KITP 02/16/16

Searching for the Tracy-Widom distribution in nonequilibrium processes

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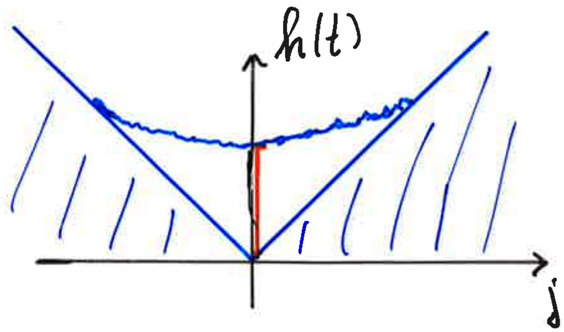
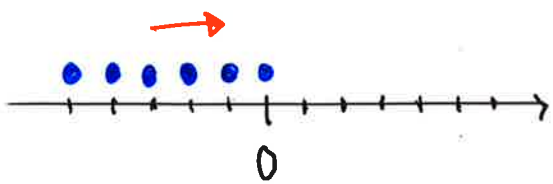
jointly with C. Mendl, Stanford

- TASEP, GUE Tracy-Widom



$$h_0(t) \cong \frac{1}{4} t + t^{1/3} \xi_{TW}$$

↝ stochastic conservation law



time-integrated current $= h_0(t)$

GOAL Tracy-Widom in nonlinear discrete wave equations

wave field $\phi(x,t)$ $\partial_t^2 \phi = \partial_x V'(\partial_x \phi)$

$$H = \int dx \left(\tfrac{1}{2} \pi^2 + V(\partial_x \phi) \right)$$

- random initial data
- assumes deterministic chaos

bridge ‖ stochastic conservation laws ‖

1 D KPZ, several components

- discrete wave eq.

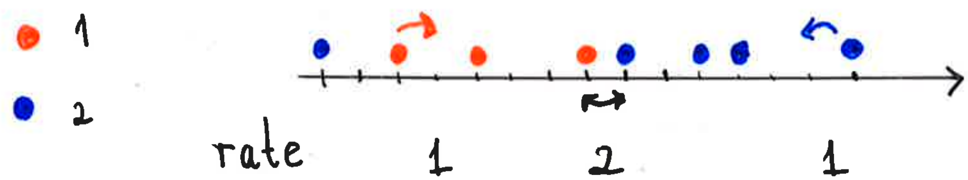
$$\left.\begin{array}{l}\text{stretch}\\ \text{momentum}\\ \text{energy}\end{array}\right\} 3$$

{ step in-between

- Leroux stochastic lattice gas 2

Leroux lattice gas (special case of AHR model)

two-component TASEP



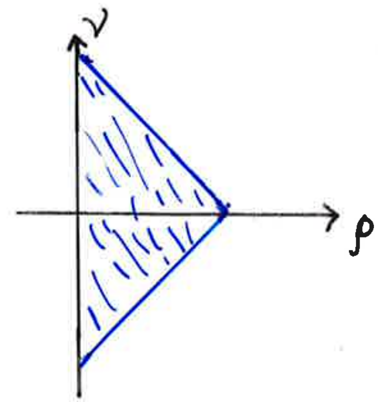
Bernoulli invariant measures

→ Euler equations

densities ρ_{-1} ρ_1

⇝ holes $1-\rho_{-1}-\rho_1$

velocity $v = \rho_1 - \rho_-$



$$\partial_t \rho - \partial_x \rho v = 0$$

$$\partial_t v - \partial_x (\rho + v^2) = 0$$

Fritz, Toth 2002

→ domain wall initial conditions
(step)

