Multi-species exclusion process and Macdonald polynomials

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Motivation

- Obtain explicit expressions for the stationary state of the multi-species asymmetric simple exclusion process using represention theory and theory of symmetric polynomials.

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- Obtain explicit expressions for the stationary state of the multi-species asymmetric simple exclusion process using represention theory and theory of symmetric polynomials.

- Obtain new explicit expressions for Macdonald polynomials using stochastic processes.

Asymmetric simple exclusion process (ASEP)

ASEP

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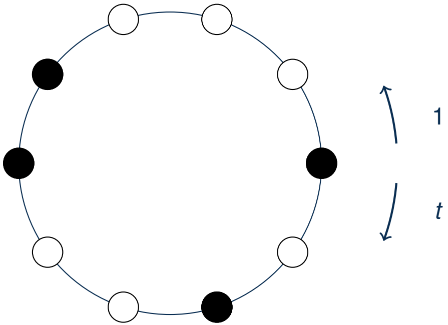
Continuous time Markov chain of hopping particles:

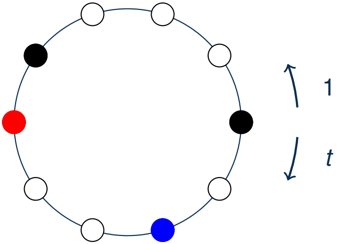


Configurations $\mu = (\mu_1, \ldots, \mu_n)$
$\mu_i \in \{0, 1\}$

Markov chain:
$01 \mapsto 10$ with rate 1
$10 \mapsto 01$ with rate t

Generalise to multi-species process

multi-species ASEP



Configurations $\mu = (\mu_1, \dots, \mu_n)$, $\qquad \mu_i \in \{0, \dots, r\}$

$$\dots \mu_i, \mu_{i+1} \dots \ \mapsto \ \dots \mu_{i+1}, \mu_i \dots \begin{cases} \text{rate } 1 & \text{if } \mu_i < \mu_{i+1} \\ \text{rate } t & \text{if } \mu_i > \mu_{i+1} \end{cases}$$

We will be interested in the stationary state

Transition matrix

Let $|\mu\rangle \in \mathbb{C}^{r+1}$ be the standard basis.

The local transition matrix between $|\dots\mu_i, \mu_{i+1}\dots\rangle$ and $|\dots\mu_{i+1}, \mu_i\dots\rangle$ is given by

$$L_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & t & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The stationary state $|\infty\rangle$ is defined by

$$\sum_{i=1}^{n} L_i\,|\infty\rangle = 0, \qquad |\infty\rangle = \sum_{\mu} f_{\mu_1,\dots,\mu_n}|\mu\rangle.$$

and we would like to know f_μ.

In the case of $r = 1$:

Theorem (Derrida,Evans, Hakim,Pasquier,)

There exist matrices A_0 and A_1 such that

$$f_{\mu_1,\dots,\mu_n} = \mathrm{Tr}\left(A_{\mu_1}\cdots A_{\mu_n}\right)$$

and

$$A_0A_1 - tA_1A_0 = (1-t)(A_0+A_1).$$

Trivial representation ($A_0 = A_1 = 2$) suffices for $r = 1$ periodic boundary conditions.

For general r (Prolhac et al) or open boundaries we need "t-bosons":

$$A_0 = \phi + 1, \qquad A_1 = \phi^\dagger + 1,$$

$$\phi\phi^\dagger - t\phi^\dagger\phi = 1 - t,$$

with infinite "Fock representation"

$$\phi^\dagger|m\rangle = |m+1\rangle, \qquad \phi|m\rangle = (1-t^m)|m-1\rangle.$$

Inhomogeneous generalisation

- The (multi-species) ASEP is a quantum integrable system (Yang-Baxter)
- There exist an integrable discrete time generalisation with spatial inhomogeneities:

Let

$$\begin{aligned} b^+ &= \frac{t(x-y)}{tx-y}, & b^- &= t^{-1}b^+, \\ c^+ &= 1-b^+, & c^- &= 1-b^-. \end{aligned} \tag{1}$$

Then for $r=1$, define a generalised local transition matrix between $|\ldots\mu_i,\mu_{i+1}\ldots\rangle$ and $|\ldots\mu_{i+1},\mu_i\ldots\rangle$ by

$$\check{R}_i(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c^- & b^+ & 0 \\ 0 & b^- & c^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad L_i = \check{R}_i(1,1)^{-1}\check{R}'_i(1,1).$$

