



ENTANGLEMENT IN STRONGLY INTERACTING MANY-BODY SYSTEMS



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SPIN GAS - DEFINITION

System of N spins with interaction Hamiltonian

$$H(t) = \sum_{a < b} g_{ab}(t) H_{ab} \quad ; \quad H_{ab} = \sum_{\mu=0}^2 h_{ab}^{\mu} \sigma_{\mu}^{(a)} \sigma_{\mu}^{(b)}$$

with $g_{ab}(t) =$ stochastic functions of time ; $a, b = 1, \dots, N$

Special cases: $g_{ab}(t) = g_{ab}$; random but time independent coupling
 → spin glasses

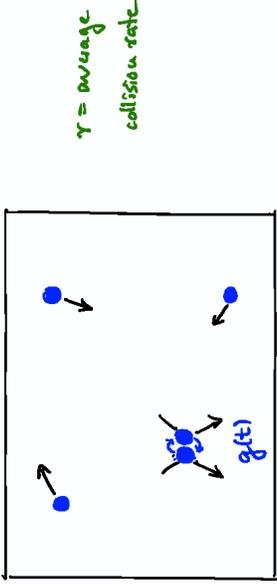
$g_{ab} = g_{a+b, b+a}$; fixed coupling ; translational symmetry
 → spin chains ; lattices

Realisations

- Spin network
prescribe interaction pattern $g_{ab}(t)$



- Boltzmann gas
interactions
 $g_{ab}(t) = g(|\vec{r}_a(t) - \vec{r}_b(t)|)$
driven by classical kinematics
(Boltzmann molecular chaos)

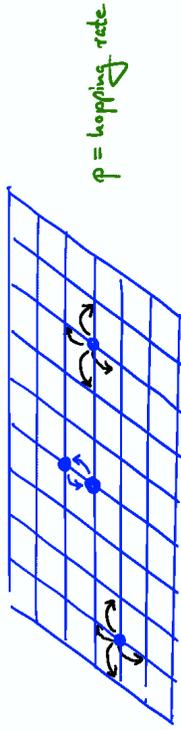


γ = average collision rate

Example: Spin-dependent collisions between ultracold atoms

- Lattice gas

- interaction between neighbouring sites.
- driven by classical hopping.



• ...

Question: What types of entangled states are generated?

- Can we classify these states?
- Do they contain "useful" entanglement?

Outline of remaining talk:

- Ising interactions: Weighted graph states (WGS) : $\propto N^2$
 - Graph states: Classification & applications
- Long-range entanglement in spin chains & lattices
- Variational method based on WGS
- Entanglement generation & decoupling in spin gas.

Ising-type interactions

$$H_{ab} = \sigma_z^{(a)} \sigma_z^{(b)} \quad \text{or}$$

$$H_{ab} = \frac{1 + \sigma_z^{(a)}}{2} \frac{1 + \sigma_z^{(b)}}{2} = |1\rangle\langle 1|_a \otimes |1\rangle\langle 1|_b$$

These commute:

$$[H_{ab}, H_{a'b'}] = 0$$

Examples: Spin-dependent elastic collisions between ultra-cold atoms in optical lattices (\rightarrow phase gates)

$$H(t) = \sum_{a < b} g_{ab}(t) H_{ab}$$

Unitary evolution:

$$U(t) = \prod_{a < b} U_{ab}(t), \quad U_{ab}(t) = e^{-i \varphi_{ab}(t) H_{ab}}$$

$$\varphi_{ab}(t) = \frac{1}{\hbar} \int_0^t dt' g_{ab}(t')$$

Action on initial state:

$$|\psi(0)\rangle = |0\rangle_X^{\otimes N}$$

Spin-polarized along x direction

$$\left[\begin{array}{l} \delta_k |s\rangle_k = (-1)^s |s\rangle_k, \quad k=x,y,z; \quad s=0,1 \\ |s\rangle_z \equiv |s\rangle, \quad \text{standard basis } z \end{array} \right]$$