$$\rho(x,0) = \begin{cases} \rho_- & x < 0 \\ \rho_+ & x \geq 0 \end{cases}$$

$$v(x,0) = \begin{cases} v_- & x < 0 \\ v_+ & x \geq 0 \end{cases}$$

INSERT Riemann problem for hyperbolic conservation laws

n components

$$\partial_t u + \partial_x j(u) = 0$$

$u \in \mathbb{R}^n$

$$\partial_t u + A(u)\, \partial_x u = 0$$

linearization

initial condition

$$u(x,0) = \begin{cases} u_- & x < 0 \\ u_+ & x \geq 0 \end{cases}$$

solution $u(x,t) = u_{dw}(x/t)$

based on $A(u)\, \psi_\alpha(u) = c_\alpha(u)\, \psi_\alpha(u)$

n vector fields on $\mathbb{R}^n$

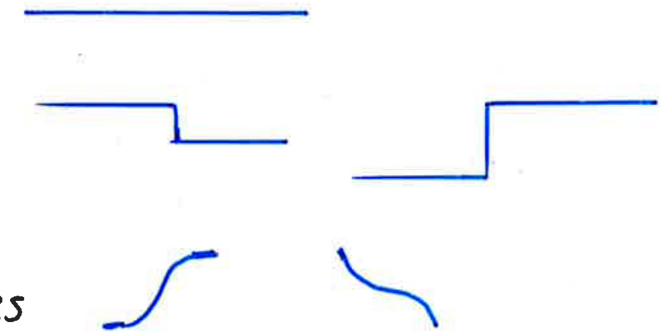
see A. Bressan
Hyperbolic Conservation Laws
Tutorial, Springer 2013

- integral curves $\frac{d}{d\tau} u = \psi_\alpha(u)$
- Rankine - Hugoinot conservation across shocks
- entropy condition (stability)

u_{dw} consists of flat pieces

shocks / contact discontinuities

rarefaction waves



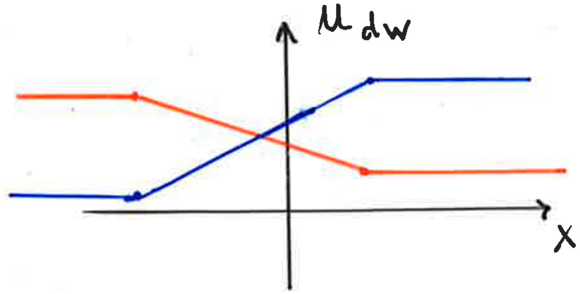
monotone

Leroux lattice gas is in Temple class

(shock and rarefaction waves coincide)

and

linear rarefaction waves

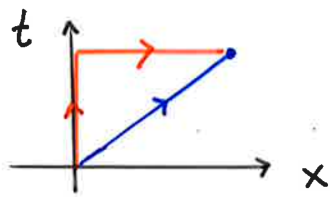


observables — time-integrated current

scalar $\quad \partial_t \eta + \partial_x \zeta = 0 \qquad (-\eta, \zeta)$ curl 0

random

potential Φ



$$\Phi(x,t) = \int_0^t ds\, \zeta(0,s) - \int_0^x dx' \cdot \eta(x',t)$$

in principle $\quad$ fix eigenvalue γ, rarefaction wave,

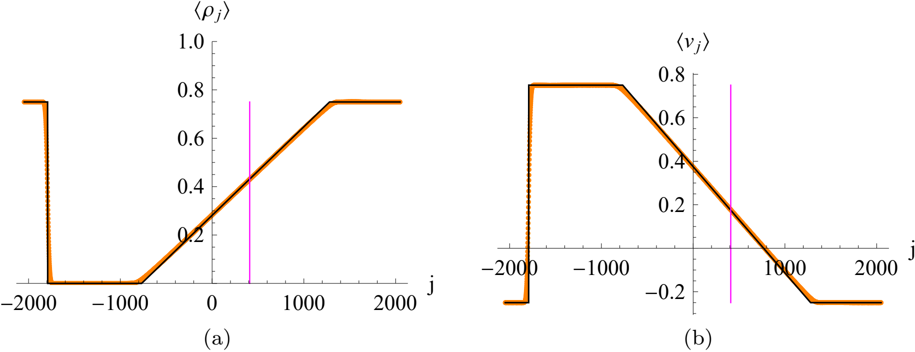
ray $\{x = vt\}$ inside wave, $\quad u_v = u_{dw}(vt,t)$

$\rightsquigarrow$ integrated current $\quad \Phi_\alpha(vt,t) \simeq t\, j_\alpha(u_v) - v\, u_{v\alpha} + \begin{cases} t^{1/3}\, \zeta_{TW} \\ ? \end{cases}$