Generalised stationary state

The generalised inhomogeneous stationary state $|\infty\rangle$ is now defined by

$$\check{R}_i(x_i, x_{i+1})\,|\infty\rangle = s_i|\infty\rangle, \qquad |\infty\rangle = \sum_{\mu} f_{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n)|\mu\rangle.$$

with quasi-periodic boundary condition

$$f_{\mu_n,\mu_1,\ldots,\mu_{n-1}}(qx_n, x_1,\ldots,x_{n-1};q,t) = q^{\mu_n} f_{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n;q,t).$$

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To solve for f_μ we assume that

$$f_{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n) = \mathrm{Tr}\left(A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)S\right)$$

Macdonald polynomials

Macdonald polynomials

Symmetric group

Let s_i $(i = 1, \ldots, n-1)$ be generators of the symmetric group S_n:

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$
$$s_i^2 = 1;$$

There exist a natural t-deformation of S_n:

$$(T_i - t)(T_i + 1) = 0, \qquad (i = 1, \ldots, n-1),$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

This is the Hecke algebra (of type A_{n-1}) and S_n is recovered when $t \to 1$.

Polynomial action

The generators s_i act naturally on polynomials:

$$s_i f(\dots, x_i, x_{i+1}, \dots) = f(\dots, x_{i+1}, x_i, \dots) \qquad i = 1, \dots n-1$$

and the t-deformation also has an action:

$$T_i = t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i).$$

Define the (non-symmetric) polynomials $f_\mu(x_1, \dots, x_n)$ by these relations:

$$\begin{aligned} T_i f_{\dots,\mu_i,\mu_{i+1},\dots} &= t\, f_{\dots,\mu_i,\mu_{i+1},\dots} \quad && \mu_i = \mu_{i+1}, \\ T_i f_{\dots,\mu_i,\mu_{i+1},\dots} &= f_{\dots,\mu_{i+1},\mu_i,\dots} \quad && \mu_i > \mu_{i+1}, \\ \omega f_{\mu_n,\mu_1,\dots,\mu_{n-1}} &= q^{\mu_n} f_{\mu_1,\dots,\mu_n}. \end{aligned}$$

- Dynamics of the multi-species inhomogeneous ASEP
- t-deformed Knizhnik-Zamolodchikov equations

Macdonald polynomial

Proposition

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_1 \geq \ldots \geq \lambda_n$*. The polynomial* P_λ *defined by*

$$P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma \in S_n}^{*} f_{\sigma \circ \lambda}(x_1, \ldots, x_n; q, t)$$

is symmetric and equal to a Macdonald polynomial.

Macdonald polynomials are (q, t) generalisations of Schur polynomials (characters of the symmetric group).

The form

$$f_\lambda(x_1, \ldots, x_n) = \mathrm{Tr}\left(A_{\lambda_1}(x_1) \cdots A_{\lambda_n}(x_n) S\right),$$

implies a matrix product for Macdonald polynomials which is a completely new way of writing these polynomials

Theorem (Cantini, dG, Wheeler)

$$P_\lambda(x_1,\ldots,x_n;q,t) = \sum_{\mu|\mu^+=\lambda} \mathrm{Tr}\left[S\prod_{i=1}^{n} A_{\mu_i}(x_i)\right],$$

where the sum is over all permutations μ of λ.

Corollary

The normalised stationary state of the multi-species ASEP is given by

$$f_{\mu_1,\ldots,\mu_n} = \frac{1}{P_{\mu^+}} \mathrm{Tr}\left[S\prod_{i=1}^{n} A_{\mu_i}(x_i)\right],$$

specialised to $q = x_1 = \ldots = x_n = 1$.

Explicit construction

For $r = \lambda_1$ write

$$\mathbb{A}(x) = (A_0(x), \ldots, A_r(x))^T,$$

as an $(r+1)$-dimensional operator valued column vector.

Lemma

The exchange relations are equivalent to

$$\check{R}(x,y) \cdot [\mathbb{A}(x) \otimes \mathbb{A}(y)] = [\mathbb{A}(y) \otimes \mathbb{A}(x)]$$

$\check{R}(x,y)$ is the $U_t(sl_{r+1})$ R-matrix of dimension $(r+1)^2$ ($r = 1$ is the 6-vertex model).

