Quantum Control of Qubits

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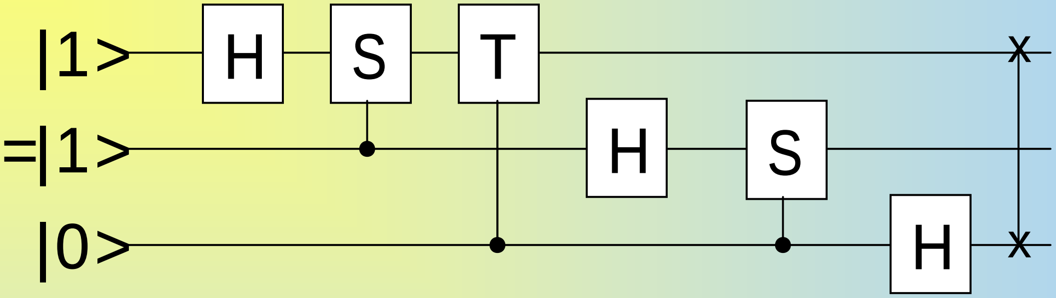
Quantum Circuit model



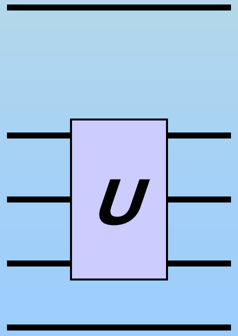
input $|6\rangle$ = $|1\rangle|1\rangle|0\rangle$

$$\sum_{k=0}^{7} e^{3\pi ik/2}\left|k\right\rangle$$ output

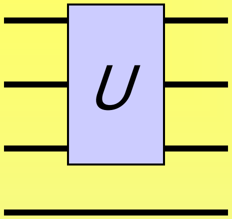
e.g., Quantum Fourier Transform on 3 qubits

wire = carrier of quantum information
(qubit {|0>, |1>}, qudit {|0>, |1>,...., |d>}

gate = time evolution of quantum information



$$\boldsymbol{U} = \mathrm{T}\left[\exp\left(-i\boldsymbol{H}(t)t_0 / \hbar\right)\right]$$



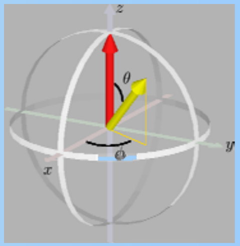
Universal sets of quantum gates

Theorem: every n-qubit unitary can be decomposed into combinations of 1-qubit and 2-qubit operations

(Barenco et al, 1995)

1. single qubit gates → SU(2), rotations on Bloch sphere

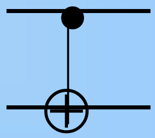
$$U(\theta,\hat{n},\phi)=e^{i\phi}\left(\cos\theta\mathbf{I}+i\sin\theta\hat{n}\cdot\sigma\right)$$



2. Two qubit gates

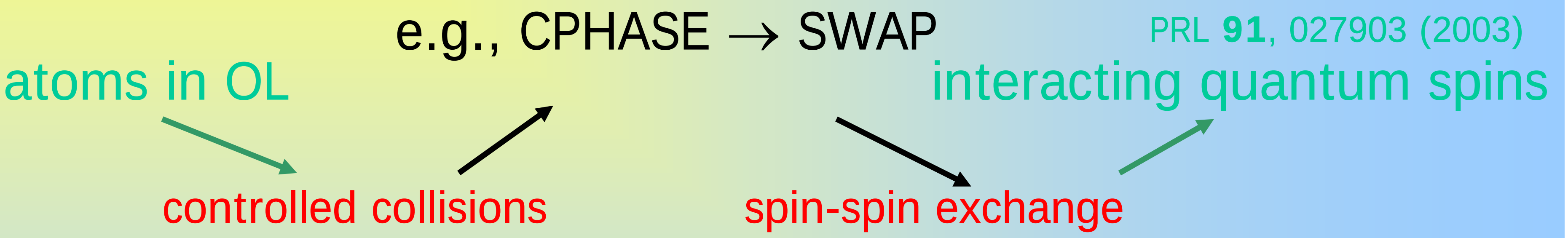
e.g., CNOT

$$CNOT=\begin{array}{c} |00\rangle\ |01\rangle\ |10\rangle\ |11\rangle \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$



Quantum simulators

- What can we implement, given a physical system?



e.g., CPHASE → SWAP

PRL **91**, 027903 (2003)

atoms in OL

interacting quantum spins

controlled collisions

spin-spin exchange

- What control fields are required? What cost?

- Can we ‘simply’ generate arbitrary quantum operations?

QIP requires ultra-high level of quantum control

- high fidelity quantum operations required for fault-tolerant quantum computation in standard model
- admissible error threshold
  - generic threshold value ~ .0001 [Aharonov, Gottesman '02]
  - scaling:

#levels recursion	#qubits	#operations
2	50	20000
3	350	4000000

- experimental fidelities ~ .01

  need #qubits, #operations 1-3 orders of magnitude larger

- [Roadmap Goal: recursion level 2 by 2012]

quantum control and robustness

- How generate gates and arbitrary quantum operations from Hamiltonians?
- Efficiency – various criteria for optimality
  - Time
  - On/off switching of interactions and external fields
  - Energy input from external fields
  - Minimal decoherence
  - All of the above together, with accurate gates....?

Algebraic approach

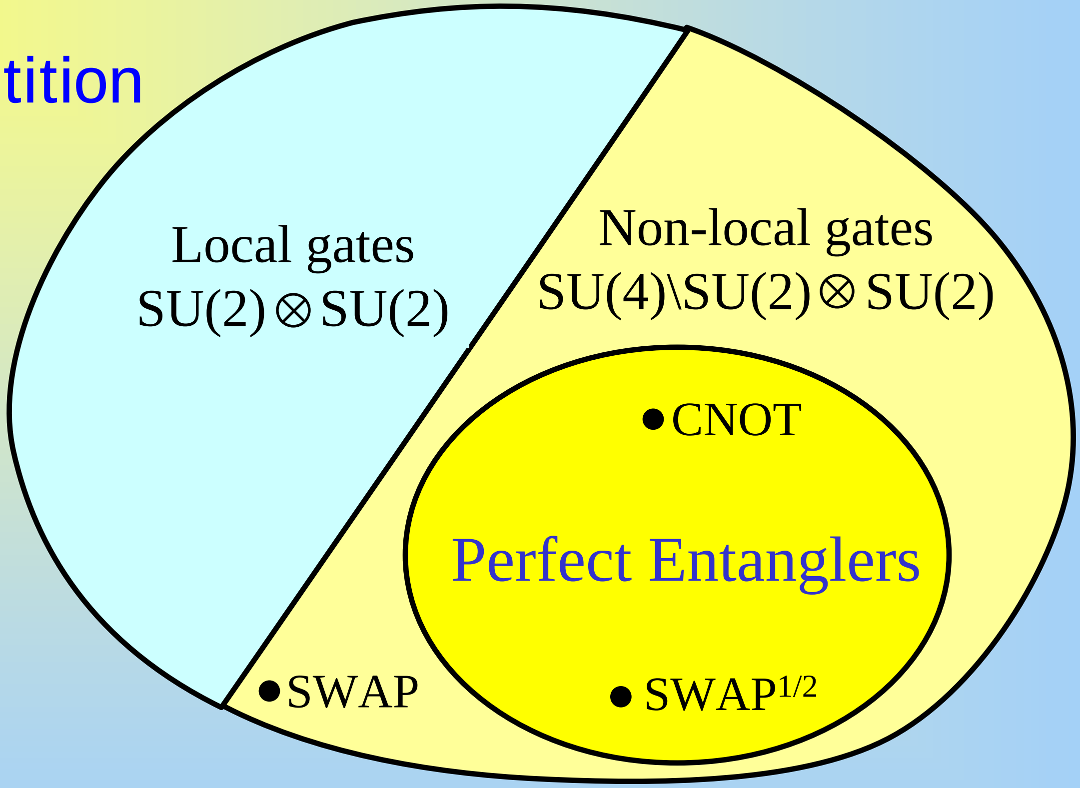
- Tunable interactions: 2-qubit gates by Weyl chamber steering
- Non-tunable interactions: algebraic decoupling for 1-qubit gates
- allows some gate optimization

add optimal control

- optimize with respect to cost function
  - time
  - energy
  - decoherence

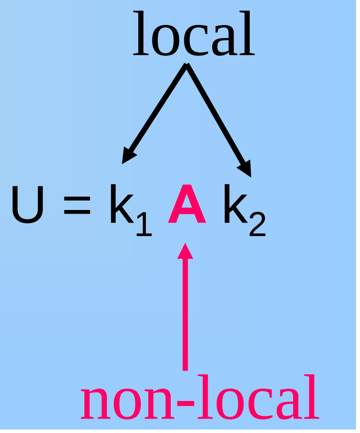
all 1- and 2-qubit gates: SU(4)

schematic partition of SU(4)



algebra su(4): σ^1_i , σ^2_i , $\sigma^1_i\sigma^2_j$, … i, j = x, y, z
Abelian subalgebra: $\sigma^1_x\sigma^2_x$, $\sigma^1_y\sigma^2_y$, $\sigma^1_z\sigma^2_z$

Cartan decomposition



su(4) algebra = $\mathfrak{k} \oplus \mathfrak{p}$

$\mathfrak{k} = span \frac{i}{2} \{\sigma_x^1, \sigma_y^1, \sigma_z^1, \sigma_x^2, \sigma_y^2, \sigma_z^2\}$ **local**

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

non-local

$\mathfrak{p} = span \frac{i}{2} \{\sigma_x^1\sigma_x^2, \sigma_x^1\sigma_y^2, \sigma_x^1\sigma_z^2, \sigma_y^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_y^1\sigma_z^2, \sigma_z^1\sigma_x^2, \sigma_z^1\sigma_y^2, \sigma_z^1\sigma_z^2\}$

$$\sigma_x^1\sigma_x^2 = \begin{pmatrix} 0 & \sigma_x^2 \\ \sigma_x^2 & 0 \end{pmatrix} = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}$$

Commutation relation, e.g.: $[\sigma_y^2, \sigma_y^1\sigma_z^2] \sim - \sigma_y^1\sigma_x^2$

Cartan decomposition: su(4) algebra = $\mathfrak{k} \oplus \mathfrak{p}$

$$\mathfrak{k} = span\,\frac{i}{2}\,\{\sigma_x^1, \sigma_y^1, \sigma_z^1, \sigma_x^2, \sigma_y^2, \sigma_z^2\}$$

$$\mathfrak{p} = span\,\frac{i}{2}\,\{\sigma_x^1\sigma_x^2, \sigma_x^1\sigma_y^2, \sigma_x^1\sigma_z^2, \sigma_y^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_y^1\sigma_z^2, \sigma_z^1\sigma_x^2, \sigma_z^1\sigma_y^2, \sigma_z^1\sigma_z^2\}$$

Maximal Abelian subalgebra

$$\mathfrak{a} = span\,\frac{i}{2}\,\{\sigma_x^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_z^1\sigma_z^2\}$$

Decomposition of a unitary transformation U in SU(4)

$$U = k_1\, \mathbf{A}\, k_2 = k_1\, exp[\quad (c_1\, \sigma_x^1\sigma_x^2 + c_2\, \sigma_y^1\sigma_y^2 + c_3\, \sigma_z^1\sigma_z^2)]\, k_2$$

local

non-local

local

Local gates and local equivalence (~)

$$\mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathrm{C\text{-}z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$U_1 \sim U_2$ if $U_1 = k_1 U_2 k_2$, where k_1 and k_2 are local gates,

e.g., CNOT ~ C-z

$$\mathrm{CNOT} = \frac{1}{\sqrt{2}}(I \otimes H) \cdot \mathrm{C\text{-}z} \cdot \frac{1}{\sqrt{2}}(I \otimes H)$$

SWAP $\nsim$ CNOT

local equivalence can be determined by evaluating 3 invariants

(Makhlin quant-ph/0002045)

Makhlin's local invariants

Given a two-qubit operation ***U***

$$m = (Q^\dagger U Q)^T (Q^\dagger U Q), \qquad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

$$G_1(U) = \frac{\mathrm{tr}^2\, m}{16 \det U}$$

$$G_2(U) = \frac{\mathrm{tr}^2\, m - \mathrm{tr}\, m^2}{4 \det U}$$

G_1: complex number

G_2: real number

3 invariants

	G_1	G_2
Local gates	1	3
CNOT	0	1
SWAP	-1	-3
$\sqrt{\mathrm{SWAP}}$	i/4	0

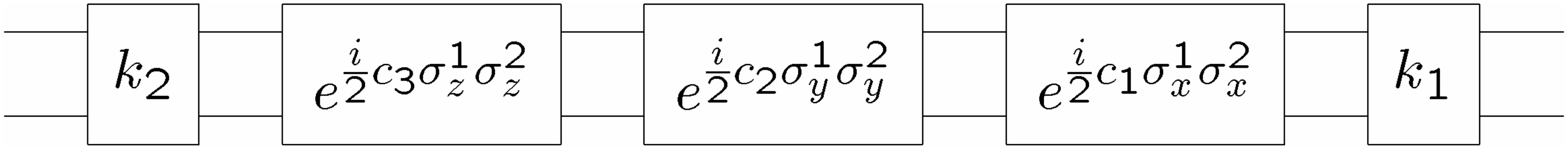
Claim: If $G_1(U_1)=G_1(U_2)$ and $G_2(U_1)=G_2(U_2)$, then $U_1 \sim U_2$

Makhlin, QIP **1**, 243 (2002)

Cartan decomposition on *su*(4)

any ***U*** ∈ **SU(4)** can be decomposed as:

$$U = e^{i\alpha} \cdot k_1 \cdot \exp\{\frac{i}{2}(c_1\sigma_x^1\sigma_x^2 + c_2\sigma_y^1\sigma_y^2 + c_3\sigma_z^1\sigma_z^2)\} \cdot k_2$$



$$G_1 = \cos^2 c_1 \cos^2 c_2 \cos^2 c_3 - \sin^2 c_1 \sin^2 c_2 \sin^2 c_3 + \frac{i}{4} \sin 2c_1 \sin 2c_2 \sin 2c_3$$

$$G_2 = 4\cos^2 c_1 \cos^2 c_2 \cos^2 c_3 - 4\sin^2 c_1 \sin^2 c_2 \sin^2 c_3 - \cos 2c_1 \cos 2c_2 \cos 2c_3$$