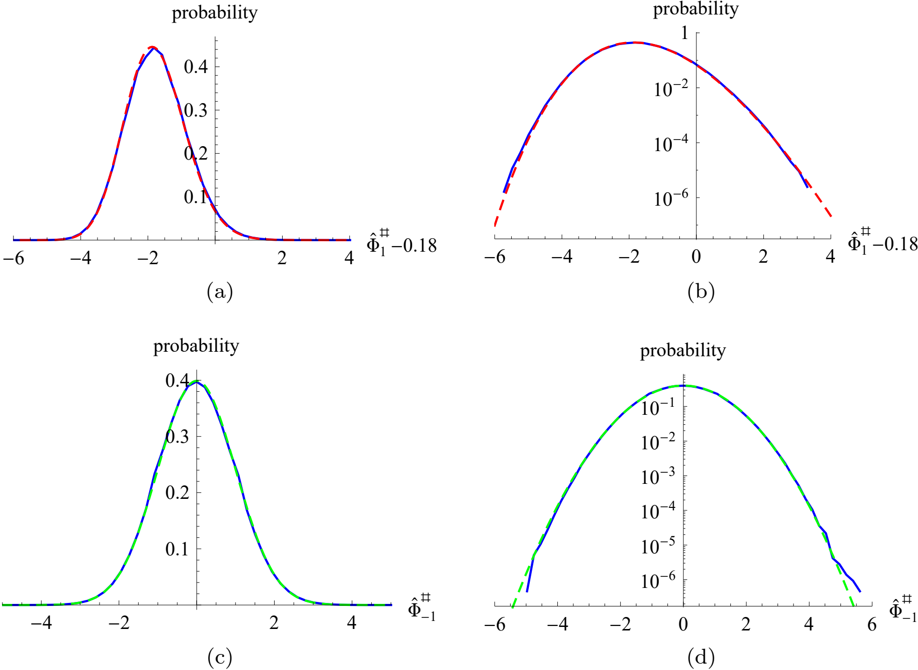
in fact: $\quad \tilde{\psi}_\gamma(u_v) \cdot \Phi(vt,t) \simeq \cdot\, t + \cdot\, t^{1/3}\, \zeta_{TW}$

$\tilde{\psi}_\gamma A = c_\gamma \tilde{\psi}_\gamma$

all other projections $\quad t^{1/2}\, \zeta_G$



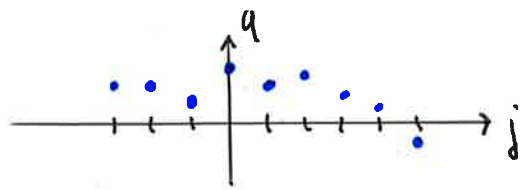
⟨ρ_j⟩
1.0
0.8
0.6
0.4
0.2
−2000
−1000
0
1000
2000
j
(a)
⟨v_j⟩
0.8
0.6
0.4
0.2
−0.2
−2000
−1000
1000
2000
j
(b)


probability
0.4
0.3
0.2
0.1
$\hat{\Phi}_1^{\sharp} - 0.18$
−6
−4
−2
2
4
(a)
probability
1
10^{-2}
10^{-4}
10^{-6}
$\hat{\Phi}_1^{\sharp} - 0.18$
−6
−4
−2
0
2
4
(b)
probability
0.4
0.3
0.2
0.1
$\hat{\Phi}_{-1}^{\sharp}$
−4
−2
2
4
(c)
probability
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\hat{\Phi}_{-1}^{\sharp}$
−6
−4
−2
0
2
4
6
(d)

- nonlinear wave equations

$$H = \sum_j \left\{ \tfrac{1}{2} p_j^2 + V(q_{j+1} - q_j) \right\}$$

lattice field theory



$$\int dx \, e^{-\beta (V(x) + P x)} < \infty$$

$$\beta > 0 \, , \quad P \in I \subset \mathbb{R}$$

$$\dot{q}_j = p_j \, , \quad \dot{p}_j = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1})$$

- stretch $\quad r_j = q_{j+1} - q_j$
- energy $\quad e_j = \tfrac{1}{2} p_j^2 + V(r_j)$

$$\frac{d}{dt} r_j = p_{j+1} - p_j$$

$$\frac{d}{dt} p_j = V'(r_j) - V'(r_{j-1})$$

$$\frac{d}{dt} e_j = p_{j+1} V'(r_j) - p_j V'(r_{j-1})$$

ALL conservation laws

$\Rightarrow$ equilibrium measures, parameters β, momentum ν, pressure P

$$P = - \langle V'(r_0) \rangle$$

(r_j, p_j) i.i.d.

$$\frac{1}{(2\pi\beta)^{1/2}} e^{-\frac{1}{2}\beta (p_j - \nu)^2} \, , \quad \frac{1}{Z} e^{-\beta (V(r_j) + P r_j)}$$

$\Rightarrow$ Euler equations: fields $\vec{u} = (r, v, \bar{e})$ $\leftarrow$ total energy

current $\vec{j} = (-v, P, vP)$ $\qquad P = P(r, \underbrace{\bar{e} - \tfrac{1}{2}v^2}_{\text{internal } e})$