Yang-Baxter algebra and Nested Matrix Product Form

More familiar is rank r Yang-Baxter algebra:

$$\check{R}(x, y) \cdot [L(x) \otimes L(y)] = [L(y) \otimes L(x)] \cdot \check{R}(x, y)$$

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Assume a solution of the following modified *RLL* relation

$$\check{R}^{(r)}(x,y) \cdot \left[\tilde{L}(x) \otimes \tilde{L}(y)\right] = \left[\tilde{L}(y) \otimes \tilde{L}(x)\right] \cdot \check{R}^{(r-1)}(x,y)$$

in terms of an $(r+1) \times r$ operator-valued matrix $\tilde{L}(x) = \tilde{L}^{(r)}(x)$.

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in terms of an $(r+1)\times r$ operator-valued matrix $\tilde{L}(x)=\tilde{L}^{(r)}(x)$.

Then

$$\mathbb{A}^{(r)}(x) = \tilde{L}^{(r)}(x)\cdot\tilde{L}^{(r-1)}(x)\cdots\tilde{L}^{(1)}(x)$$

Solves the algebra

$$\check{R}(x,y)\cdot[\mathbb{A}(x)\otimes\mathbb{A}(y)] = [\mathbb{A}(y)\otimes\mathbb{A}(x)]$$

Zipper proof

$$\check{R}^{(r)}(x,y)\cdot\left[\tilde{L}^{(r)}(x)\otimes\tilde{L}^{(r)}(y)\right]\cdot\left[\tilde{L}^{(r-1)}(x)\otimes\tilde{L}^{(r-1)}(y)\right]$$
$$=\left[\tilde{L}^{(r)}(y)\otimes\tilde{L}^{(r)}(x)\right]\cdot\check{R}^{(r-1)}(x,y)\cdot\left[\tilde{L}^{(r-1)}(x)\otimes\tilde{L}^{(r-1)}(y)\right]$$
$$=\left[\tilde{L}^{(r)}(y)\otimes\tilde{L}^{(r)}(x)\right]\cdot\left[\tilde{L}^{(r-1)}(y)\otimes\tilde{L}^{(r-1)}(x)\right]\cdot\check{R}^{(r-2)}(x,y)$$

Rank 1 solution

Explicitly

$$\check{R}^{(r)}(x,y)\cdot\left[\tilde{L}(x)\otimes\tilde{L}(y)\right]=\left[\tilde{L}(y)\otimes\tilde{L}(x)\right]\cdot\check{R}^{(r-1)}(x,y)$$

for $r=1$ is given by

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & c^- & b^+ & 0 \\ \hline 0 & b^- & c^+ & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\cdot\left[\begin{pmatrix} 1 \\ x \end{pmatrix}\otimes\begin{pmatrix} 1 \\ y \end{pmatrix}\right]=\left[\begin{pmatrix} 1 \\ y \end{pmatrix}\otimes\begin{pmatrix} 1 \\ x \end{pmatrix}\right].$$

Rank 2 solution

$$\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^- & 0 & b^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^- & 0 & 0 & 0 & b^+ & 0 & 0 \\ \hline 0 & b^- & 0 & c^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^- & 0 & b^+ & 0 \\ \hline 0 & 0 & b^- & 0 & 0 & 0 & c^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^- & 0 & c^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \cdot \left[\begin{pmatrix} 1 & \phi^\dagger \\ xk & 0 \\ x\phi & x \end{pmatrix} \otimes \begin{pmatrix} 1 & \phi^\dagger \\ yk & 0 \\ y\phi & y \end{pmatrix}\right] =$$

$$\left[\begin{pmatrix} 1 & \phi^\dagger \\ yk & 0 \\ y\phi & y \end{pmatrix} \otimes \begin{pmatrix} 1 & \phi^\dagger \\ xk & 0 \\ x\phi & x \end{pmatrix}\right] \cdot \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & c^- & b^+ & 0 \\ \hline 0 & b^- & c^+ & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

We construct a solution for $\mathbb{A}$ in the following way:

$$\mathbb{A}(x) = \tilde{L}^{(2)}(x) \cdot \tilde{L}^{(1)}(x) = \begin{pmatrix} 1 & \phi^\dagger \\ xk & 0 \\ x\phi & x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 + x\phi^\dagger \\ kx \\ x\phi + x^2 \end{pmatrix}.$$

Example:

$$f_{001122}(x_1, \ldots, x_6; q = t^u, t) = \mathrm{Tr}\left[A_0(x_1)A_0(x_2)A_1(x_3)A_1(x_4)A_2(x_5)A_2(x_6)S\right],$$

$$\begin{aligned} A_0(x) &= 1 + x\phi^\dagger, \\ A_1(x) &= xk, \\ A_2(x) &= x\phi + x^2, \end{aligned}$$

