KPZ and Airy limits of Hall-Littlewood random plane partitions

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Overview

Physical system/ combinatorial object

Plane partitions π.

Probabilistic model

$\mathbb{P}_{HL}^{r,t}(\pi) \propto r^{|\pi|} A_\pi(t)$ - introduced by [Vuletić '09].

Algebraic framework and observables

Macdonald difference operators [Borodin-Corwin '14] and [Borodin-Corwin-Gorin-Shakirov '14]. Formulas for t-Laplace transform.

Asymptotic analysis

Saddle point method. GUE Tracy-Widom and KPZ fluctuations.

Contents

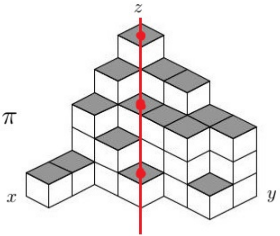
Plane partitions

$$\pi = (\pi_{i,j}),\ i,j \in \mathbb{N}, \pi_{i,j} \geq \max(0, \pi_{i,j+1}, \pi_{i+1,j}), |\pi| = \sum_{i,j} \pi_{i,j} < \infty.$$

$$\lambda(t) = \lambda^t = (\pi_{i,i+t}) \ \text{ for } i \geq \max(0,-t)\,(t \in \mathbb{Z})\,. \text{ Important quantity: } \lambda_1'(t).$$

$$\pi = \cdots \prec \lambda^{-2} \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \lambda^2 \succ \cdots, \quad .$$

5	4	4	3	3
4	3	3	3	1
3	2	1		
1				
1				



$\lambda^0 = \lambda(0) = (5,3,1)$

Connected components

5	5	4	3	3	3
4	4	3	3	3	3
4	3	3	3	3	3
3	3	3	2	2	1
3	3	3	2	2	
1	1	1			

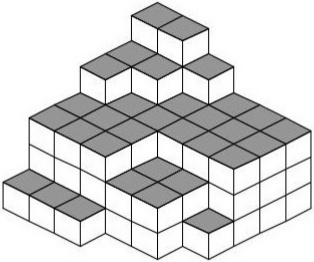


Figure: A plane partition and its 3-d Young diagram. In this example the number of components $k(\pi) = 7$.

Levels and border components

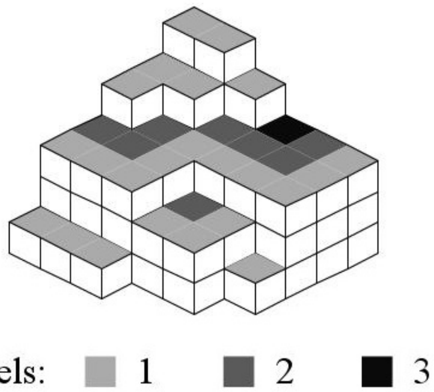
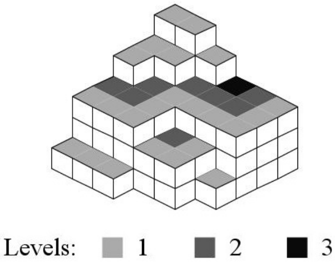


Figure: Border components and their levels.

The polynomials $A_\pi(t)$



For each connected component C define the sequence $(n_1, n_2, \ldots)$, where n_i is the number of i-level border components of C.

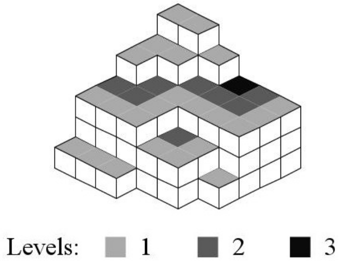
$$P_C(t) := \prod_{i \geq 1} (1 - t^i)^{n_i}.$$

Let $C_1, C_2, \ldots C_{k(\pi)}$ be the connected components of π.

$$A_\pi(t) := \prod_{i=1}^{k(\pi)} P_{C_i}(t).$$

For the example
$A_\pi(t) = (1-t)^7(1-t^2)^3(1-t^3)$.