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extensive - intensive

$$r = \frac{1}{Z}\int dx\, x\, e^{-\beta(V(x) + Px)}$$

$$e = \frac{1}{2\beta} + \frac{1}{Z}\int dx\, V(x)\, e^{-\beta(V(x)+Px)}$$

$(P, \beta) \mapsto (r, e)$ Inverse

$P(r,e)$, $\beta(r,e)$

$\Rightarrow$

$$\partial_t \vec{u} + \partial_x \vec{j}(\vec{u}) = 0$$

Riemann problem linearized A, eigenvalues $0, \pm c$

- eigenvalue 0

contact discontinuity

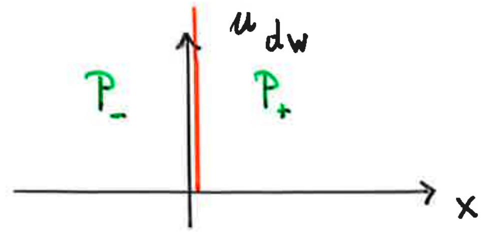
$$\frac{d}{d\tau} x(\tau) = \psi_0(x(\tau))$$

$$\rightsquigarrow \quad \frac{d}{d\tau}\tau = \partial_e P(\tau, e)$$

$$\frac{d}{d\tau} v = 0$$

$$\frac{d}{d\tau} = -\partial_\tau P(\tau, e)$$

Hamiltonian

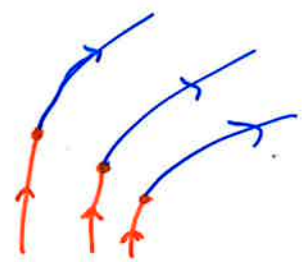


$P_+ = P_-$

- eigenvalues $\pm c$

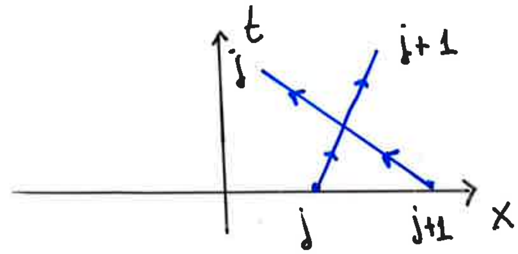
shocks, rarefaction waves

$$\frac{d}{d\tau}\vec{x}(\tau) = \psi_{\sigma c}(\vec{x}(\tau)), \quad \sigma = \pm 1$$



molecular dynamics

$$V(x) = \begin{cases} 0 & x \geq 0 \\ \infty & x < 0 \end{cases}$$



integrable

non-integrable [alternating mass] odd j: m_1 even j: m_0, $k = \frac{m_1}{m_0}$

$$\rightsquigarrow \begin{pmatrix} p_j' \\ p_{j+1}' \end{pmatrix} = \frac{1}{1+k} \begin{pmatrix} 1-k & 2k \\ 2k & 1-k \end{pmatrix} \begin{pmatrix} p_j \\ p_{j+1} \end{pmatrix}$$