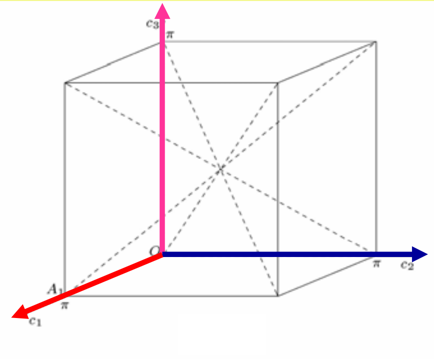
→ invariants are periodic in c_1, c_2, c_3

Geometric Theory of Non-local Gates

Cartan decomposition

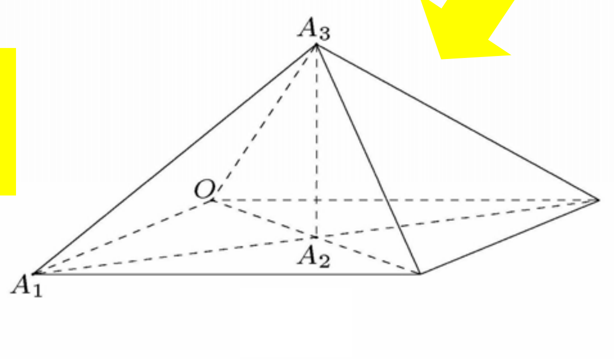
$$U = k_1 A k_2 = k_1 \, exp[\frac{i}{2}(c_1 \sigma_x^1\sigma_x^2 + c_2 \sigma_y^1\sigma_y^2 + c_3 \sigma_z^1\sigma_z^2)] \, k_2$$

c_1, c_2, c_3 periodic → 3-Torus



cartesian representation

symmetry reduction



→ one-to-one correspondence between the points inside the tetrahedron and local equivalence classes (except on base)

J. Zhang et al., PRA **67**, 042313 (2003)

Local invariants

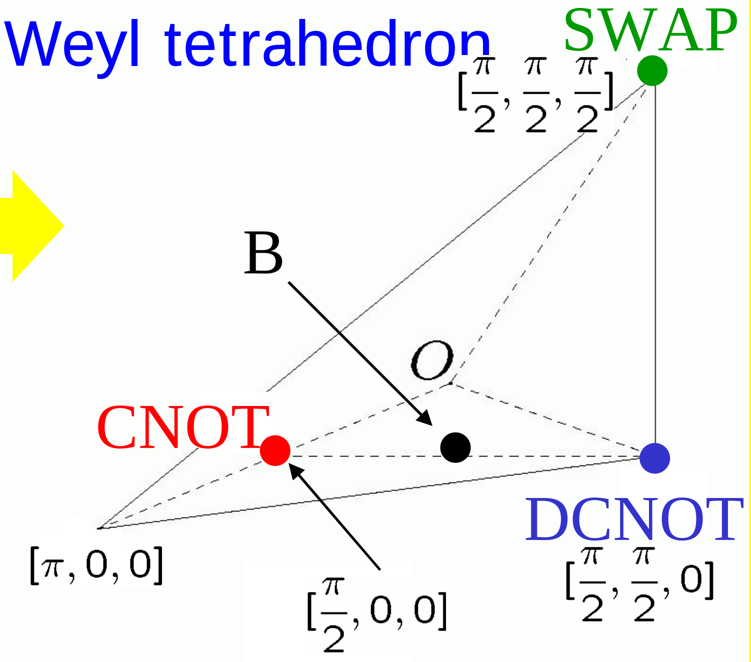
$\mathbf{G_1} = \quad cos\, c_1\, cos\, c_2\, cos\, c_3$

$\mathbf{G_2} = \quad sin\, c_1\, sin\, c_2\, sin\, c_3$

$\mathbf{G_3} = 2\,(cos^2 c_1 + cos^2 c_2 + cos^2 c_3) - 3$

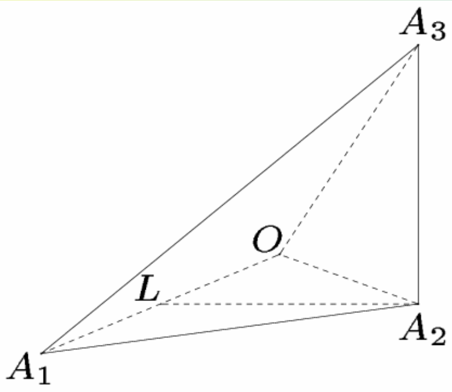
G_1,G_2 and G_3 are invariant on permuting c_1, c_2, and c_3 with/without sign flips

Weyl tetrahedron



Implications of geometric analysis

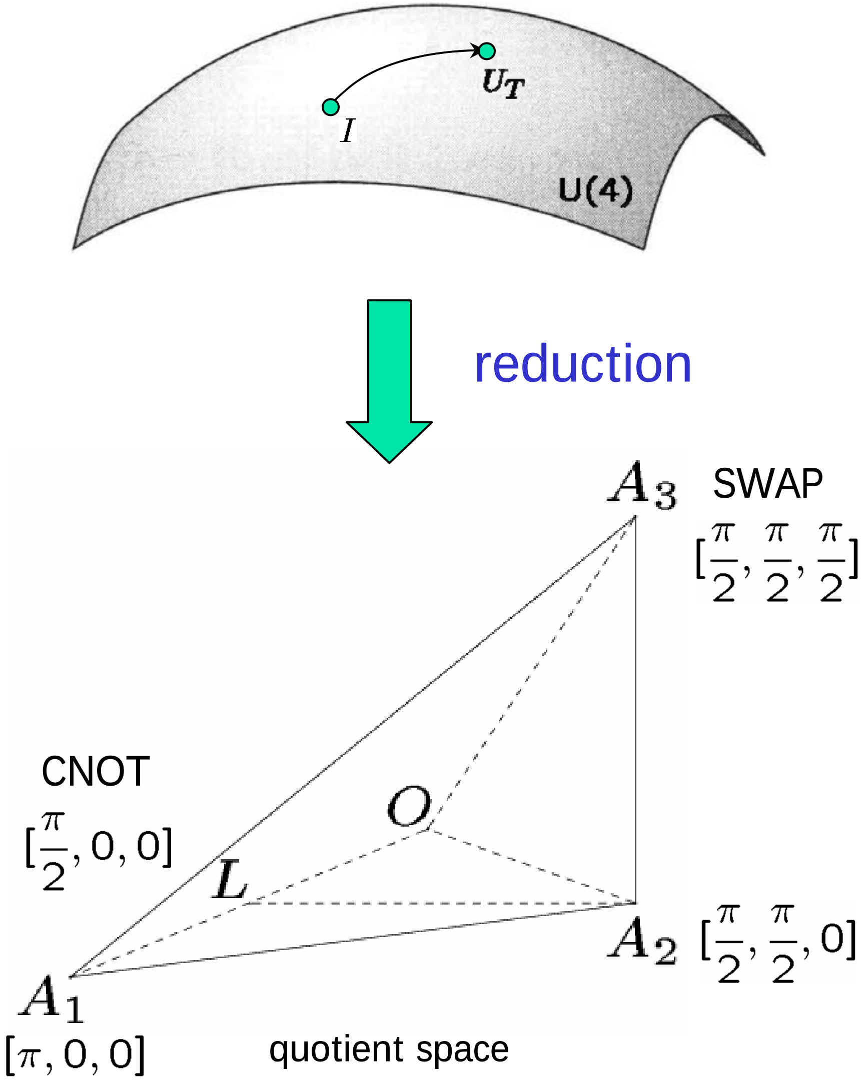
Tetrahedral representation of local equivalence classes



Applications:

- physical generation of non-local gates, arbitrary 2-qubit operations
- optimally efficient quantum circuits
- characterization of perfect entanglers

Generation of non-local gates as a steering problem in the Weyl tetrahedron



15 dimensional control problem on U(4)

$$\dot{U} = -iH(v)U, \quad U(0) = I$$

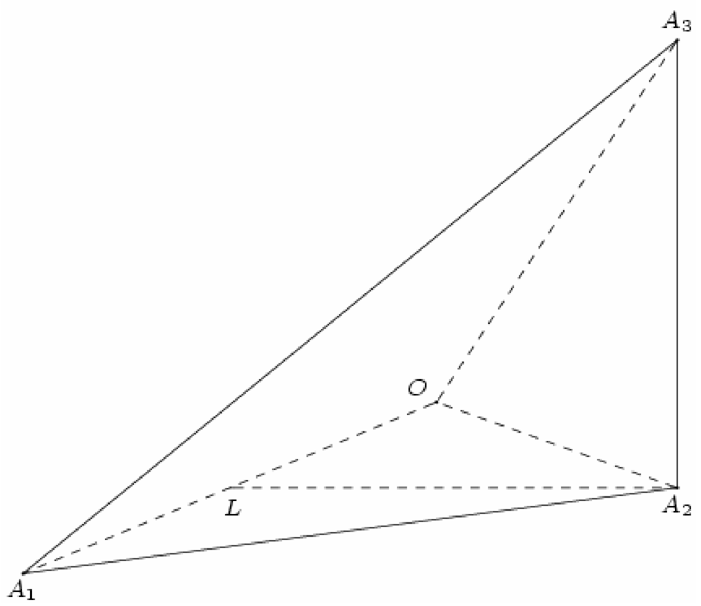
3 dimensional steering problem in Weyl tetrahedron

Weyl tetrahedron trajectory

System dynamics: $\dot{U} = -iH(v)U$

For any t, $U(t)$ determines a point in the tetrahedron via the Makhlin invariants for the non-local equivalence classes, i.e.,

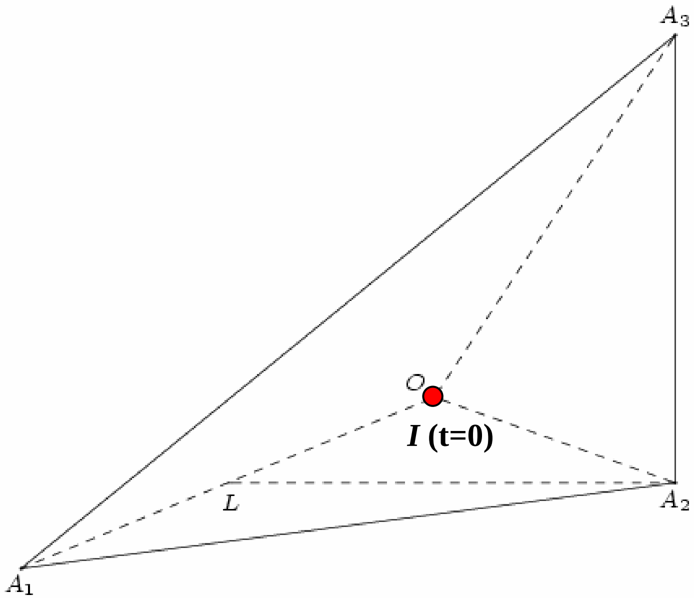
$$U(t) \rightarrow G_i(t) \rightarrow c_i(t)$$



Weyl tetrahedron trajectory

$$\dot{U} = -iH(v)U$$

$$t = 0, \quad U(0) = I$$

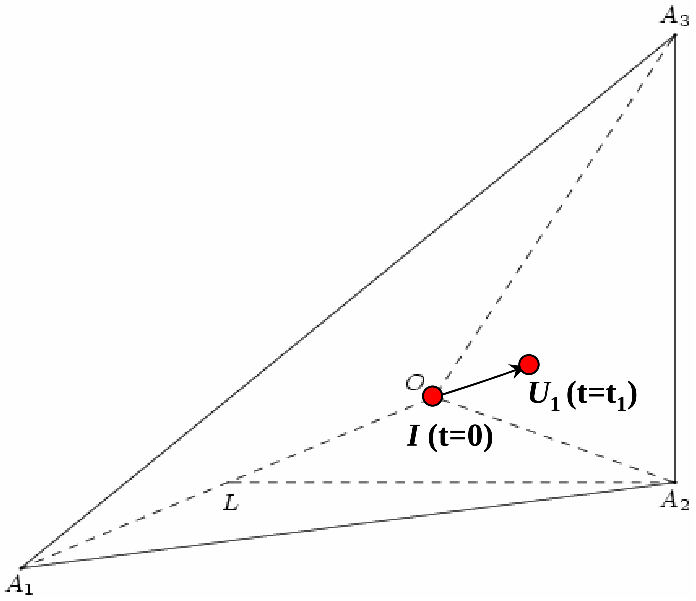


Weyl tetrahedron trajectory

$$\dot{U} = -iH(v)U$$

$$t = 0, \quad U(0) = I$$

$$t = t_1, \quad U(t_1) = U_1$$



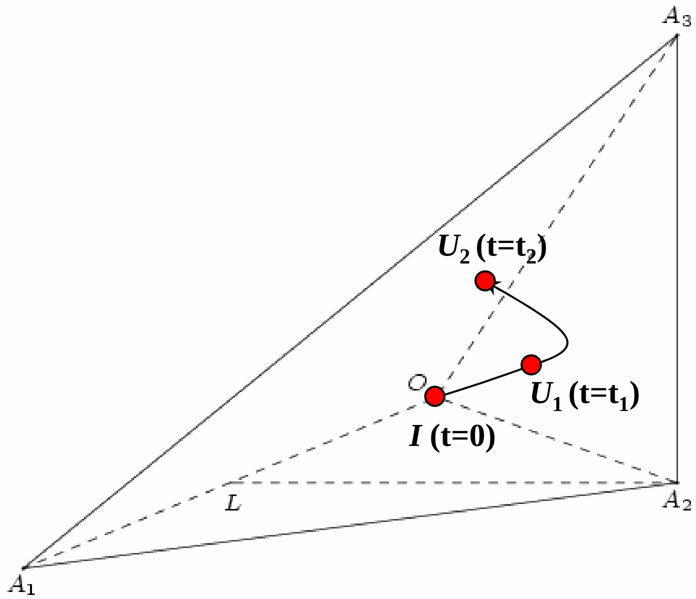
Weyl tetrahedron trajectory

$$\dot{U} = -iH(v)U$$

$$t = 0, \quad U(0) = I$$

$$t = t_1, \quad U(t_1) = U_1$$

$$t = t_2, \quad U(t_2) = U_2$$

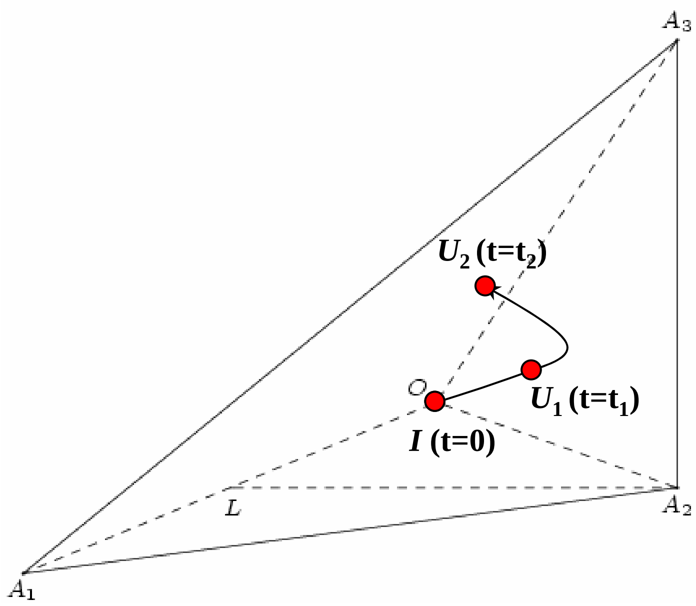


Weyl tetrahedron trajectory

$$\dot{U} = -iH(v)U$$

$$t = 0, \quad U(0) = I$$
$$t = t_1, \quad U(t_1) = U_1$$
$$t = t_2, \quad U(t_2) = U_2$$