The measure $\mathbb{P}^{r,t}_{HL}$



$r \in (0,1)$ and $t \in (-1,1)$
Define $\mathbb{P}^{r,t}_{HL}$ so that

$$\mathbb{P}^{r,t}_{HL}(\pi) \propto r^{|\pi|} A_\pi(t).$$

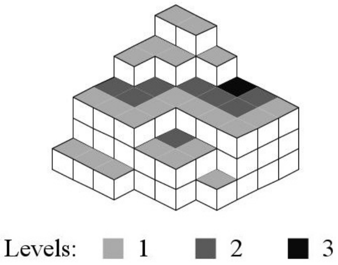
$\mathbb{P}^{r,t}_{HL}$ first considered in [Vuletić '09].

$$\sum_\pi r^{|\pi|} A_\pi(t) = \prod_{n=1}^{\infty} \left(\frac{1 - tr^n}{1 - r^n} \right)^n.$$

Setting $Z(r,t) := \prod_{n=1}^{\infty} \left(\frac{1-tr^n}{1-r^n} \right)^n$, we have

$$\mathbb{P}^{r,t}_{HL}(\pi) := Z(r,t)^{-1} r^{|\pi|} A_\pi(t).$$

The measure $\mathbb{P}_{HL}^{r,t}$, when $t = 0$ and $t = -1$



$$\mathbb{P}_{HL}^{r,t}(\pi) = Z(r,t)^{-1} r^{|\pi|} A_\pi(t).$$

- $t = 0$: $A_\pi(t) = 1$
- $t = -1$: $A_\pi(t) = 0$ or $2^{k(\pi)}$

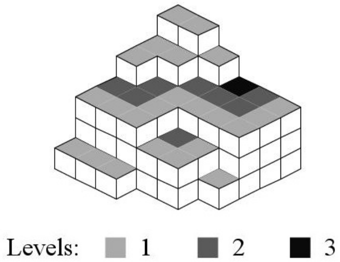
Case $t = 0$. Typical size $\sim \frac{1}{1-r} = N(r)$.

1. Schur process [Okounkov-Reshetikhin '03].
2. Limit shape as $r \to 1^-$ [Cerf-Kenyon '01].
3. Fluctuations converge to the Airy process [Ferrari-Spohn '03].

Case $t = -1$ stidied by [Vuletić '07].

1. Measure supported on *strict plane partitions*.
2. Shifted Schur process.
3. Limiting correlation kernel, strong evidence for limit shape.
4. Edge fluctuations: unknown.

The measure $\mathbb{P}^{r,t}_{HL}$, when $t \in (0,1)$



$$\mathbb{P}^{r,t}_{HL}(\pi) = Z(r,t)^{-1} r^{|\pi|} A_\pi(t).$$

Interested in the case $t \in (0,1)$. Consider two limiting regimes as $r \to 1^-$.

Case $t \in (0,1)$ - fixed, while $r \to 1^-$.

1. Volume and hence limit shape should change.
2. Limit of bottom slice behaves as if $t = 0$.
3. Fluctuations converge to Tracy-Widom.

Case $t, r \to 1^-$.

1. Different limit is expected.
2. Limit shape stays the same.
3. Fluctuations converge to the Hopf-Cole solution of the KPZ.

Limit shape

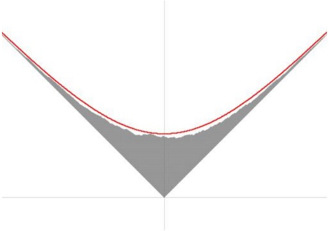


Figure: $t = 0$.

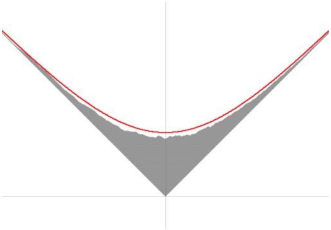


Figure: $t = 0.2$.

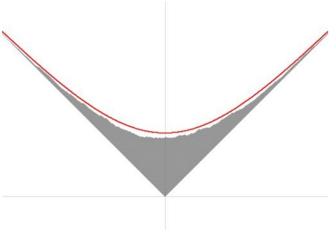


Figure: $t = 0.4$.