S has the form

$$S = k^u = \mathrm{diag}\{1, t^{-u}, t^{-2u}, \ldots\} = \mathrm{diag}\{1, q^{-1}, q^{-2}, \ldots\}.$$

Example

$$f_{001122}(x_1,\ldots,x_6;q=t^u,t) =$$
$$\mathrm{Tr}\left[\left(1+x_1\phi^\dagger\right)\left(1+x_2\phi^\dagger\right)x_3kx_4kx_5\left(\phi+x_5\right)x_6\left(\phi+x_6\right)S\right]$$
$$= x_3x_4x_5x_6\,\mathrm{Tr}\left[\left(x_5x_6k^2+(x_1+x_2)(x_5+x_6)\phi^\dagger k^2\phi + x_1x_2(\phi^\dagger)^2k^2\phi^2\right)S\right],$$

where other terms involving unequal powers of $\phi^\dagger$ and a have zero trace.

Normalising with $\mathrm{Tr}(k^2S)$ we finally get

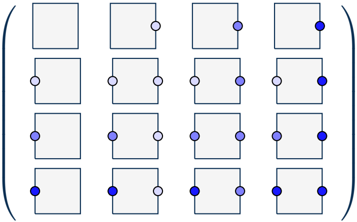
$$f_{001122}(x_1,\ldots,x_6;q=t^u,t) = x_3x_4x_5^2x_6^2$$
$$+ x_3x_4x_5x_6(x_1+x_2)(x_5+x_6)t^2\frac{\mathrm{Tr}\,\phi^\dagger\phi k^2S}{\mathrm{Tr}\,k^2S} + x_1x_2x_3x_4x_5x_6t^4\frac{\mathrm{Tr}(\phi^\dagger)^2\phi^2k^2S}{\mathrm{Tr}\,k^2S}$$

General construction and sum rules

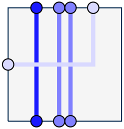
General construction

Starting from RLL=LLR

$L^{(3)}(x) =$



$$= \begin{pmatrix} 1 & \phi_1^\dagger & \phi_2^\dagger & \phi_3^\dagger \\ xk_3k_2\phi_1 & xk_3k_2 & 0 & 0 \\ xk_3\phi_2 & xk_3\phi_2\phi_1^\dagger & xk_3 & 0 \\ x\phi_3 & x\phi_3\phi_1^\dagger & x\phi_3\phi_2^\dagger & x \end{pmatrix}$$

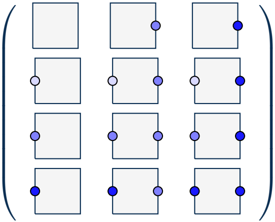


corresponds with $L^{(3)}_{1,0} = k_3k_2\phi_1$,

Trivialising ϕ_1

$$\phi_1 = \phi_1^\dagger = 1, \qquad k_1 = 0.$$

$\tilde{L}^{(3)}(x) =$

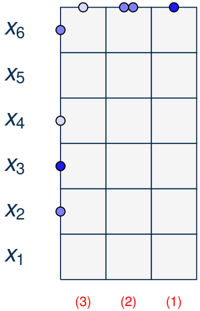


$$= \begin{pmatrix} 1 & \phi_2^\dagger & \phi_3^\dagger \\ xk_3k_2 & 0 & 0 \\ xk_3\phi_2 & xk_3 & 0 \\ x\phi_3 & x\phi_3\phi_2^\dagger & x \end{pmatrix}.$$

Combinatorial rule

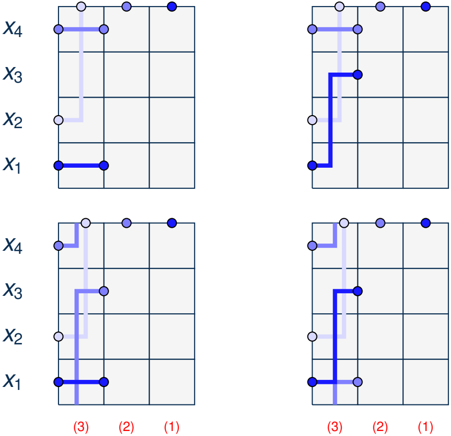
For $r = 3$ and $\lambda = (0, 2, 3, 1, 0, 2)$, the matrix product can be represented in the following way:

$$\mathrm{Tr}(A_0(x_1)A_2(x_2)A_3(x_3)A_1(x_4)A_0(x_5)A_2(x_6)S) =$$



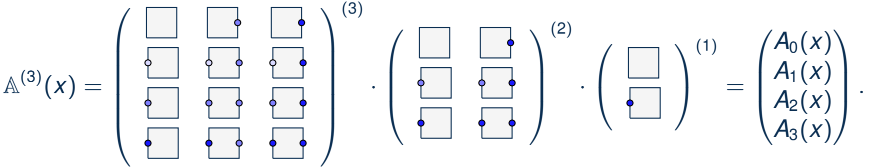
Column by column transition

With $\lambda = (3, 1, 0, 2)$. We obtain the following four terms:

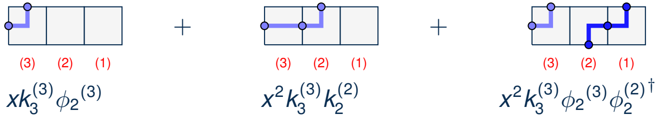


Solution for rank 3

$$\mathbb{A}^{(3)}(x) = \begin{pmatrix} 1 & \phi_2^\dagger & \phi_3^\dagger \\ xk_3k_2 & 0 & 0 \\ xk_3\phi_2 & xk_3 & 0 \\ x\phi_3 & x\phi_3\phi_2^\dagger & x \end{pmatrix}^{(3)} \cdot \begin{pmatrix} 1 & \phi_2^\dagger \\ xk_2 & 0 \\ x\phi_2 & x \end{pmatrix}^{(2)} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^{(1)} = \begin{pmatrix} A_0(x) \\ A_1(x) \\ A_2(x) \\ A_3(x) \end{pmatrix}.$$



$A_2(x) =$



Summation formula

A corollary is the following new summation formula.

Theorem

Let $\lambda[k]$ be a partition obtained from λ by replacing all parts of size $\leq k$ with 0.

$$P_\lambda(x_1,\ldots,x_n;q,t) = \sum_{\sigma\in S_\lambda} T_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} \left(\sum_{\sigma\in S_{\lambda[i]}} C_i \binom{\lambda[i-1]}{\sigma\circ\lambda[i]} T_\sigma \circ x_{\lambda[i]} \circ \right) 1$$

with coefficients that satisfy $C_i(\lambda,\mu)=0$ if any $0<\lambda_k<\mu_k$, and

$$C_i(\lambda,\mu) \equiv C_i \binom{\lambda_1\cdots\lambda_n}{\mu_1\cdots\mu_n} = \prod_{j=i+1}^{r} \left(q^{(j-i)a_j(\lambda,\mu)} \prod_{k=1}^{b_j(\lambda,\mu)} \frac{1-t^k}{1-q^{j-i}t^{\lambda'_i-\lambda'_j+k}} \right),$$

otherwise.

Specialisations

- Monomial symmetric polynomials ($t = 1$)

$$P_\lambda(x_1,\ldots,x_n;q,1) = \sum_{\sigma\in S_\lambda} s_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} x_{\lambda[i]} = \sum_{\sigma\in S_\lambda} \sigma \circ \left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) = m_\lambda(x_1,\ldots,x_n),$$

- Hall–Littlewood polynomials ($q = 0$)

$$P_\lambda(x_1,\ldots,x_n;t) = \sum_{\sigma\in S_\lambda} T_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} x_{\lambda[i]} = \sum_{\sigma\in S_\lambda} T_\sigma \circ \left(\prod_{i=1}^{n} x_i^{\lambda_i}\right).$$

Specialisations

- q–Whittaker polynomials ($t = 0$)

$$P_\lambda(x_1, \ldots, x_n; q, 0) = \sum_{\sigma \in S_\lambda} D_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} \left(\sum_{\sigma \in S_{\lambda[i]}} C_i \binom{\lambda[i-1]}{\sigma \circ \lambda[i]} D_\sigma \circ x_{\lambda[i]} \circ \right) 1$$

with coefficients that satisfy $C_i(\lambda, \mu) = 0$ if any $0 < \lambda_k < \mu_k$, and $C_i(\lambda, \mu) = \prod_{j=i+1}^{r} q^{(j-i)a_j(\lambda,\mu)}$ otherwise, and where each D_σ is now composed of the divided-difference operators

$$D_i = (x_i/x_{i+1} - 1)^{-1}(1 - s_i), \qquad 1 \le i \le n-1.$$

Conclusion

- Explicit construction of (matrix product) stationary state of a multi-species inhomgeneous exclusion process
- Use Yang-Baxter integrability, representation theory, theory of multi-variable polynomials
- New explicit formulas for Macdonald polynomials using ideas from stochastic processes