As time evolves, we can obtain a continuous trajectory in the Weyl tetrahedron

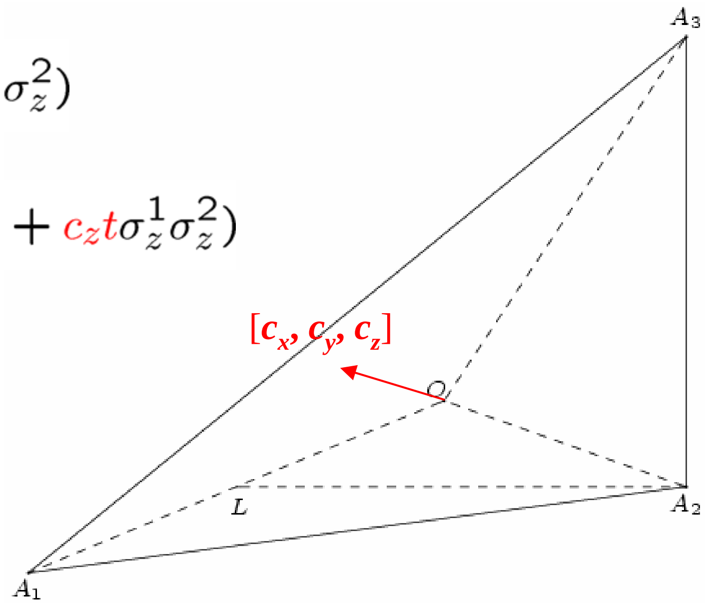


Pure nonlocal Hamiltonian

Consider $H = -\frac{1}{2}(c_x\sigma_x^1\sigma_x^2 + c_y\sigma_y^1\sigma_y^2 + c_z\sigma_z^1\sigma_z^2)$

$$U(t) = \exp(-iHt) = \exp\frac{i}{2}(c_x t\sigma_x^1\sigma_x^2 + c_y t\sigma_y^1\sigma_y^2 + c_z t\sigma_z^1\sigma_z^2)$$

$\Rightarrow$ $[c_x, c_y, c_z]t$



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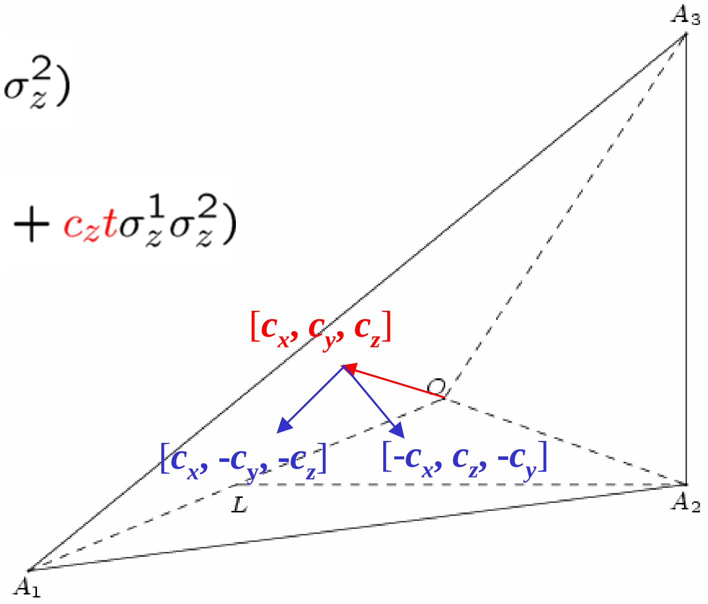
⇒ $[c_x, c_y, c_z]t$



$k \cdot \exp(-iHt) \cdot k^\dagger$, where $k \subset$ Weyl group

⇔ Reflections of $[c_x, c_y, c_z]$ w.r.t. diagonal planes

New directions: $[c_x, -c_y, -c_z]$, $[-c_x, c_z, -c_y]$, $[c_x, c_z, c_y]$, $[-c_x, -c_z, c_y]$, …

Steering a Weyl chamber trajectory

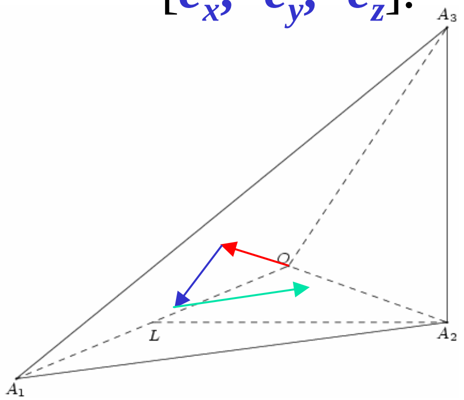
Piece two segments together:

$$\underbrace{k \cdot \exp(-iHt_2) \cdot k^{\dagger}}_{\Downarrow} \cdot \underbrace{\exp(-iHt_1)}_{\Downarrow}$$

$$[c_x, -c_y, -c_z]t_2 + [c_x, c_y, c_z]t_1$$

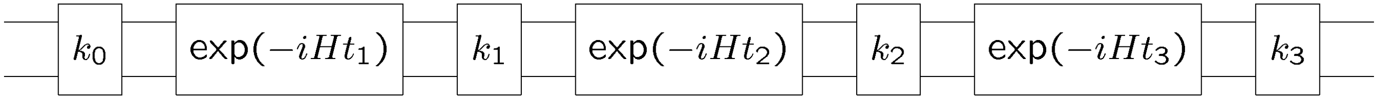
we can reach anywhere in the plane spanned by [***c_x, c_y, c_z***] and [***c_x, -c_y, -c_z***].

Changing direction twice suffices:



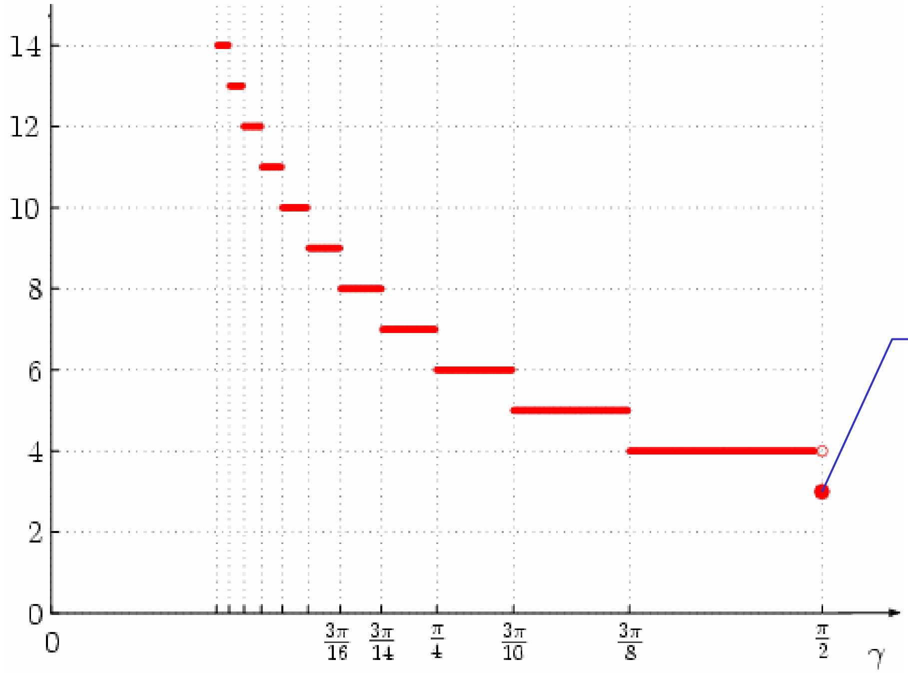
the trajectory defines a quantum circuit

→ Theorem: the following circuit can implement any two-qubit gate



Minimum bound for Controlled-Unitary

For a Controlled-U gate $e^{\gamma \frac{i}{2}\sigma_z^1\sigma_z^2}$, minimum applications needed to implement any arbitrary two-qubit gate is $\left\lceil \frac{3\pi}{2\gamma} \right\rceil$.



CNOT

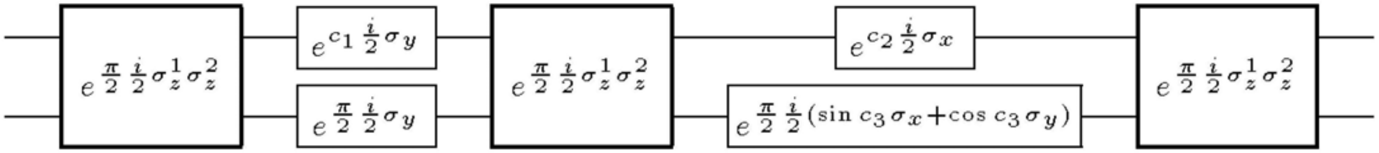
The most efficient among Controlled-U

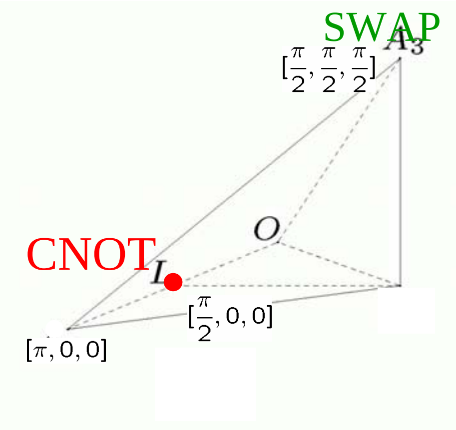
J. Zhang et al, PRA **69** 042309 (2004)

Quantum Circuits for arbitrary 2-qubit operations

CNOT

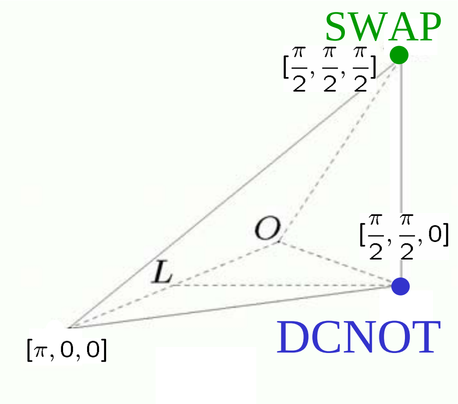
3 applications suffice

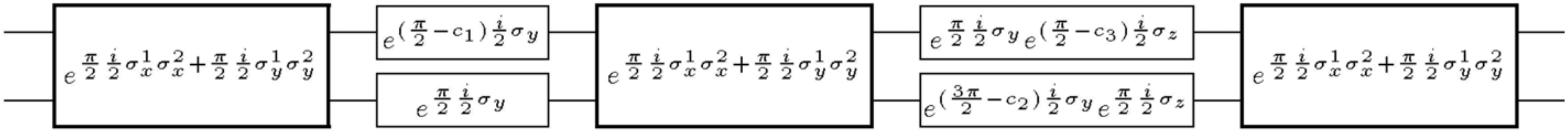




Double-CNOT

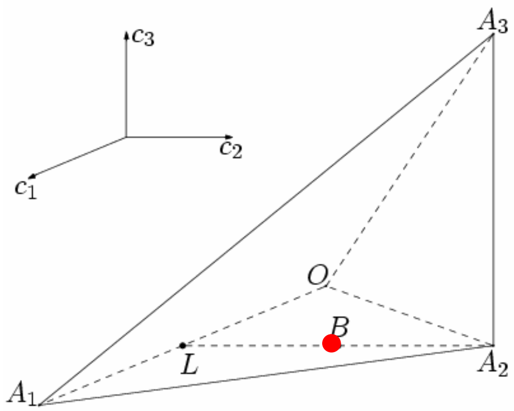
3 applications suffice

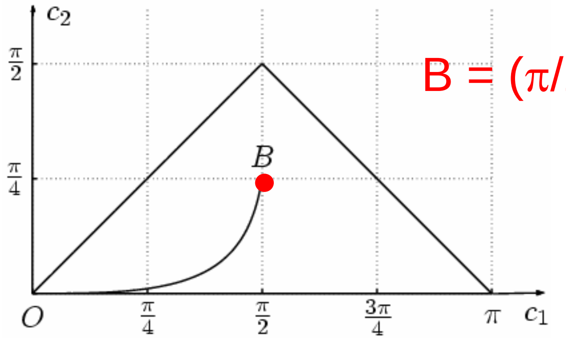




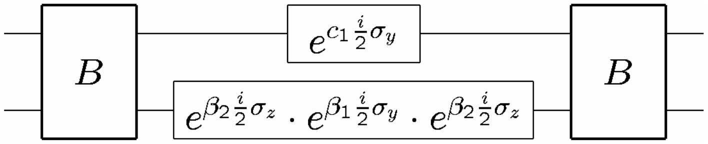
J. Zhang, J. Vala, S.Sastry, K.B. Whaley
PRA **69**, 042309 (2004)