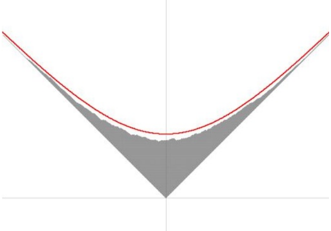


Figure: $t = 0.6$.

Convergence to Tracy-Widom

$$N(r) := \frac{1}{1-r} \quad , \alpha(\tau) := \left[\frac{4}{\cosh^2(\tau/4)}\right]^{-1/3} \quad r \in (0,1) \text{ and } \tau \in \mathbb{R}.$$

Theorem

Consider the measure $\mathbb{P}_{HL}^{r,t}$ on plane partitions, with $t \in (0,1)$ fixed. Then for all $\tau \in \mathbb{R}/\{0\}$ and $x \in \mathbb{R}$ we have

$$\lim_{r \to 1^-} \mathbb{P}_{HL}^{r,t}\left(\frac{\lambda_1'(\lfloor \tau N(r) \rfloor) - 2N(r)\log(1+e^{-|\tau|/2})}{\alpha(\tau)^{-1} N(r)^{1/3}} \leq x\right) = F_{GUE}(x).$$

F_{GUE} is the GUE Tracy-Widom distribution [Tracy-Widom '94].

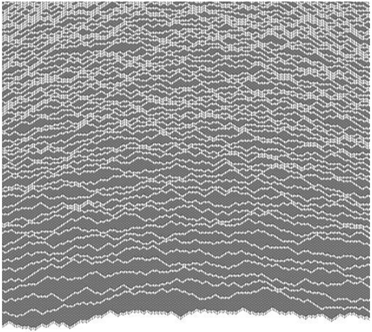


Figure: $t = 0.4$

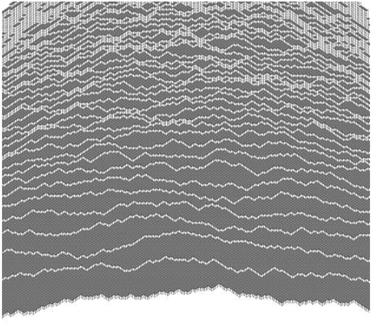


Figure: $t = 0.8$

Convergence to KPZ

$$N(r) := \frac{1}{1-r}\ , \alpha(\tau) := \left[\frac{4}{\cosh^2(\tau/4)}\right]^{-1/3} \quad r \in (0,1) \text{ and } \tau \in \mathbb{R}.$$

$$\xi_r := (-\log t)\left(\lambda_1'(\lfloor \tau N(r) \rfloor) - 2N(r)\log(1+e^{-|\tau|/2})\right) - \log(1-t)$$

Theorem

Suppose $\kappa > 0$, $T = 2\kappa^3\alpha^{-3}$ are fixed and $\frac{-\log t}{(1-r)^{1/3}} = \kappa$. Then for all $\tau \in \mathbb{R}/\{0\}$ and $x \in \mathbb{R}$ we have

$$\lim_{r\to 1^-} \mathbb{P}_{HL}^{r,t}(\xi_r \leq x) = \mathbb{P}\left(\mathcal{F}(T,0) + T/4! \leq x\right).$$

$\mathcal{F}(T,X)$ is the Hopf-Cole solution to the Kardar-Parisi-Zhang equation with initial data $\log \mathcal{Z}_0(X)$ and $\mathcal{Z}_0(X) = \mathbf{1}_{\{X=0\}}$.