$$|\psi(0)\rangle = \frac{1}{2^{N/2}} \sum_{s_1, \dots, s_N=0,1} |s_1, \dots, s_N\rangle = \frac{1}{2^{N/2}} \sum_{s \in \{0,1\}^N} |s\rangle$$

$$U_{ab}(t) |s\rangle = e^{-i \varphi_{ab}(t) S_a S_b} |s\rangle$$

phase gate between spin a and b

State of spin gas at $t > 0$:

$$U(t) |0\rangle_X^{\otimes N} = \frac{1}{2^{N/2}} \sum_s e^{-i s^T \cdot T(t) \cdot s} |s\rangle \equiv |T(t)\rangle$$

$$T(t) = \begin{pmatrix} 0 & \varphi_{12} & \dots & \varphi_{1N} \\ \varphi_{21} & 0 & & \vdots \\ \vdots & & \ddots & \varphi_{N-1,N} \\ \varphi_{N1} & \varphi_{N2} & \dots & \varphi_{N,N-1} & 0 \end{pmatrix}$$

adjacency matrix of graph

$$\frac{N(N-1)}{2} \quad \text{real parameter}$$

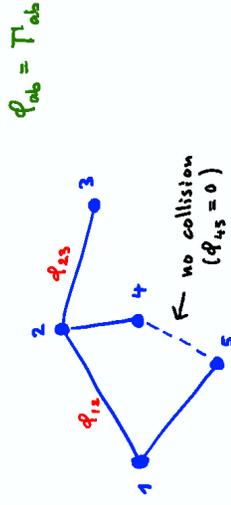
They summarize effective "collision history" of the gas

Weighted graph Γ



Weighted graph state $|\Gamma\rangle$

$$|\Gamma(t)\rangle = 2^{-M/2} \sum_{\mathbf{s}} e^{-is \cdot \Gamma(t) \cdot \mathbf{s}} |s\rangle$$



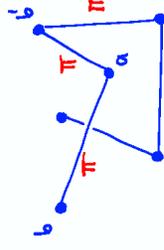
Question: What can we say about the class of these states, and their entanglement properties?

Special subclass: $\phi_{ab} \in \{0, \pi\}$ "binary collisions"

- These are the so-called **graph states** (or **stabilizer states**).
- They can be described by a set of **eigenvalue equations**:

$$K^{(a)} |\Gamma\rangle = |\Gamma\rangle \quad ; \quad a=1, \dots, N \quad ; \quad \hat{T}_{ab} = T_{ab} / \pi$$

$$K^{(a)} = \sigma_x^{(a)} \prod_{b \sim a} [\sigma_z^{(b)}]^{\hat{T}_{ab}}$$



- Have been classified regarding the entanglement properties (Schmidt measure)

Hahn, Eisert, HJ, Phys. Rev. A 69, 062311 (2004)

Vanden Nest, Dehaene, de Moor, PRA 70, 034302 (2004)

ibid. quant-ph/0411115 (2004)

Review Article: Hein et al. quant-ph/0602096.

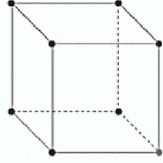
Examples :



Bell state



GHZ state



Graph code



2D Cluster state



One-way quantum computer
Raussendorf et al. PRL 86 (2001)

Some of (our) recent work relating to graph states :

Raussendorf et al. PRA 68, 022312 (2003) : Graph states as **algorithmic resources**

Hein et al. PRA 69, 062311 (2004) : **Classification of GS** - "how many ; how entangled"

Dür et al. PRL 92, 180403 (2004) : **Lifetime** of multi-partite entanglement

Gühne et al. PRL 95, 120405 (2005) : All graph states violate **local realism** !