J. Zhang et al., PRL **93**, 020502 (2004)





B = (π/2, π/4,0)

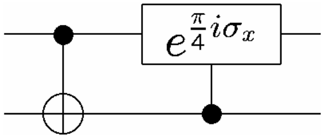


$$\sim\ e^{c_1\frac{i}{2}\sigma_x^1\sigma_x^2}\cdot e^{c_2\frac{i}{2}\sigma_y^1\sigma_y^2}\cdot e^{c_3\frac{i}{2}\sigma_z^1\sigma_z^2}$$

Two applications of the B gate suffice to implement any arbitrary two-qubit gate: explicit solutions for β_1 and β_2 as functions of c_2 and c_3

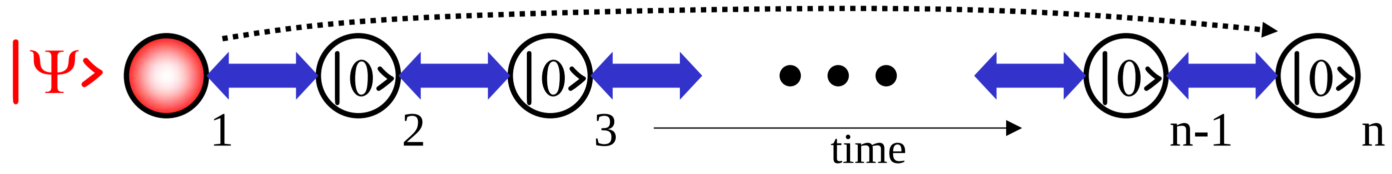
on the computational basis B acts as:

$$|m\rangle\otimes|n\rangle\longrightarrow e^{\frac{\pi}{4}i\sigma_x(m\oplus n)}|m\rangle\otimes|m\oplus n\rangle$$



Example: SWAP gate and quantum wire

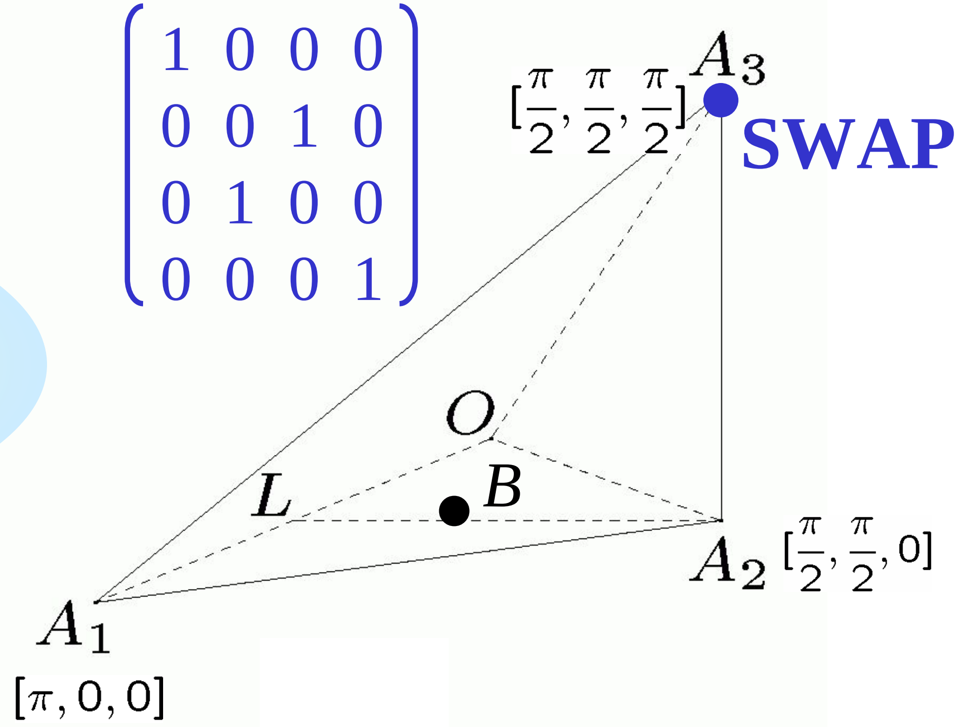
qubit state transfer via a sequence of SWAP gates:



Oskin et al.,
Proc. ISCA 2002

SWAP $[(c_0|0>_k+c_1|1>_k)|0>_{k+1}] = |0>_k (c_0|0>_{k+1}+c_1|1>_{k+1})$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



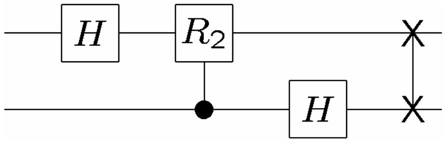
only two B gates needed

compared to three CNOTs

for each step of the wire

Example: QFT on two qubits

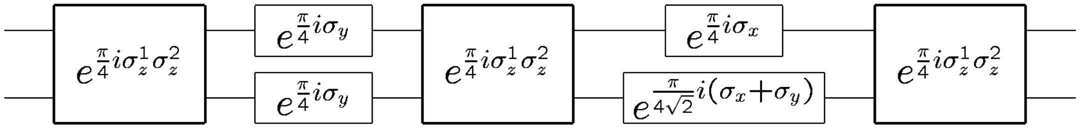
computer science implementation of two-qubit QFT:



FIVE CNOT gates in total

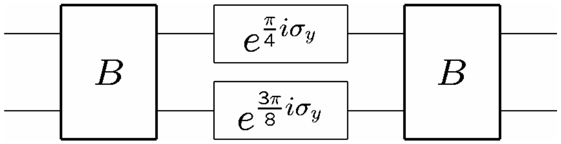
in matrix representation: $\mathrm{QFT}_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} \sim [\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}]$

three CNOT gates can implement this*:



$\sim \quad [\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}]$

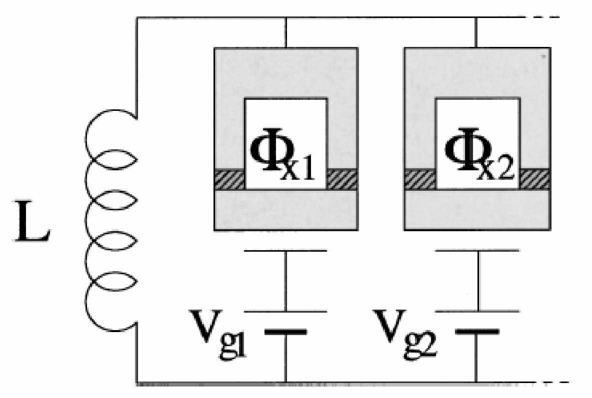
B gate needs only two applications:



in n-qubit case, the B gate is slightly better than the CNOT gate

* J. Zhang et al., quant-ph/0308167

I. Josephson junction charge-coupled qubits



Y. Makhlin et al., RMP **73**, 357 (2001)

tune $B_z(V_g(t))$ $E_J(\Phi_x(t))$

charge qubit with tunable coupling: for 2 qubits

$$H=\sum_{i=1,2}\frac{1}{2}B_z\sigma_z^i-\sum_{i=1,2}\frac{1}{2}E_J^i\sigma_x^i-\frac{E_J^1E_J^2}{E_L}\sigma_y^1\sigma_y^2$$

switch E_J^i independently
→ 1 qubit operations

switch E_J^1, E_J^2 together
→ 2 qubit operations

$$E_J\sim 100\,mK,\ E_L\sim 1-100\,mK$$

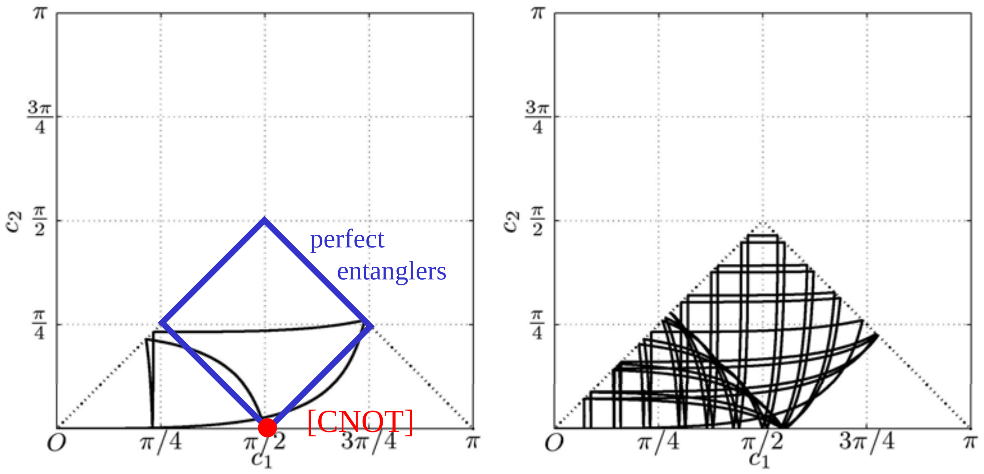
interaction between Josephson junction qubits:

$$H_{int} = -(\alpha E_L/2)\,(\sigma_x^1 + \sigma_x^2) + \alpha^2 E_L\, \sigma_y^1 \sigma_y^2 \qquad (\alpha = E_J/E_L)$$

curvature translation

Weyl chamber trajectory

$$c_1(t) = \alpha^2 E_L t - \omega(\alpha, t),$$
$$c_2(t) = \alpha^2 E_L t + \omega(\alpha, t),$$
$$c_3(t) = 0.$$

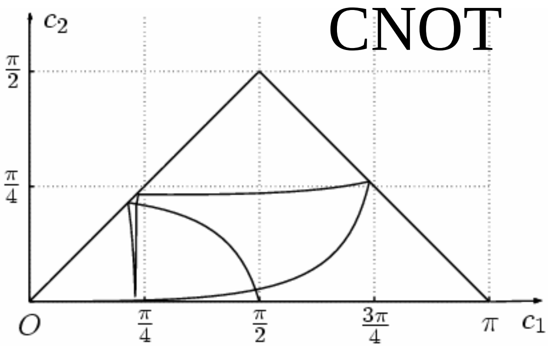


$\alpha = 1.1991,\ t = 2.7309$ $\alpha = 1.1991,\ t = 20$

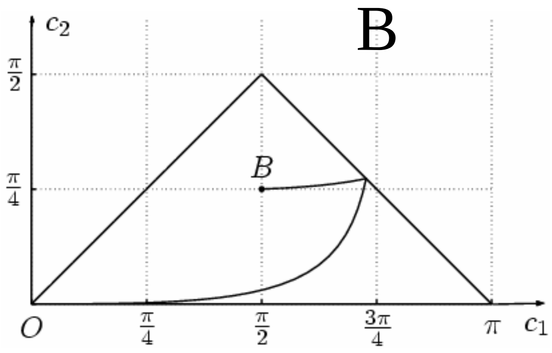
time optimal parameters for CNOT, α = 1.1991, t=2.73

J. Zhang et al., PRA **67**, 042313 (2003) scaled parameters $E_J = \alpha E_L$, $E_L = 1$:

tune α to implement various gates in minimum time



$\alpha = 1.1191,\ t = 2.7309$



$\alpha = 1.1436,\ t = 1.5013$

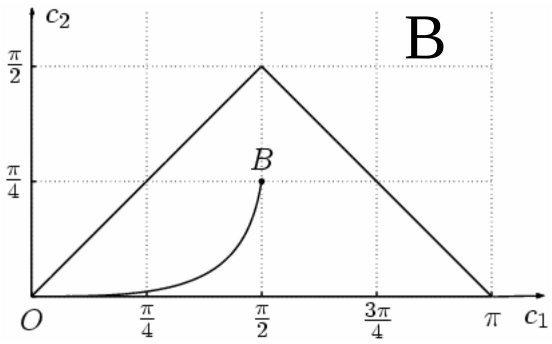
$\alpha > 1$

- time optimal solution for CNOT has $\alpha > 1$
- no CNOT solution for $\alpha < 1$
- B gate has solution for all α regimes