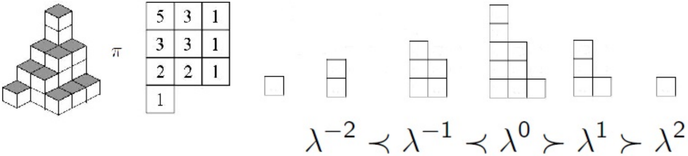
$\mathbb{P}_{HL}^{r,t}$: special case $t = 0$ (part 1)

Fix $r \in (0,1)$ and set $t = 0$. Then we have $\mathbb{P}_{HL}^{r,t}(\pi) \propto r^{|\pi|}$.
Key observation: above can be realized as a *Schur process* [Okounkov-Reshetikhin '03].
Let $(\lambda, \mu) = (\lambda^k, \mu^k)$ be sequences of partitions. Define their weight as

$$\mathcal{W}(\lambda, \mu) := \prod_{k \in \mathbb{Z}} S_{\lambda^k/\mu^{k-1}}(x_k) S_{\lambda^k/\mu^k}(y_k).$$

$S_{\lambda/\mu}$ are skew Schur polynomials in one variable, which satisfy $S_{\lambda/\mu}(x) = \mathbf{1}_{\lambda \succ \mu} x^{|\lambda|-|\mu|}$ $(0^0 = 1)$.
Assuming $x_i, y_i \in [0,1)$ and $\Pi(X;Y) = \sum_{\lambda,\mu} \mathcal{W}(\lambda,\mu) \in (0,\infty)$, we have that $\mathbb{P}_{X,Y}(\lambda,\mu) = \frac{\mathcal{W}(\lambda,\mu)}{\Pi(X;Y)}$ defines a probability measure.
Remark: Above is a special case of Macdonald processes.
Recall: $\pi = \cdots \prec \lambda^{-2} \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \lambda^2 \succ \cdots$, .

$\mathbb{P}_{HL}^{r,t}$: special case $t = 0$ (part 2)



Idea is to pick x_k, y_k suitably so that the projection of the law $\mathbb{P}_{X,Y}$ on λ^k matches the measure of π, i.e. $\propto r^{|\pi|}$.

$$x_n = r^{-n-1/2}, y_n = 0 \text{ if } n \le -1; y_n = r^{n+1/2}, x_n = 0 \text{ if } 0 \le n.$$

One can easily check that the above construction satisfies

- $\mu^n = \lambda^n$ for $n < 0$ and $\mu^n = \lambda^{n+1}$ for $n \geq 0$,
- $\cdots \prec \lambda^{-2} \prec \lambda^{-1} \prec \lambda^{0} \succ \lambda^{1} \succ \lambda^{2} \succ \cdots$, $\lambda^t = \varnothing$ $|t| >> 1$,
- $\mathbb{P}_{X,Y}(\lambda) \propto r^{\sum_k |\lambda^k|} = r^{|\pi|}$,

Important: $\mathbb{P}_{X,Y}(\lambda^k = \lambda) \propto S_\lambda(a, ar, ar^2, \cdots) S_\lambda(a, ar, ar^2, \cdots)$, where $a = r^{(1+|k|)/2}$.

$\mathbb{P}_{HL}^{r,t}$: general case $t \in (0,1)$ (part 1)

Fix $t \in (0,1)$. Replace Schur with Hall-Littlewood polynomials:

$$\mathcal{W}(\lambda, \mu) := \prod_{k \in \mathbb{Z}} P_{\lambda^k/\mu^{k-1}}(x_k; t) Q_{\lambda^k/\mu^k}(y_k; t).$$

Specialize variables in the same way as before. Changes:

$$S_{\lambda/\mu}(x) = \mathbf{1}_{\lambda \succ \mu} x^{|\lambda|-|\mu|} \to \mathbf{1}_{\lambda \succ \mu} x^{|\lambda|-|\mu|} \psi_{\lambda/\mu}(t) = P_{\lambda/\mu}(x; t).$$

$$S_{\lambda/\mu}(y) = \mathbf{1}_{\lambda \succ \mu} y^{|\lambda|-|\mu|} \to \mathbf{1}_{\lambda \succ \mu} y^{|\lambda|-|\mu|} \phi_{\lambda/\mu}(t) = Q_{\lambda/\mu}(y; t).$$

$\phi_{\lambda/\mu}(t)$ and $\psi_{\lambda/\mu}(t)$ are explicit (but technical) integer polynomials in t.

$$\phi_{\lambda/\mu}(t) = \prod_{i \in I} (1 - t^{m_i(\lambda)}) \quad \text{and} \quad \psi_{\lambda/\mu}(t) = \prod_{j \in J} (1 - t^{m_j(\mu)}).$$

if $\lambda \succ \mu$ otherwise both expressions equal 0. $m_i(\lambda) = |\{\lambda_j = i\}|$, $I = \{i \in \mathbb{N} : \lambda'_{i+1} = \mu'_{i+1} \text{ and } \lambda'_i > \mu'_i\}$ and $J = \{j \in \mathbb{N} : \lambda'_{j+1} > \mu'_{j+1} \text{ and } \lambda'_j = \mu'_j\}$.