Calderbank et al. PRL 95, 180502 (2005) : Entanglement & Non-Markovian
Hartmann et al. PRA (2005) : **decoherence in spin gases.**

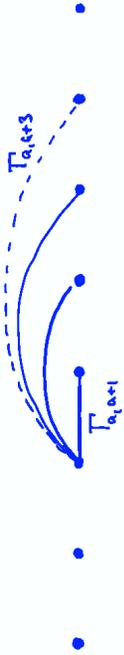
Anders et al. quant-ph/0602230 : **Ground-state approximations of strongly interacting systems in arbitrary dimensions**
- Yesterday -

Review article : Hein et al. quant-ph/0602096.
Proceedings of **Varenna school (2005)**

Back to larger class: $\varphi_{ab} \in [0, 2\pi)$ arbitrary collisional phases

- These have no simple stabilizers: $K^{(n)} \neq$ product of Paulis $\left(\begin{smallmatrix} 0 & 0 \\ \infty & n \end{smallmatrix}\right)$
- Look at special examples for T !

① 1-dim chain with prescribed distance dependence of coupling:



$$T_{ab} = \varphi(|a-b|) = \text{const.} \begin{cases} |a-b|^{-x} & ; x > 0 \\ e^{-x|a-b|} & ; x > 0 \end{cases}$$

Calculate von Neumann entropy

$$S_L = -\text{tr} \rho_L \log \rho_L$$

of reduced state ρ_L of a block of neighboring spins.



$$T_{ab} \approx \frac{3\pi}{10} |a-b|^{-x}$$

$x > 1$: S_L saturates

$x < \frac{1}{2}$: S_L seems to grow unboundedly

Correlation length & entanglement length diverges in both cases.

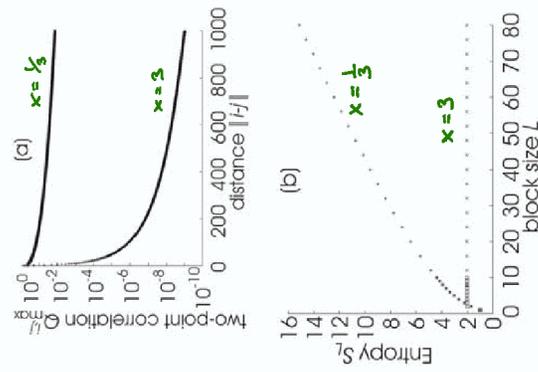


FIG. 1: Spin chain with $N = 10^5$, $t = 0.3\pi$ and $f(\nu_{ik}) = r^{-1/2}$, $r = 3$ respectively. (a) Semi-logarithmic plot of maximal two point correlation Q_{ij}^{max} between two particles as a function of the distance $||i-j||$. (b) Exact values of the entropy of entanglement S_L and upper bounds for growing block size L .

W. Dür, L. Hartman, M. Hein, M. Lewenstein and H.-J. Briegel, Phys. Rev. Lett. 94, 097203 (2005).

Similar situations can be found in d=2,3:

- Bi-partite entanglement can grow like **volume** L^d of a block

$$S_L \propto L^d$$

- DMRG and related methods, based on **matrix product states**, cannot be applied in such situations (assuming bounded amount of entanglement)
- Improved methods (Verstraete, Cirac cond-mat/0407066) using **PEPS** are still restricted to situations, where

$$S_L < D^{d-1}$$

- We use **weighted graph states WGS** as a **variational basis** to approximate ground states of critical systems in 2D, 3D.

→ Anders, Plenio, Dür, Verstraete, HJJB, quant-ph/0602230

Anders, Plenio, Dür, Verstraete, HJJB, quant-ph/0602230

② **Weighted graph states (WGS)** as a **variational set**.