- realistic SC circuit, $\alpha \leq 1$

- no CNOT from single application of H_{int}

- but B can be implemented directly

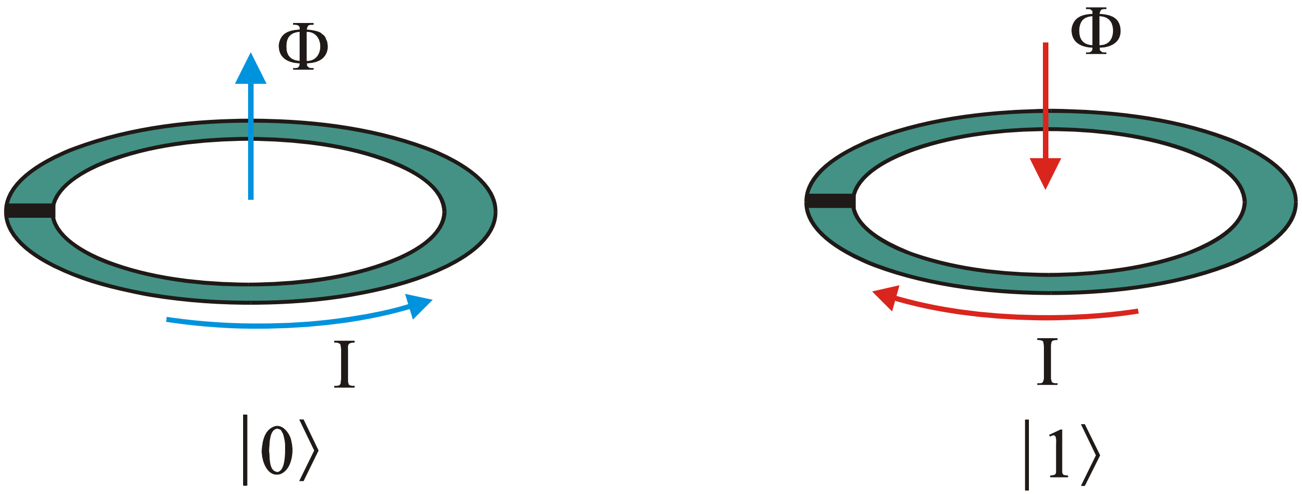


$\alpha = 0.5969,\ t = 3.3066$

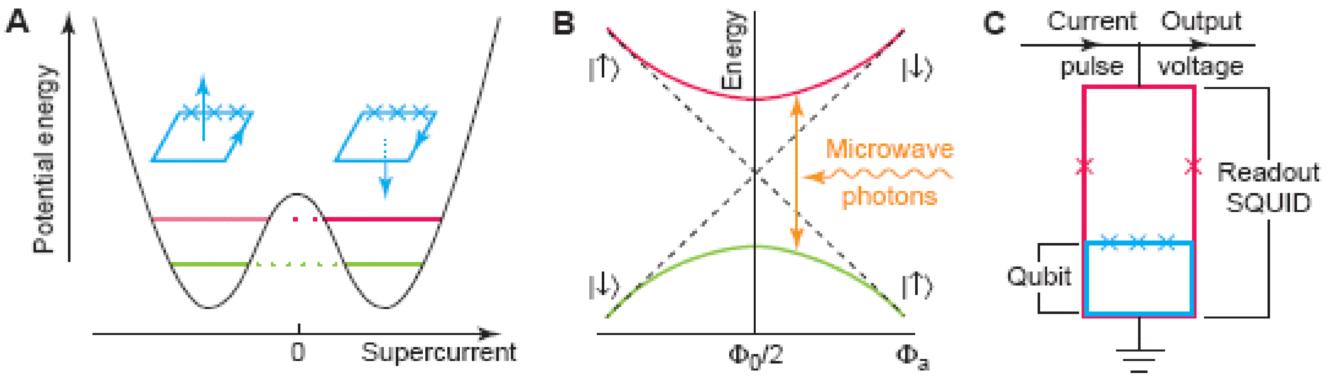
$\alpha < 1$

Zhang et al. PRL **93**, 020502 (2004)

II. inductively coupled SC flux qubits



single flux qubit $H_i = \frac{1}{2}\left[\varepsilon_i(t)\sigma_z^{(i)} + \Delta_i\sigma_x^{(i)}\right]$

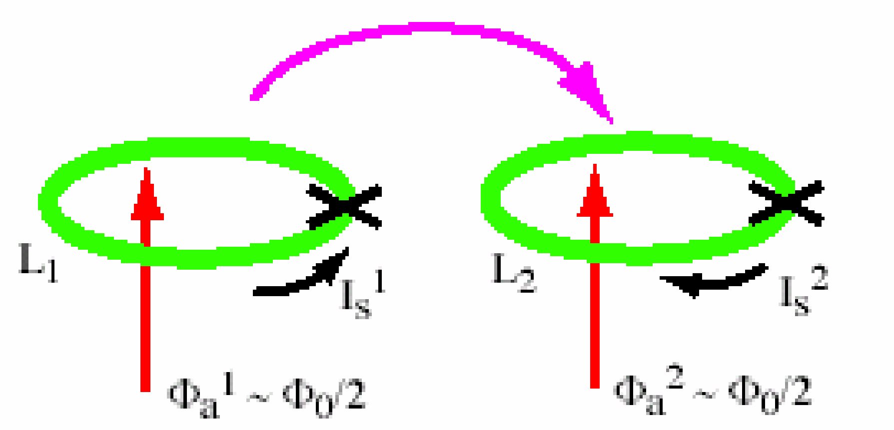


$\Phi_a = \Phi_0 / 2$

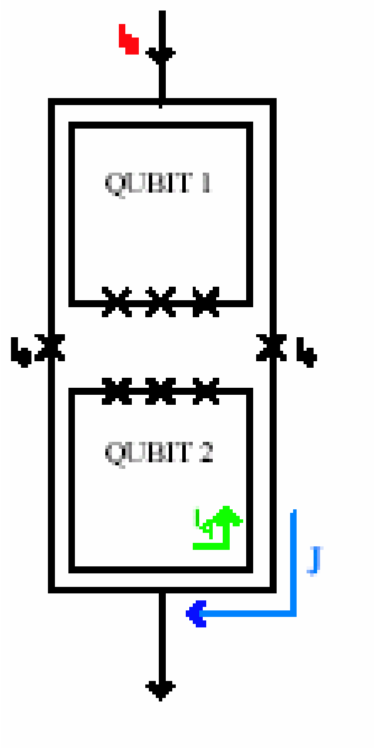
Flux qubit: Chiorescu et al. Science (2003)

inductive coupling:

natural interaction via flux:
e.g., screening flux of qubit 1
changes flux bias ε of qubit 2
→ $\sigma_z^{(1)}\sigma_z^{(2)}$ interaction



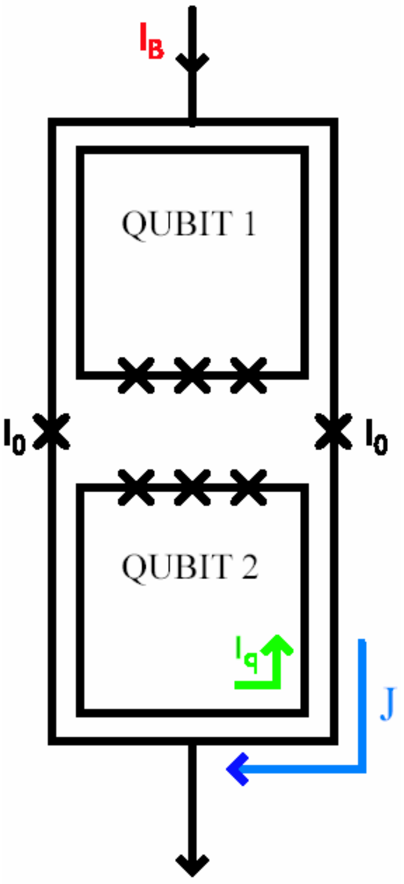
$$H = \frac{1}{2}\sum_{i=1,2}\left(\varepsilon_i(t)\sigma_z^{(i)} + \Delta_i\sigma_x^{(i)}\right) + K\sigma_z^{(1)}\sigma_z^{(2)}$$

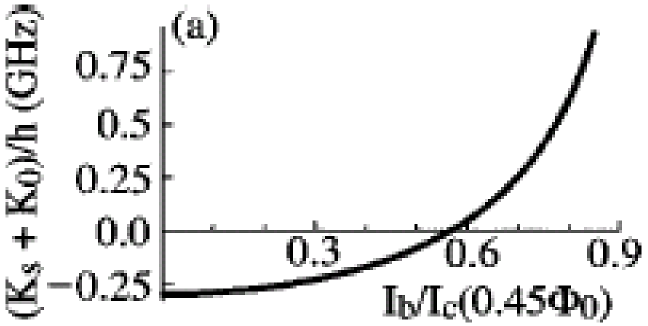


- coupling via mutual inductance: K fixed
- new - magnetic flux J in the outer loop couples the two qubits: K tunable and can be switched off

B. Plourde, T. Robinson, F. Wilhelm, J. Clarke, et al.

Entangling operation with variable inductance





Switch magnetic flux in outer loop on/off with bias current, couples two qubits

$$H = \frac{1}{2}\varepsilon\sum_{i=1,2}\left(\varepsilon_i(t)\sigma_z^{(i)} + \Delta_i\sigma_x^{(i)}\right) + K(t)\sigma_z^{(1)}\sigma_z^{(2)}$$

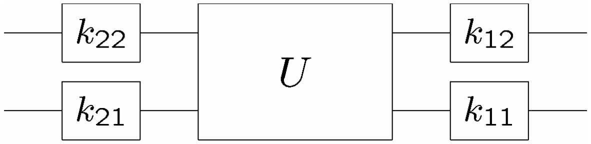
$$\varepsilon_i(t) = \varepsilon_i^{(0)} + A_i\cos(\omega_i t + \phi_i) + \delta\varepsilon_i^{xtalk}(t)$$

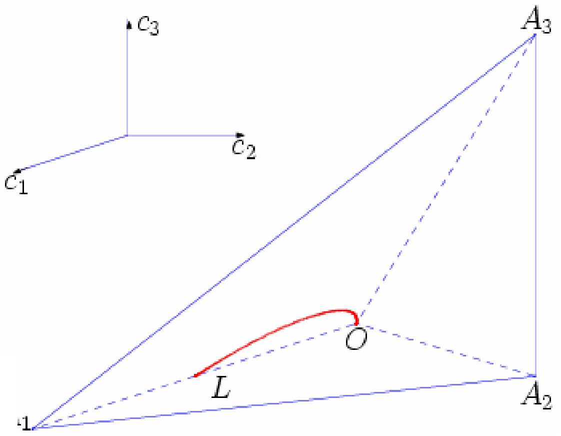
2-qubit operations: Weyl trajectory

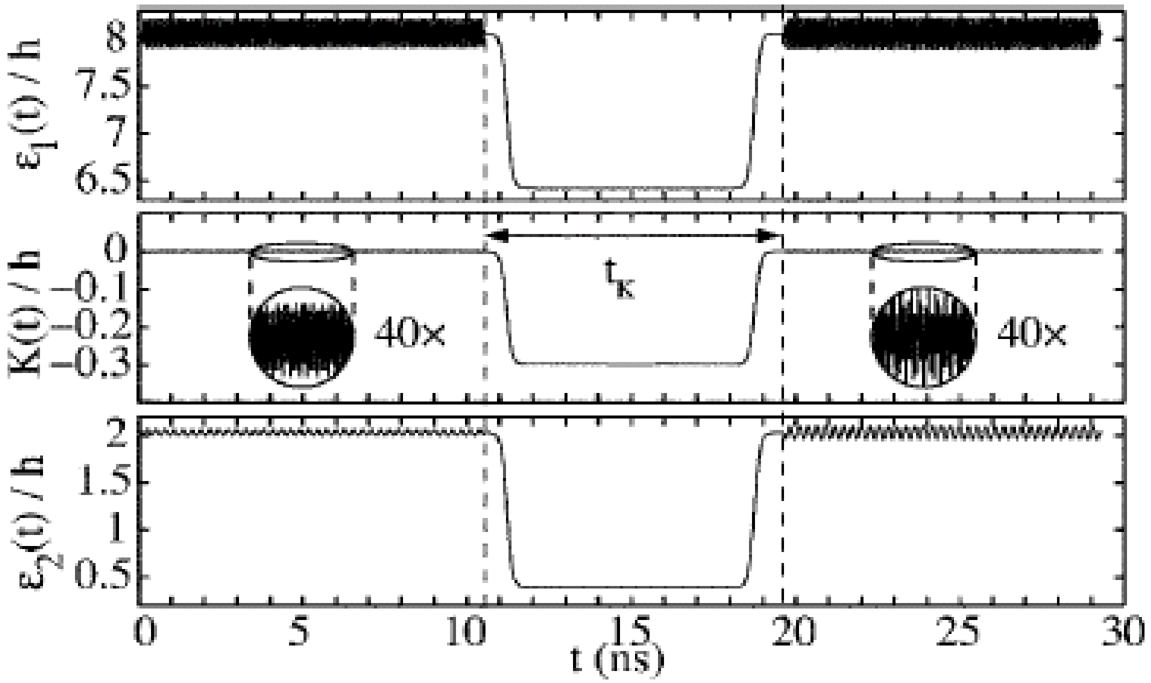
1-qubit operations: external control fields ω_i (off resonant)

Implementation of CNOT

CNOT =



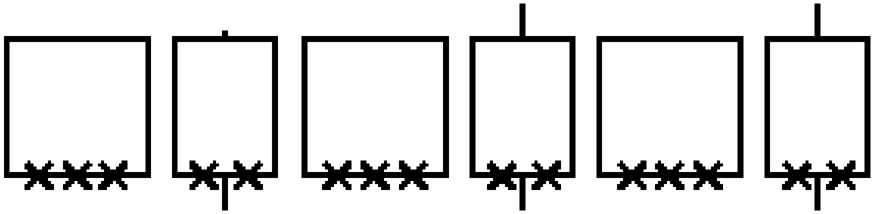




$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CNOT

scale up →



...

B. Plourde et al, PRB **70** 140501 (2004)

Non-tunable interactions:
how generate 1-qubit gates?

direct approach: exact algebraic decoupling of two-qubit Hamiltonian

$$H = \frac{\omega_1}{2}(\cos\phi_1\sigma_x^1 + \sin\phi_1\sigma_y^1) + \frac{\omega_2}{2}(\cos\phi_2\sigma_x^2 + \sin\phi_2\sigma_y^2) + \frac{J}{2}\sigma_z^1\sigma_z^2$$

J is the always-on and untunable coupling strength, ω_j and ϕ_j the amplitudes and phases of the external control fields

Target: generate any arbitrary single-qubit operation in each qubit

J. Zhang and K. B. Whaley, PRA **73** 022306 (2006)

Simplified problem

It is easy to prove that to implement any arbitrary one-qubit operation, we only need to generate an arbitrary local unitary operation:

$$e^{-i\gamma_1\sigma_x/2} \otimes e^{-i\gamma_2\sigma_x/2}$$

from the Hamiltonian:

$$H_1 = \frac{\omega_1}{2}\sigma_x^1 + \frac{\omega_2}{2}\sigma_x^2 + \frac{J}{2}\sigma_z^1\sigma_z^2$$

Now observe that $i\sigma_x^1/2$, $i\sigma_x^2/2$, and $i\sigma_z^1\sigma_z^2/2$ generate the following Lie algebra:

$$\mathfrak{k}_1 = \frac{i}{2}\{\sigma_x^1,\ \sigma_x^2,\ \sigma_z^1\sigma_y^2,\ \sigma_y^1\sigma_z^2,\ \sigma_y^1\sigma_y^2,\ \sigma_z^1\sigma_z^2\}$$

It is straightforward to show that $\mathfrak{k}_1$ satisfies the same commutation relations as so(4), where so(4) denotes the Lie algebra formed by all the 4x4 real skew symmetric matrices.