$\mathbb{P}^{r,t}_{HL}$: general case $t \in (0,1)$ (part 2)

As before the projection of the measure on λ is supported on

- $\cdots \prec \lambda^{-2} \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \lambda^2 \succ \cdots$.
- $\lambda^t = \varnothing$ for all $|t| >> 1$.

The obtained measure: $\mathbb{P}_{X,Y;t}(\pi) \propto B_\pi(t) r^{|\pi|}$.
What is remarkable is that $B_\pi(t) = A_\pi(t)$ [Vuletić '09].
An important byproduct:

$$\mathbb{P}^{r,t}_{HL}(\lambda^k = \lambda) \propto P_\lambda(a, ar, ar^2, \cdots; t) Q_\lambda(a, ar, ar^2, \cdots; t),$$

where $a(k) = r^{1+|k|/2}$.
Strategy is to study the measure
$\mathbb{P}(\lambda) \propto P_\lambda(x_1, \cdots, x_N; t) Q_\lambda(y_1, \cdots, y_N; t)$ then specialize x_i, y_i and send $N \to \infty$.

General philosophy

We have the *Cauchy identity*

$$\sum_{\lambda \in \mathbb{Y}} P_\lambda(X) Q_\lambda(Y) = \Pi(X;Y) = \prod_{i,j=1}^{N} \frac{1 - t x_i y_j}{1 - x_i y_j}.$$

1. Find an operator D, diagonalized by the Hall-Littlewood polynomials $P_\lambda(X)$. I.e. $D P_\lambda(X) = d_\lambda P_\lambda(X)$.
2. Apply it k times to both sides of the Cauchy identity to get

$$\sum_{\lambda} d_\lambda^k P_\lambda(X) Q_\lambda(Y) = D^k \Pi(X;Y).$$

3. Divide both sides by $\Pi(X;Y)$ and get

$$\mathbb{E}_{X,Y}[d_\lambda^k] = \frac{D^k \Pi(X;Y)}{\Pi(X;Y)}.$$

Hall-Littlewood difference operator

Fix $N \in \mathbb{N}$ and consider the space of functions in N variables.

Definition

For $t \in \mathbb{R}$ set $D_N^1 := \sum_{i=1}^N \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{0,x_i}$, where $(T_{0,x_i} F)(x_1, ..., x_N) = F(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_N)$.

Key fact [Macdonald '95]:
$D_N^1 P_\lambda(x_1, ..., x_N; t) = \left(\frac{1 - t^{N - \lambda_1'}}{1 - t} \right) P_\lambda(x_1, ..., x_N; t)$.
Remark: This operator is a special case of the Macdonald difference operators.
Set $\mathcal{D}_N := \left[\frac{(t-1) D_N^1 + 1}{t^N} \right]$. Then we have

$$\mathcal{D}_N P_\lambda(x_1, ..., x_N; t) = t^{-\lambda_1'} P_\lambda(x_1, ..., x_N; t).$$

Observables (part 1)

$$\mathbb{E}_{X,Y}[d_\lambda^k] = \frac{\mathcal{D}_N^k \Pi(X;Y)}{\Pi(X;Y)},\ d_\lambda = t^{-\lambda_1'},\ \Pi(X;Y) = \prod_{i,j=1}^{N} \frac{1-tx_iy_j}{1-x_iy_j}.$$