WGS form a $O(N^2)$ family of N -spin states with following properties:

1. They form an (incomplete) basis
 2. Can **efficiently** calculate expectation values of **localized** observables such as energy
 - * 3. They may contain **arbitrary large amount entanglement**
 - * 4. Defined on arbitrary graphs (→ **higher dimensional systems**)
- * **Different from DMRG & generalisations**

• Variational set:

$$|\Psi_{\Gamma, d, U}\rangle \propto U \sum_s e^{-is^T \Gamma s / 2 + d^T s} |s\rangle$$

local unitaries $U = \otimes_a U_a$ $3N$
 adjacency matrix of graph $N(N-1)/2$
 complex deformation vector $d = (d_1, \dots, d_N)$ $2N$

• Superpositions:

$$|\Psi\rangle \propto \sum_{i=1}^m \alpha_i |\Psi_{\Gamma, d^{(i)}, U}\rangle$$

$N(N-1)/2 + 3N + 2(N+1)m = O(N^2)$ parameters.
 only difference!

• Local observables: $A = \sum_{a < b} A_{ab} + \sum_a A_a$ (can be generalized)

→ Need only reduced states ρ_a, ρ_{ab}

For single WGS ($m=1$) we obtain

$$\rho_{12} = (U_1 \otimes U_2) \left(\sum_{s,t} \tau_{s,t} |s\rangle \langle t| \right) (U_1 \otimes U_2)^\dagger$$

With

$$r_{s,t} = e^{-i\gamma} \prod_{c=3}^N \left(1 + e^{d_c + d_c^* - i \sum_{e=1}^2 (s_e - t_e)} \Gamma_{ec} \right)$$

$$\gamma = \sum_{a,b=1}^2 (d_a s_a s_b - t_a t_b) + \sum_{a=1}^2 (d_a s_a + d_a^* t_a)$$

→ ρ_{12} can be calculated in $O(N)$ steps?

- For superpositions, need $O(m^2 N)$ steps

- Expectation values of observables: $A = \sum_{a < b} A_{ab} + \sum_a A_a$

Energy, long-range interactions: $O(m^2 N^3)$

short-range interactions: $O(m^2 N^2)$

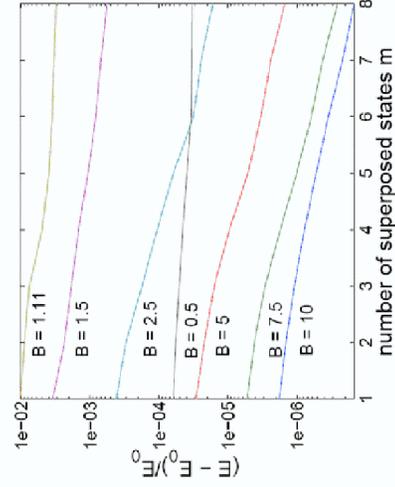
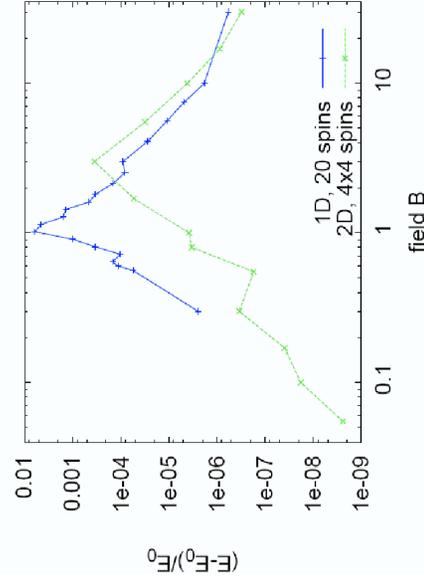
↑
no symmetry assumption

- Symmetry assumptions: (reduce variational set, $m=1$)

