N cannot be realized by braiding Fibonacci anyons,

Hence approximation is unavoidable.

Same for any unitary TQFT (Freedman, W.)

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