Proposition

Let $x_i, y_i \in (0,1)$ and $y_i < \epsilon <<_k 1$ for all i. Then we have

$$\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right] = \frac{1}{(2\pi\iota)^k}\int_{C_{0,1}}\cdots\int_{C_{0,k}} \prod_{1\le a<b\le k} \frac{z_a-z_b}{z_a-z_bt^{-1}} \prod_{i=1}^{k}\left[\prod_{j=1}^{N}\frac{(z_i-x_jt^{-1})(1-z_iy_j)}{(z_i-x_j)(1-tz_iy_j)}\right]\frac{dz_i}{z_i},$$

where $C_{0,a}$ are positively oriented contours encircling $x_1,\ldots,x_N$ and 0 and contained in a circle of radius ϵ^{-1} around 0. In addition, $C_{0,a}$ contains $t^{-1}C_{0,b}$ for $a<b$.

Strategy: show by induction on k that $\mathcal{D}_N^k\Pi(X;Y) =$

$$\frac{1}{(2\pi\iota)^k}\int_{C_{0,1}}\cdots\int_{C_{0,k}} \prod_{1\le a<b\le k} \frac{z_a-z_b}{z_a-z_bt^{-1}}\Pi(X;Y) \prod_{i=1}^{k}\left[\prod_{j=1}^{N}\frac{(z_i-x_jt^{-1})(1-z_iy_j)}{(z_i-x_j)(1-tz_iy_j)}\right]\frac{dz_i}{z_i}.$$

Proof (sketch): similar to [Borodin-Corwin '14]

Write RHS as $\frac{1}{(2\pi\iota)^k}\int_{C_{0,1}}\cdots\int_{C_{0,k}}\prod_{1\le a<b\le k}\frac{z_a-z_b}{z_a-z_bt^{-1}}F_k(z_1,\dots,z_k)$.

1. As function of z_k poles at: tz_i for $i<k$; $t^{-1}y_i^{-1}$; x_i and 0.
2. Only poles at x_i and 0 are contained in $C_{0,k}$ and sum of residues equals $\prod_{1\le a<b\le k-1}\frac{z_a-z_b}{z_a-z_bt^{-1}}\mathcal{D}_NF_{k-1}(z_1,\dots,z_{k-1})$.
3. In particular, we get $\mathcal{D}_N^k\Pi(X;Y)=$ $\frac{1}{(2\pi\iota)^{k-1}}\int_{C_{0,1}}\cdots\int_{C_{0,k-1}}\prod_{1\le a<b\le k-1}\frac{z_a-z_b}{z_a-z_bt^{-1}}\mathcal{D}_NF_{k-1}(z_1,\dots,z_{k-1})$.
4. By linearity of $\mathcal{D}_N$ can switch order of $\mathcal{D}_N$ and integrals and reduce to base case $k=1$.
5. Base case is (again) an application of the Residue Theorem.

Residue-book-keeping formula

- Idea: combine above formulas with different k in some generating series (similarly to $\sum_k \mathbb{E}[X^k]u^k/k! = \mathbb{E}[\exp(uX)]$).
- Problem: contours are distinct and become too large. Allowable y's depend on k.

Following result is proved in [Borodin-Bufetov-Corwin '15 (also appeared earlier in [Borodin-Corwin '14]).

$$\frac{1}{(2\pi\iota)^k}\int_{C_{0,1}} \cdots \int_{C_{0,k}} \prod_{1\le a<b\le k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^{k}\left[\prod_{j=1}^{N} \frac{(z_i - x_j t^{-1})(1-z_i y_j)}{(z_i - x_j)(1-tz_i y_j)}\right]\frac{dz_i}{z_i} =$$

$$(-1)^k t^{\frac{k(k-1)}{2}} k_{t^{-1}}! \sum_{\lambda\vdash k} \frac{(1-t^{-1})^k}{(2\pi\iota)^{l(\lambda)}} \frac{1}{m_1!m_2!\dots} \int_{C_{0,k}} \cdots \int_{C_{0,k}} \det\left[\frac{1}{w_i t^{-\lambda_i} - w_j}\right]_{i,j=1}^{l(\lambda)} \times$$

$$\prod_{j=1}^{l(\lambda)} G(w_j)G(w_j t^{-1})\cdots G(w_j t^{1-\lambda_j})dw_1\cdots dw_{l(\lambda)}.$$

In the above formula $G(w) = \prod_{j=1}^{N} \frac{w - x_j t^{-1}}{w - x_j}\frac{1-y_j w}{1-ty_j w}$.