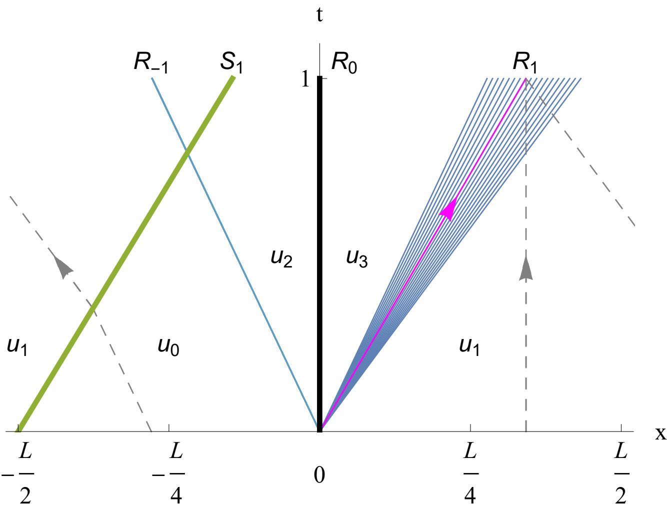
good mixing $k = 3$

lattice size $N = 4096$

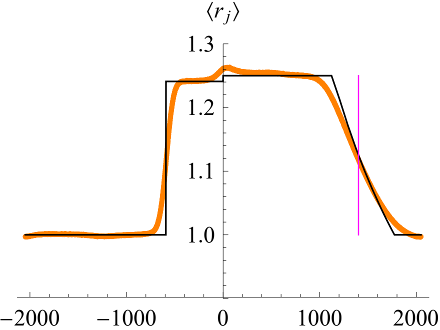
dynamics up to $t = 2000$

collision to collision

search algorithm

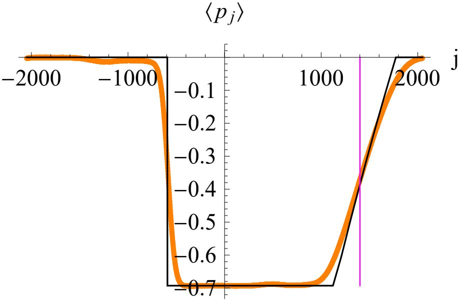


t
R_{-1}
S_1
R_0
R_1
1
u_2
u_3
u_1
u_0
u_1
x
$-\frac{L}{2}$
$-\frac{L}{4}$
0
$\frac{L}{4}$
$\frac{L}{2}$



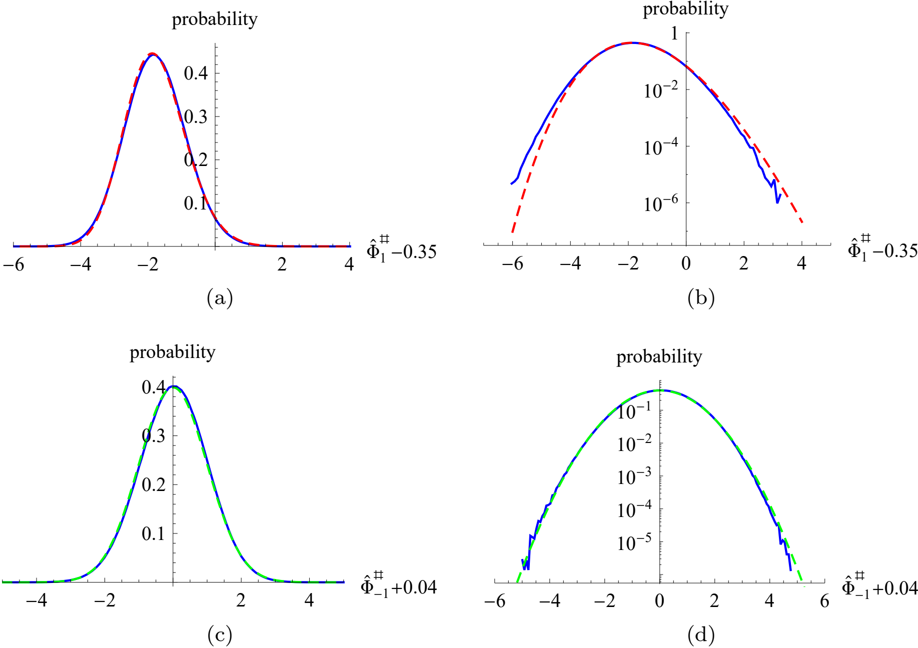
⟨r_j⟩
1.3
1.2
1.1
1.0
j
−2000
−1000
0
1000
2000

(a)



⟨p_j⟩
j
−2000
−1000
1000
2000
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7

(b)


probability
0.4
0.3
0.2
0.1
$\hat{\Phi}_1^{\sharp} - 0.35$
−6
−4
−2
2
4
(a)
probability
1
10^{-2}
10^{-4}
10^{-6}
$\hat{\Phi}_1^{\sharp} - 0.35$
−6
−4
−2
0
2
4
(b)
probability
0.4
0.3
0.2
0.1
$\hat{\Phi}_{-1}^{\sharp} + 0.04$
−4
−2
2
4
(c)
probability
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\hat{\Phi}_{-1}^{\sharp} + 0.04$
−6
−4
−2
0
2
4
6
(d)

outlook

- domain wall inital conditions

Tracy-Widom GOE curved: projection

- Leroux stochastic lattice gas
  2 conservation laws
- alternating mass hard point chain
  hamiltonian
  3 conservation laws

- GOE ? flat: lattice gas ✓ (not done)
  hamiltonian tricky

- Baik-Rains ? stationary:

look up Mendl, HS 2015 "current"

✓ current integrated along sound peak
  ∥ + projection $t^{1/3}$ BR ∥

BUT current integrated along heat peak
  ∥ + projection $t^{3/10}$ Gauss ∥

otherwise $\sqrt{t}$ Gauss