A number of symmetries can be imposed, trading speed for accuracy. For translational invariant Hamiltonians,

use e.g. $T_{ab} = f_n(|a-b|)$: reduction down to

$$O(L), N = L^d$$



Ising model:

$$H = - \sum_{\langle n_1, n_2 \rangle} \sigma_z^{(n_1)} \sigma_z^{(n_2)} - B \sum_a \sigma_x^{(a)}$$

quant-ph/0602230

FIG. 1: (Color online.) (a) Relative deviation from exact ground state energy for Ising chain with $N = 20$ (blue) and 4×4 2D lattice (green) with periodic boundary conditions as function of magnetic field B (calculated using BFGS minimization with symmetrized phases, $m \leq 6$). (b) 1D Ising chain with $N = 20$. Improvement of relative deviation from ground state energy as function of number of superposed states m for various field values B (calculated using Rayleigh minimization without symmetrized phases).

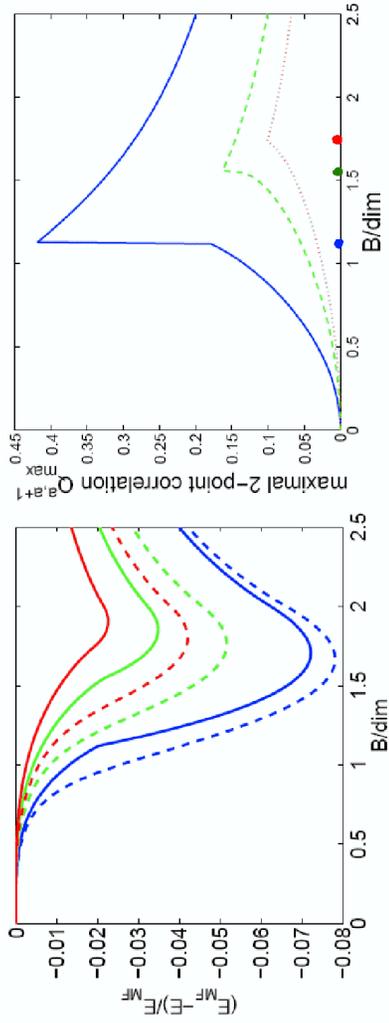


FIG. 2: (Color online.) Ising model in 1D (blue) with $N = 30$, 2D (green) with $N = 30 \times 30 = 900$ and 3D (red) with $N = 30 \times 30 \times 30 = 27000$ spins arranged as chain, square, and cubic lattice, respectively, for fully symmetric ansatz states with $\varphi_{ab} = f(|x_a - x_b|)$; $d_a = 1$ as function of magnetic field B/dim , where dim is dimension of lattice. (a) Relative deviation of ground state energy $(E_{MF} - E)/E_{MF}$ per bond from to mean field approximation E_{MF} (solid), and of Anderson bound $(E_{MF} - E_A)/E_{MF}$ (dashed). Translational invariance is reduced by using $U_1 \neq U_2$ (alternating). (b) maximal two-point correlation $Q_{max}^{a,a+1}$ for nearest neighbors.

Ising in 1D, 2D, 3D
using fully symmetric
ansatz states.

- 1D: 30
- 2D: 30x30
- 3D: 30x30x30

Calsamiglia et al. PRL 95, 180502 (2005)

③ Spin gases:

Random matrices T ; $P_t(T)$

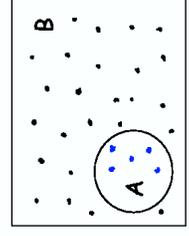
Probability distribution $P_t(T)$ depends on specific model:

a) Boltzmann gas: molecular chaos; Maxwell-Boltzmann distribution;
hard sphere model; $\delta f_{rel} \propto \frac{1}{|v_{rel}|}$; ...

b) Lattice gas: highly correlated collision partners possible
(numerical treatment only)

Subsystem:

$$N = N_A + N_B$$



$$T = \begin{pmatrix} T_{AA} & T_{AB} \\ T_{AB}^T & T_{BB} \end{pmatrix}$$

Equilibrium state:

$$P_{\text{eq}}(T_{\text{th}}) = \frac{1}{Z_{\text{IT}}} \text{ uniformly distributed phases (weights) of a fully connected graph.}$$

$$\rightarrow N_A \gg \langle S(S_A) \rangle > N_A - 1$$

Expected entanglement grows like "volume" of subsystem A!