Observables (part 2)

Proposition

Let $x_i, y_i \in (0,1)$ and C_0 be the positively oriented circle of radius t^{-1} around the origin. Then we have

$$\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right] = (t^{-1}-1)^k k_t! \sum_{\lambda \vdash k} \frac{1}{(2\pi\iota)^{l(\lambda)}} \frac{1}{m_1(\lambda)! m_2(\lambda)! \cdots} \int_{C_0} \cdots \int_{C_0}$$

$$\prod_{j=1}^{l(\lambda)} \prod_{i=1}^{N} \frac{1-x_i(w_j t)^{-1}}{1-x_i(w_j t)^{-1} t^{\lambda_j}} \times \prod_{j=1}^{l(\lambda)} \prod_{i=1}^{N} \frac{1-y_i(w_j t)t^{-\lambda_j}}{1-y_i(w_j t)} \det\left[\frac{1}{w_i t^{-\lambda_i} - w_j}\right]_{i,j=1}^{l(\lambda)} \prod_{i=1}^{\ell(\lambda)} dw_i,$$

where $k_t! = \frac{(1-t)(1-t^2)\cdots(1-t^k)}{(1-t)^k}$.

Advantages: contours are all the same, formula holds for all y_i not just very small. Sum both sides of above proposition as generating series with coefficients $u^k/k_t!$.

Combining observables

Proposition

Let $x_i, y_i \in (0,1)$. Then for $u \notin \mathbb{R}^+$ one has that

$$\mathbb{E}_{X,Y}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right] = \det(I + K_u^N)_{L^2(C_0)}.$$

The contour C_0 is the positively oriented circle of radius t^{-1}, centered at 0, and the operator K_u^N is defined in terms of its integral kernel

$$K_u^N(w;w') = \frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty} ds\Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g^N_{w,w'}(t^s),$$

where we choose the principal branch of the logarithm and

$$g^N_{w,w'}(t^s) = \frac{1}{wt^{-s}-w'}\prod_{j=1}^{N}\frac{(1-x_j(wt)^{-1})(1-y_j(wt)t^{-s})}{(1-x_j(wt)^{-1}t^s)(1-y_j(wt))}.$$

A word on asymptotics

1. Above work gives Fredholm determinant formulas for the t-Laplace transforms of certain random variables.
2. If $t \in (0,1)$ is fixed and we let $r \to 1^-$ the t-Laplace transform converges to an indicator function.
3. If $r, t \to 1^-$ the t-Laplace transform converges to the usual Laplace transform.
4. Can find explicit expressions for the limiting Fredholm determinants and match them with existing formulas for F_{GUE} and the Hopf-Cole solution to the KPZ with narrow wedge initial data.

Thank you!

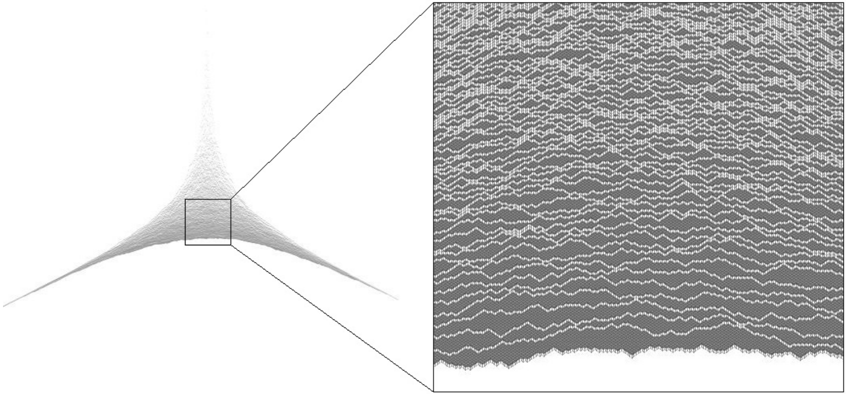


Figure: $t = 0.4$