Lie algebra isomorphism

Let

$$\epsilon_x^1 = \frac{\sigma_x^1 - \sigma_x^2}{4},\ \ \epsilon_y^1 = \frac{\sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2}{4}, \epsilon_z^1 = \frac{\sigma_z^1\sigma_y^2 - \sigma_y^1\sigma_z^2}{4},$$

$$\epsilon_x^2 = \frac{\sigma_x^1 + \sigma_x^2}{4},\ \ \epsilon_y^2 = \frac{\sigma_y^1\sigma_y^2 - \sigma_z^1\sigma_z^2}{4}, \epsilon_z^2 = \frac{\sigma_z^1\sigma_y^2 + \sigma_y^1\sigma_z^2}{4},$$

we have the following commutation relations:

$[\cdot,\cdot]$	$i\epsilon_x^1$	$i\epsilon_y^1$	$i\epsilon_z^1$	$i\epsilon_x^2$	$i\epsilon_y^2$	$i\epsilon_z^2$
$i\epsilon_x^1$	0	$-i\epsilon_z^1$	$i\epsilon_y^1$	0	0	0
$i\epsilon_y^1$	$i\epsilon_z^1$	0	$-i\epsilon_x^1$	0	0	0
$i\epsilon_z^1$	$-i\epsilon_y^1$	$i\epsilon_x^1$	0	0	0	0
$i\epsilon_x^2$	0	0	0	0	$-i\epsilon_z^2$	$i\epsilon_y^2$
$i\epsilon_y^2$	0	0	0	$i\epsilon_z^2$	0	$-i\epsilon_x^2$
$i\epsilon_z^2$	0	0	0	$-i\epsilon_y^2$	$i\epsilon_x^2$	0

Therefore, $\mathfrak{k}_1$ is isomorphic to $\mathrm{su}(2)\otimes\mathrm{su}(2)$. This isomorphism allows simplification for the generation of single-qubit operation, because it provides an algebraic way to decouple the entangling Hamiltonian into two unentangled single-qubit Hamiltonians.

Two sub-problems

We can now rewrite the Hamiltonian as

$$H_1 = (\omega_1 - \omega_2)\epsilon_x^1 + J\epsilon_y^1 + (\omega_1 + \omega_2)\epsilon_x^2 - J\epsilon_y^2$$

and the target operation as

$$k_2 = e^{-i\gamma_1\sigma_x/2} \otimes e^{-i\gamma_2\sigma_x/2} = e^{-i((\gamma_1-\gamma_2)\epsilon_x^1+(\gamma_1+\gamma_2)\epsilon_x^2)}$$

Now the original problem of generating $e^{-i\gamma_1\sigma_x/2} \otimes e^{-i\gamma_2\sigma_x/2}$ from

$$H_1 = \frac{\omega_1}{2}\sigma_x^1 + \frac{\omega_2}{2}\sigma_x^2 + \frac{J}{2}\sigma_z^1\sigma_z^2$$

becomes generating $e^{-i((\gamma_1-\gamma_2)\sigma_x^1/2+(\gamma_1+\gamma_2)\sigma_x^2/2)}$ from

$$\frac{\omega_1 - \omega_2}{2}\sigma_x^1 + \frac{J}{2}\sigma_y^1 + \frac{\omega_1 + \omega_2}{2}\sigma_x^2 - \frac{J}{2}\sigma_y^2$$

We only need to implement the following two one-qubit operations:

(1) Generate $e^{-i(\gamma_1-\gamma_2)\sigma_x^1/2}$ from the Hamiltonian $(\omega_1 - \omega_2)\sigma_x^1/2 + J\sigma_y^1/2$; and

(2) Generate $e^{-i(\gamma_1+\gamma_2)\sigma_x^2/2}$ from the Hamiltonian $(\omega_1 + \omega_2)\sigma_x^2/2 - J\sigma_y^2/2$.

One-qubit sub-operations: optimal control

Consider a general one-qubit system:

$$i\dot{U} = \left(\frac{\omega(t)}{2}\sigma_x + \frac{J}{2}\sigma_y\right)U, \quad U(0) = I$$

Solution of ω(t) to exactly implement a target one-qubit operation $U_T = e^{-i\gamma/2\sigma_x}$ is possible with simultaneous minimization of a cost function

$$J = \int_0^T L(\omega(t))\, dt$$

Time optimal $J = \int_0^T 1dt$

Energy optimal $J = \frac{1}{2}\int_0^T \omega^2(t)dt$

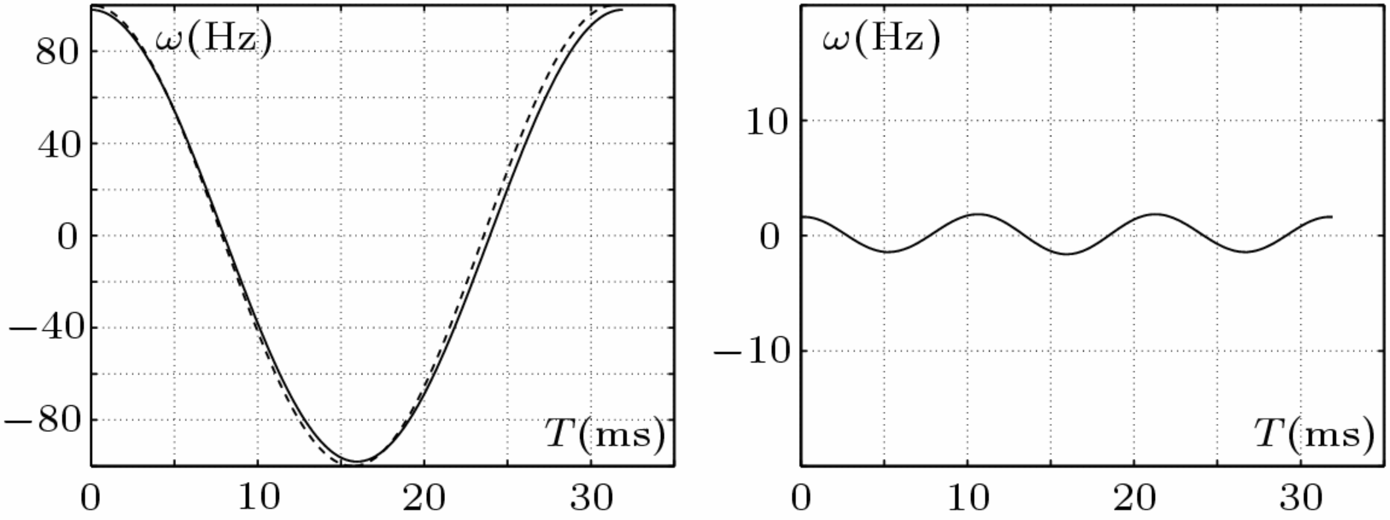
Also have analytic approximate implementation with $\omega = \frac{\gamma J}{n\pi}\cos\left(Jt\right),\ T = 2n\pi / J$

Simple example:

Let J=200 Hz in the Hamiltonian and $e^{-i\pi/4\sigma_x^1}$ be the desired target 1-qubit operation. Choosing $\phi_1 = \phi_2 = 0$ and n=1, we obtain an approximate solution $\omega_1 = 100\cos 200t, \quad \omega_2 = 0.$ The corresponding pulse time is T=31.4 ms.

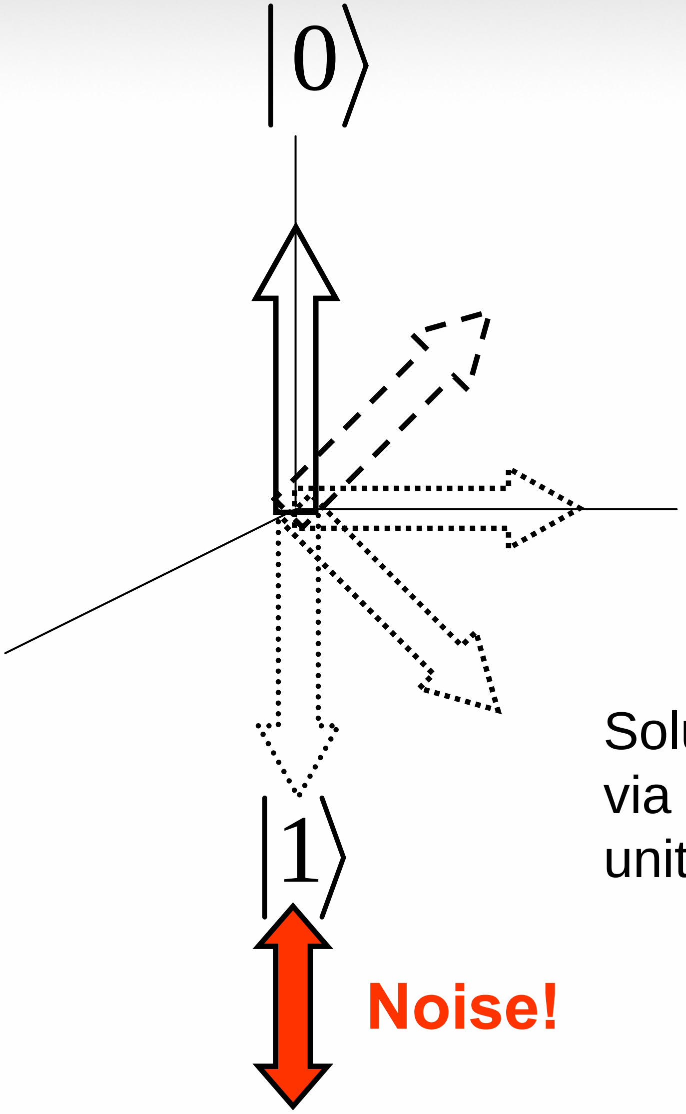
Numerical optimization via the maximization of the fidelity leads to the improved solution parameters $\omega_1 = 98.062\cos 196.900t, \quad \omega_2 = 0,$

with corresponding pulse time T=31.911 ms and fidelity error 4.104 x 10^{-11}.



Control functions that generate $e^{-i\pi/4\sigma_x^1}$. (A) Dashed line: **approximate control**; solid line: fidelity optimized control; (B) the difference between fidelity optimized control and minimum energy control

Optimal control of 1-qubit operations subject to random telegraph noise



$$H = a(t)\sigma_x + \eta(t)\sigma_z$$

Bounded control
$a \leq a_{\max}$

Random telegraph noise

Solution for given noise correlation time
via numerical optimization of
unitary quantum trajectory formulation of fidelity

M. Möttönen, R. de Sousa, J. Zhang,
K. B. Whaley, PRA **73**, 022332 (2006)

Summary of Part I

- Geometric approach to non-local gates
  - steering approach to generation of 2-qubit gates
  - analytic construction of quantum circuits
- How implement an arbitrary 2-qubit operation?
  - starting from given Hamiltonian
    - → steering in Weyl chamber (tetrahedron)
  - starting from given gate, e.g., CNOT
    - → gate B is optimal, only 2 applications
- Constraints
  - physical feasibility of H
  - minimal switchings, time optimization
- Algebraic decoupling for 1-qubit operations when non-tunable interactions present
  - optimization with respect to general cost function
  - time, energy, decoherence ...

→ broad route to optimal feasible control of coupled qubits ...

Part II

- Geometric approach to n -local gates
  - steering approach generation of 2-qubit gates
  - analytic constructi of quantum circuits
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    - → steering in Weyl chamber (tetrahedron)
  - starting from given gate, e.g., CNOT
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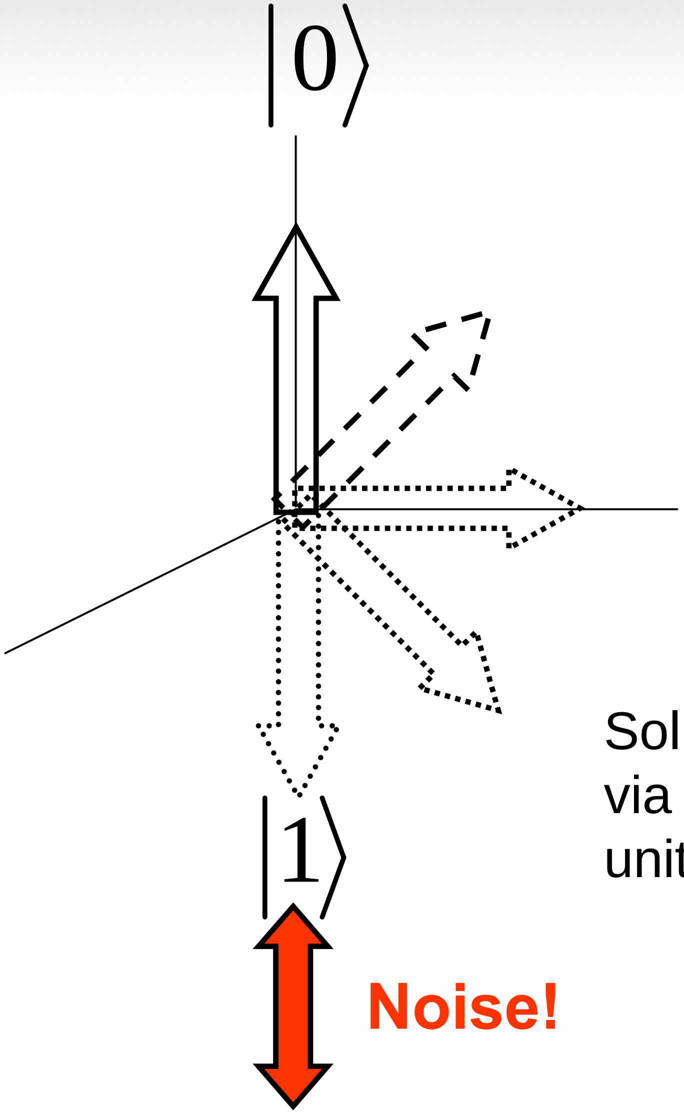
Optimal control of 1-qubit operations subject to random telegraph noise