quenched average

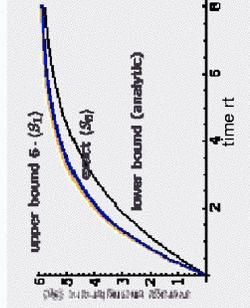
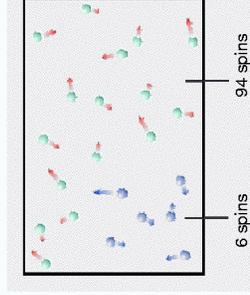
Time dependence: (assume large collisional phases)

$$\langle S_A(t) \rangle \geq -\log_2 \left[\frac{1}{2^N} \sum_{Z_A=0}^{N_A} \binom{N_A}{Z_A} (1 + e^{-tZ_A/(N-1)})^{N_B} \right]$$

lowers bound to entropy

→ short-time and long-time approximations

Calzavaglia et al. PRL 95, 180502 (2005)



Entanglement production rate for $t \rightarrow 0$:

large collisional phases (low temperature): $\frac{d\langle S_A \rangle}{dt} \propto \sqrt{t}$

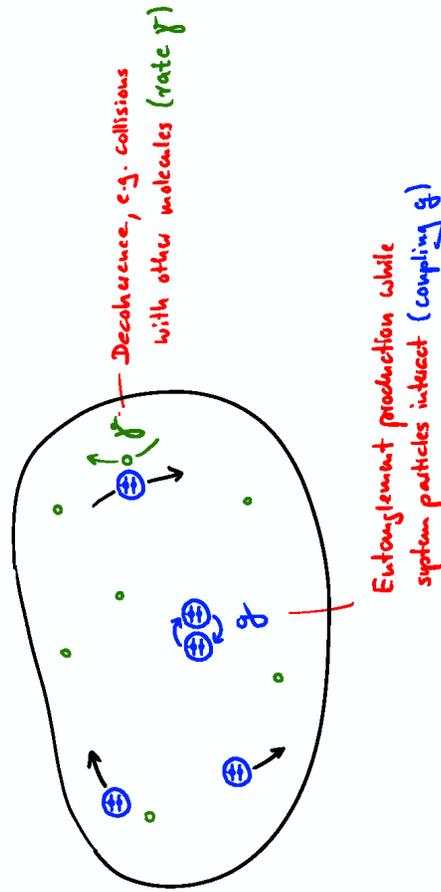
Small collisional phases (high temperature): $\frac{d\langle S_A \rangle}{dt} \propto \frac{1}{\sqrt{t}}$

- This system represents a natural semi-quantal extension of a Boltzmann gas.
- Study interesting thermodynamic questions, e.g.

Entanglement \leftrightarrow H-Theorem

Hartmann et al., quant-ph/0512217
 "Entanglement in noisy open quantum systems at high T "

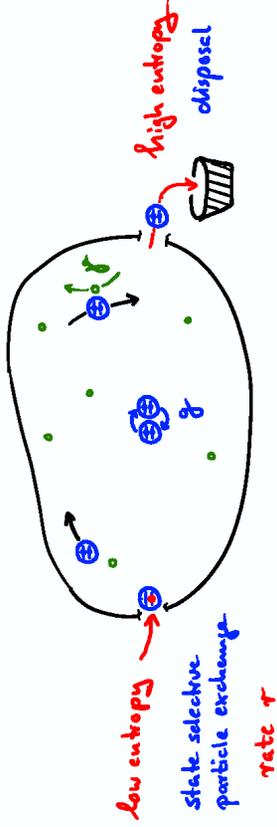
Decoherence: Can any entanglement survive if you have coupling to other degrees of freedom of an environment?