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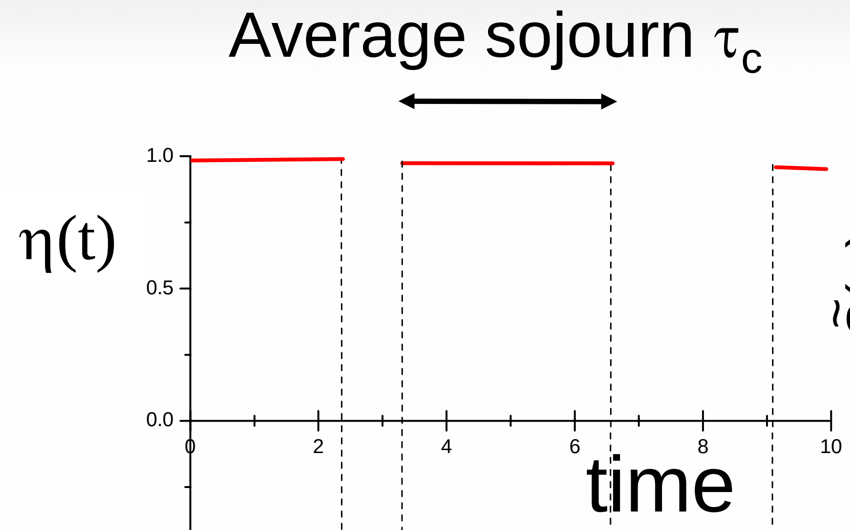
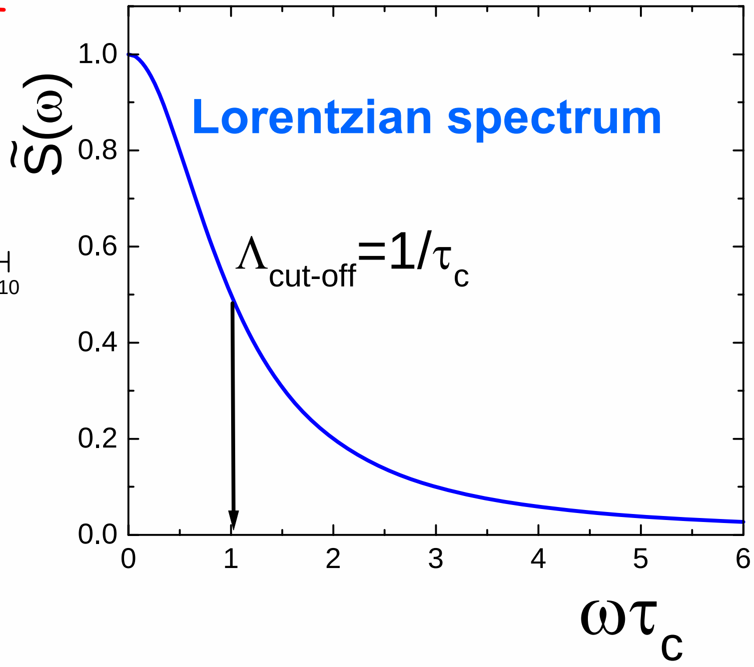
Random telegraph noise





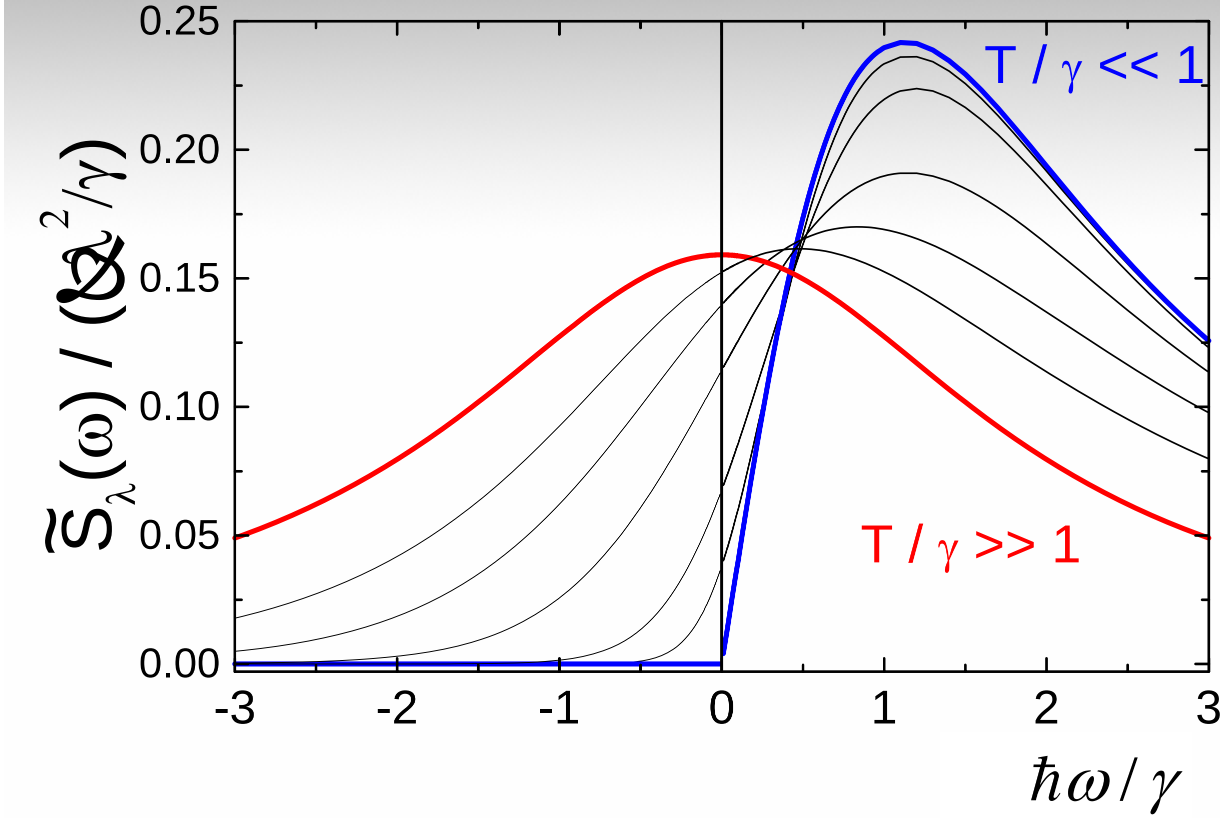
Random telegraph noise (RTN)

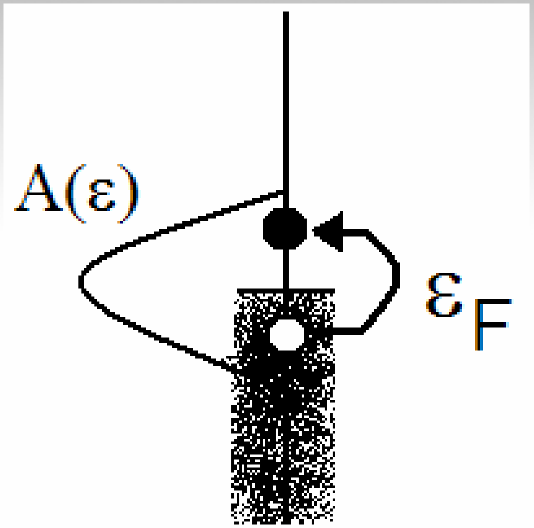
- Described by the correlation time τ_c and the noise strength Δ
- The noise amplitude jumps between values Δ and $-\Delta$
- Probability of no jumps in time t is $\boxed{p_0(t) = e^{-t/\tau_c}}$
- Jump time instants and the noise amplitude are given by

$$\boxed{t_i = \sum_{j=1}^{i} -\tau_c \ln(p_j)}$$

$$\boxed{\eta(t) = (-1)^{\sum_i \Theta(t-t_i)} \eta(0)}$$

Trapping center noise on charge qubits





R. de Sousa, K.B. Whaley, F.K. Wilhelm, J. von Delft, PRL **95**, 247006 (2005)

- Effective only if trap energy level is close to Fermi level;
- High temperature, $k_BT>>\gamma$: Lorentzian spectrum, semiclassical RTN;
- Low temperature, $k_BT<<\gamma$: **QUANTUM REGIME** f-noise, Ohmic

System dynamics

- Let *k* index the sample paths of RTN
- The dynamics of the system density matrix is given by an average over all different noise samples as

$$\boxed{\rho(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} U_k \rho_0 U_k^\dagger}$$

$$\boxed{U_k = \mathcal{T} e^{-i \int_0^t d\tau [a(\tau)\sigma_x + \eta_k(\tau)\sigma_z]/\hbar}}$$

Example operations

Bit flip |0> → |1>

$$\rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \rho_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Fidelity

$$\phi(\rho_t, \rho_0) = \mathrm{tr}\{\rho_t^\dagger \rho(T)\}$$

$$\phi(\rho_t, \rho_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{tr}\{\rho_t^\dagger U_k \rho_0 U_k^\dagger\}$$

NOT gate = target gate U_t

- Fidelity

$$\Phi(U_t) = \frac{1}{4\pi} \int_{c_x^2+c_y^2+c_z^2=1} \mathrm{d}\Omega\, \phi(U_t \rho_0 U_t^\dagger, \rho_0)$$

$$\rho_0 = (I + c_x\sigma_x + c_y\sigma_y + c_z\sigma_z)/2$$

Gradient ascent pulse engineering (GRAPE)

- Optimizes the fidelity with respect to the control pulse by a gradient method
- Solution not unique
- We used a constant control pulse as an initial condition
- Convergence of fidelity much faster than convergence of the pulse shape
- Compare with standard pulse sequences for correction of systematic (static) error, CORPSE and SCORPSE

Composite pulse sequences

- π-pulse: $a_\pi(t) = a_{\max}, \quad \text{for} \quad t \in [0, \pi\hbar/a_{\max}]$
- Compensation of off-resonence with a pulse sequence

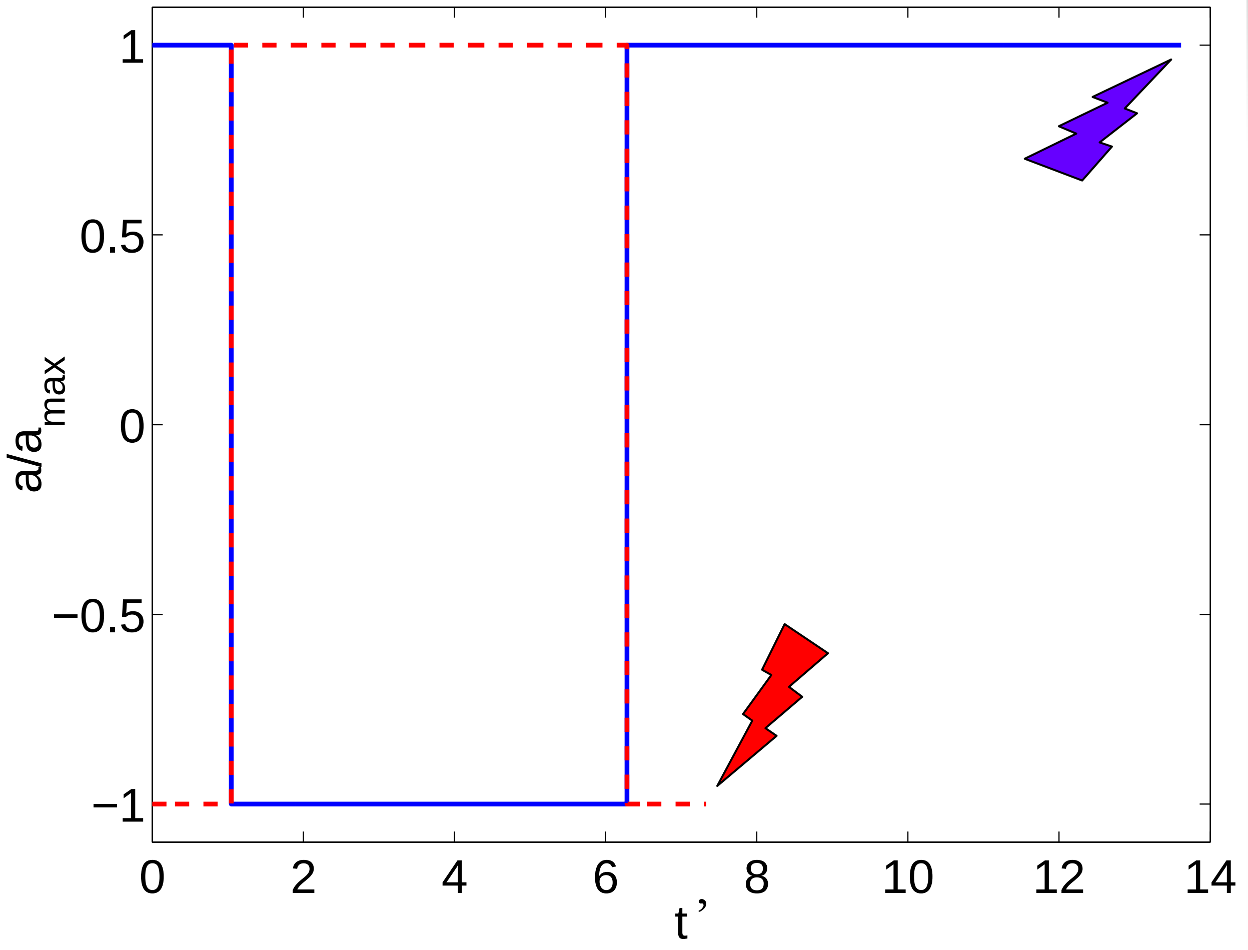
CORPSE: $a_{\rm C}(t) = \begin{cases} a_{\max}, & \text{for} \;\; 0 < t' < \pi/3 \\ -a_{\max}, & \text{for} \;\; \pi/3 \le t' \le 2\pi \\ a_{\max}, & \text{for} \;\; 2\pi < t' < 13\pi/3 \end{cases}$