System far away from thermodynamic equilibrium:

e.g. due to collisions or high temperature

Entanglement may be sustained, despite of decoherence, given replacement of particles, at a certain rate, with fresh ones, in state with low entropy:



$$\rho_{\text{form}} = \begin{cases} |X\rangle\langle X| \\ \rho |X\rangle\langle X| + (\rho - \rho) |X\rangle\langle X| \end{cases}$$

N.B.: System far-away from TD equilibrium.

Master equation:

$$\dot{\rho} = (\delta\rho)_u + (\delta\rho)_{\text{misc}} + (\delta\rho)_{\text{part. exchange}} \equiv (\mathcal{L}\rho) \delta t$$

Coarse graining / time scales ...

$$\dot{\rho} = -\frac{i}{\hbar} [H_{\text{eff}}, \rho] + \mathcal{L}_{\text{misc}} \rho + \tau \sum_a (|X_a\rangle\langle X_a| \rho - \rho) \quad \text{Lindblad form}$$

Hartmann et al., quant-ph/0512217

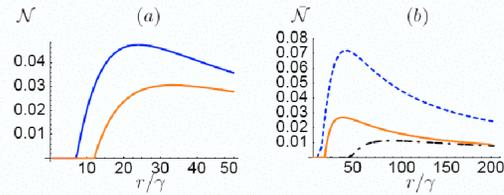


FIG. 2: (a) Negativity for 2-qubit system with XYZ-interaction and magnetic field, $H = g(0.7\sigma_x^{(1)}\sigma_x^{(2)} + 0.3\sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)} + 0.5(\sigma_x^{(1)} + \sigma_x^{(2)}))$, as a function of the reset rate r at $g = 10\gamma$. The noise is described by \mathcal{L}_{noise} of equation 1 with $C = B/2$ and $\gamma = C/10$. The upper curve corresponds to zero temperature ($s = 0$), the lower one to infinite temperature ($s = 1/2$) of the bath. Curves for any finite temperature lie in between. (b) Average negativity of (i) 5-qubits that have all pairwise interacted with each other (dashed), (ii) the reduced density matrix for any two qubits in a 5-qubit setting (dashed-dotted), and (iii) Poissonian mixture of reduced density matrices corresponding to fluctuating particle number, see text, (solid) as a function of the reset rate r for Ising interaction and dephasing, with $g = 5\gamma$.

Conclusions:

Weighted graph states are a useful tool to study strongly correlated systems from QI perspective



The Innsbruck team, some time ago ---

<http://www.uibk.ac.at/c7/c705/c705226/qigpeople.html>

Appendix

Steady-state entanglement

- measure by negativity $N_a = (\|\rho^T\| - 1)/2 = \begin{cases} 1/2 & \text{Bell state} \\ 0 & \text{separable state} \end{cases}$

- find:

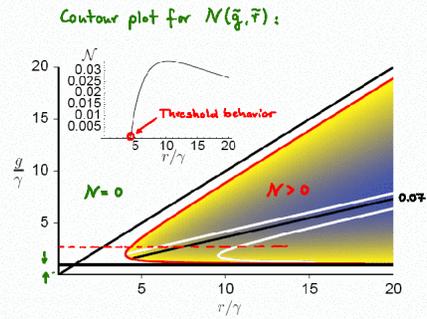


Hartmann, Dürr, HJB, quant-ph/05-- (2005)

$$= \mathcal{N}(\tau/\gamma, \theta/\gamma) \quad \text{Function of 2 parameters}$$

Plots

N.B.: Discuss first limiting cases $r=0$, $r \rightarrow \infty$



- No entanglement for $g/\gamma < 1$
- For $g/\gamma > 1$. Threshold at $r = 4g$:
 - $r \geq 4g$ equilibrium state entangled
 - $r \leq 4g$ equilibrium state separable