$t' \;=\; a_{\max} t/\hbar$

- short CORPSE: $a_{\rm SC}(t) = \begin{cases} -a_{\max}, & \text{for} \;\; 0 < t' < \pi/3 \\ a_{\max}, & \text{for} \;\; \pi/3 \le t' \le 2\pi \\ -a_{\max}, & \text{for} \;\; 2\pi < t' < 7\pi/3 \end{cases}$

Best short pulse sequences correcting for systematic static error

CORPSE and short CORPSE



Composite pulses and numerical optimization

CORPSE and SCORPSE: Cummins, Llewellyn, Jones, PRA **67**, 042308 (2003)

$$U_\pi = \mathrm{e}^{i\left(+\frac{\pi}{2}\sigma_x+\eta\sigma_z\right)} \quad\Rightarrow\quad \text{Fidelity} \approx 1-\left(\frac{\eta}{a_{\max}}\right)^2$$

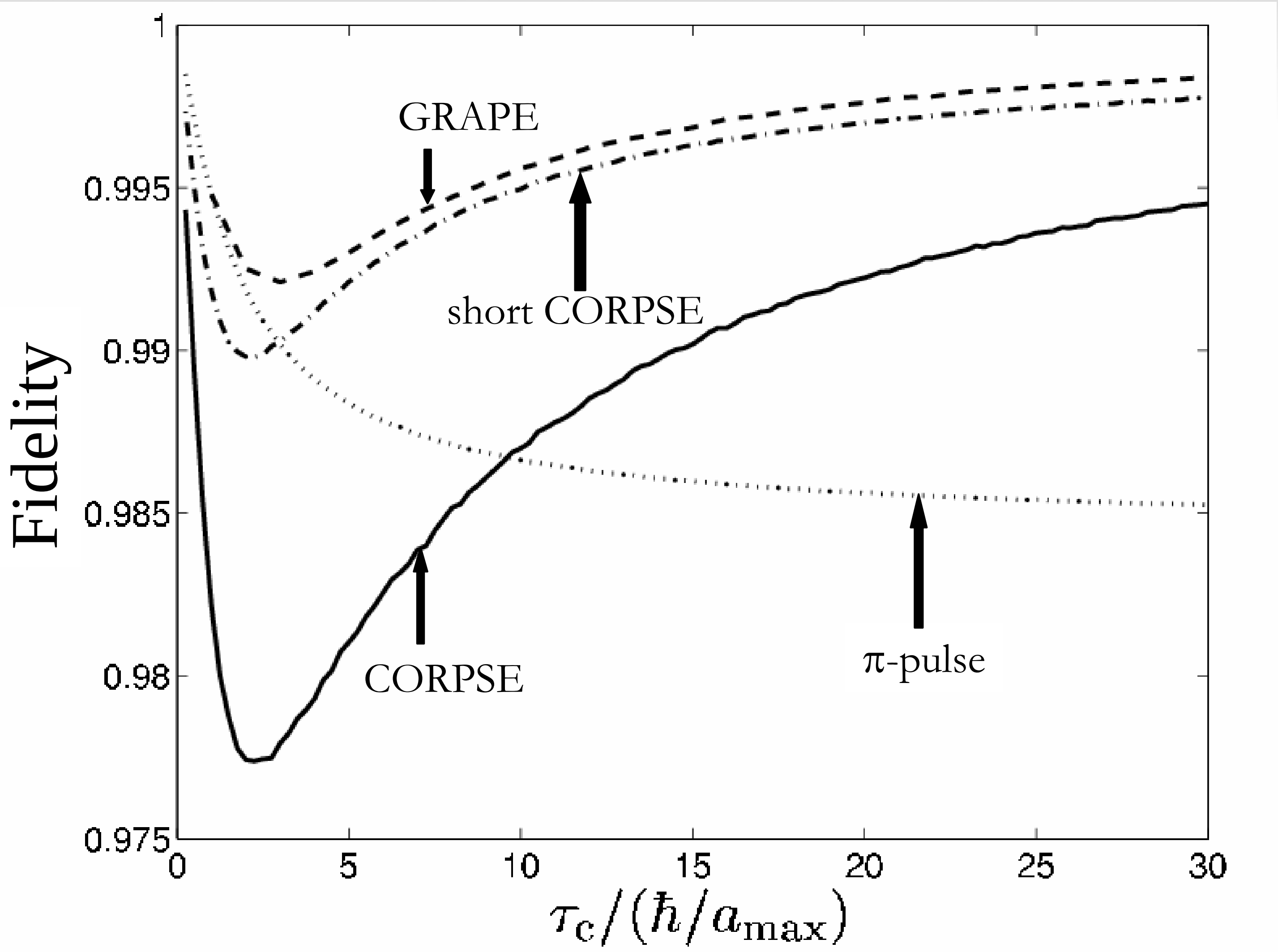
$$U_{\text{CORPSE}} = \mathrm{e}^{i\left(+\frac{7\pi}{6}\sigma_x+\eta\sigma_z\right)}\mathrm{e}^{i\left(-\frac{5\pi}{6}\sigma_x+\eta\sigma_z\right)}\mathrm{e}^{i\left(+\frac{\pi}{6}\sigma_x+\eta\sigma_z\right)} \Rightarrow \text{Fidelity} \approx 1-0.065\left(\frac{\eta}{a_{\max}}\right)^4$$

$$U_{\text{Short CORPSE}} = \mathrm{e}^{i\left(-\frac{\pi}{6}\sigma_x+\eta\sigma_z\right)}\mathrm{e}^{i\left(+\frac{5\pi}{6}\sigma_x+\eta\sigma_z\right)}\mathrm{e}^{i\left(-\frac{\pi}{6}\sigma_x+\eta\sigma_z\right)} \Rightarrow \text{Fidelity} \approx 1-2.7\left(\frac{\eta}{a_{\max}}\right)^4$$

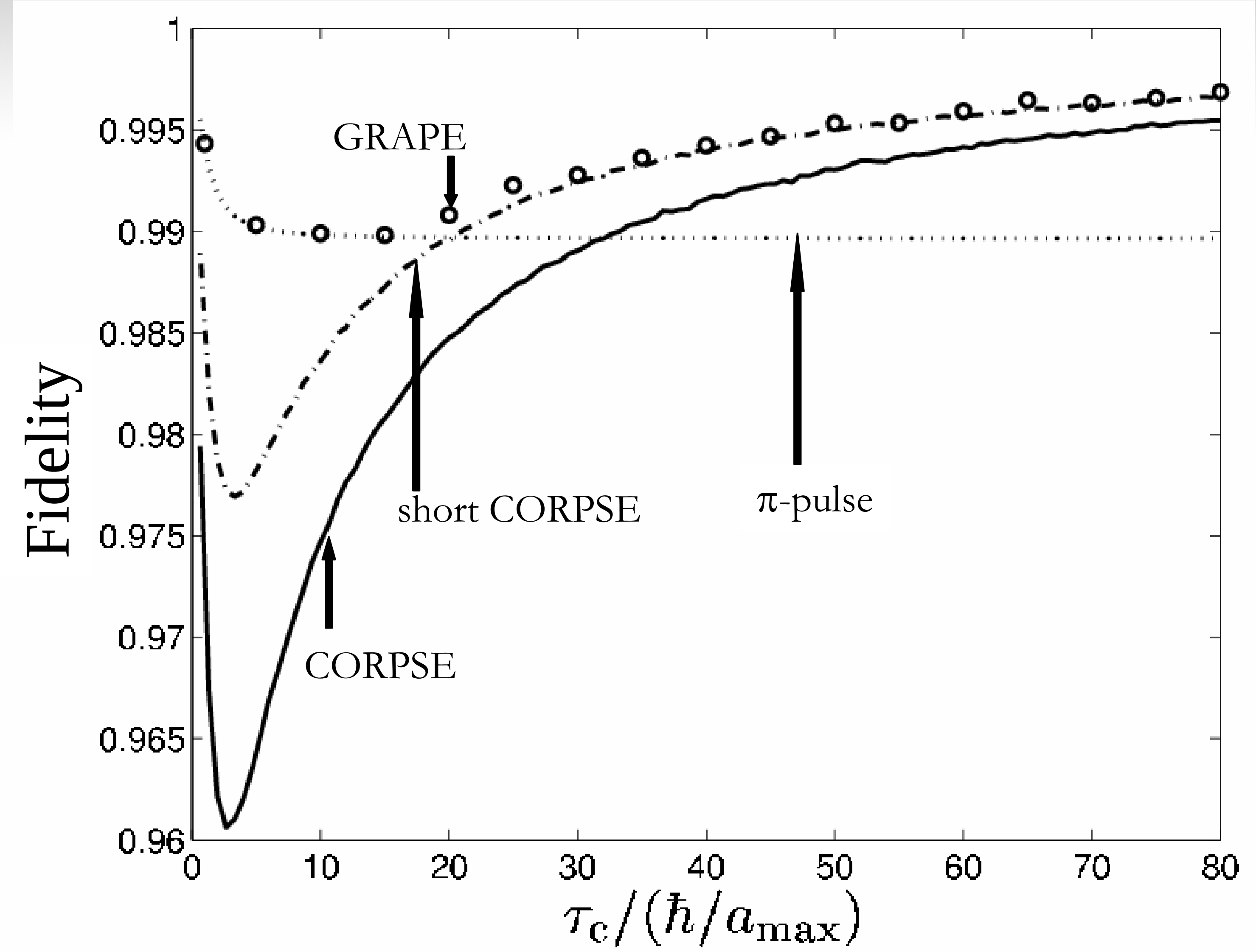
Large number of composite pulses ⇒ Numerical optimization (Gradient Ascent Pulse Engineering)

N. Khaneja et al, J. Magn. Reson. **172**, 296 (2005)

Fidelity vs noise correlation time for the state transformation |0> → |1>



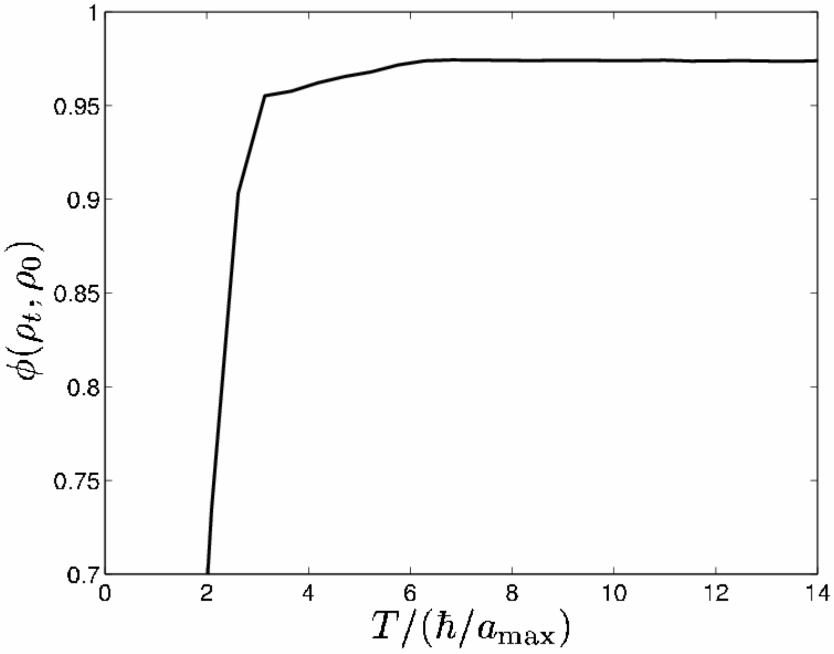
Fidelity vs noise correlation time for NOT gate



Optimized operation times

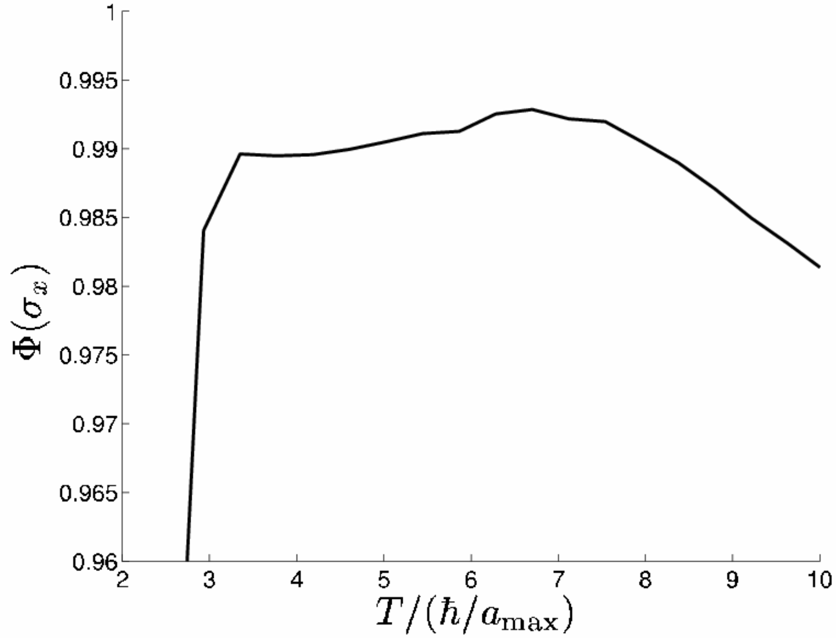
Bit flip |0> → |1>

Δ=0.25 a_{max} $\tau_c/(\hbar/a_{\rm max})$=5



NOT gate

Δ=0.125 a_{max} $\tau_c/(\hbar/a_{\rm max})$ =30



OVERALL SUMMARY:

- High fidelity quantum gate operations from Hamiltonians
- Weyl chamber steering for 2-qubit (non-local) gates
- Algebraic decoupling for 1-qubit (local) gates in presence of untunable interactions
- Efficiency issues – implementation specific
- Decoherence suppression using bounded controls for broad range of noise correlation times
- current/future: combined methodologies for optimal feasible quantum control tailored to specific qubit systems