Ultraviolet-regulated theory of non-linear diffusion

KITP, Santa Barbara
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Dispersion of eigenmodes

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]
How do nonlinear quantum fluctuations and statistical fluctuations modify diffusion when taking into account the slowest UV-mode ($\tau \neq 0$)?

$\tau = 0$: Long time tails?  Transport renormalized?

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

[Kovtun, Yaffe; PRD (2003)]  [Kovtun, Moore, Romatschke; PRD (2011)]
Motivation

Ultracold atom measurements: *Bad metallic transport in a cold atom Fermi-Hubbard system*

\[ \sigma = 1/\rho = \chi_c D \]
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Diffusion coefficient modified by quantum-statistical fluctuations (e.g. near critical points) [Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]
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Hydrodynamics as an effective field theory  
[Jensen, Kaminski, Kouvun, Meyer, Ritz, Yarom.; PRL (2012)]  
[Banerjee et al. JHEP (2012)]  
[Crossley, Glorioso, Liu; JHEP (2017)]  
[Haehl, Loganayagam, Rangamani; JHEP (2015)]
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Hydrodynamics as an effective field theory

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BUT: hydrodynamics needs to be regulated by UV-mode(s) to be *causal and stable*  
(e.g. Mueller-Israel-Stewart theory, BDNK, ...)

[Hiscock & Lindblom; PRD (1985)]  
[Bemfica, Disconzi, Noronha; PRD (2018)]  
[PRX (2022)]  
[Hoult, Kovtun; JHEP (2020)]  
[Kovtun; JHEP (2019)]  
[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]
**Method**

**linear diffusion**
linearized in hydrodynamic fields
(e.g., energy density $\mathcal{E} \sim$ temperature $T$)

**nonlinear diffusion**
via effective action from exponentiated e.o.m.
Martin-Siggia-Rose formalism (MSR)
write stochastic differential equations as a field theory formulated using path integrals

[Martin, Siggia, Rose; PRA (1973)]

**nonlinear diffusion**
via effective action via Schwinger-Keldysh formalism (SK)
effective field theory for dissipative hydrodynamics

[Crossley, Glorioso, Liu; JHEP (2017)]

**MSR is used here.**
[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]
adding one regulating UV-mode charge diffusion

**SK was used in**
[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]
no regulating UV-mode heat diffusion
Method

Dispersion of eigenmodes in complex frequency plane

Consider one conserved charge $n$

\[
\partial_t n + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} + D \nabla n = 0
\]

Fick’s law of diffusion:

\[
\partial_t n - D \nabla^2 n = 0
\]
Method

Dispersion of eigenmodes in complex frequency plane

Consider one conserved charge $n$

$$\partial_t n + \nabla \cdot J = 0,$$
$$J + D \nabla n = 0$$

Fick’s law of diffusion:

$$\partial_t n - D \nabla^2 n = 0$$

Fourier transform $n(t, x) \propto e^{-i\omega t + ikx}$ $n(\omega, k)$

to read off eigen-frequency:

$$\omega = -iDk^2$$

diffusion mode

Differential equation turned into algebraic equation by relations like $\partial_t e^{-i\omega t} = -i\omega e^{-i\omega t}$
Method

Consider one conserved charge $n$ and relaxation time $\tau$:

\[ \partial_t n + \nabla \cdot J = 0, \quad [\tau \partial_t J + J + D \nabla n] = 0 \]

Fick’s law of diffusion (UV-regulated):

\[ \tau \partial_t^2 n + \partial_t n - D \nabla^2 n = 0. \]

This was used to analyze experiment.  
[Brown et al.; Science (2018)]
Method

Dispersions of eigenmodes in complex frequency plane

Consider one conserved charge $n$ and \textbf{relaxation time} $\tau$:

$$
\partial_t n + \nabla \cdot J = 0, \quad \tau \partial_t J + J + D \nabla n = 0
$$

\textbf{Fick's law of diffusion (UV-regulated)}:

$$
\tau \partial^2_t n + \partial_t n - D \nabla^2 n = 0.
$$

Fourier transform to read off eigen-frequencies:

$$
\omega_{1,2} = -\frac{i}{2\tau} \left(1 \mp \sqrt{1 - 4\tau Dk^2}\right)
$$

diffusion mode and 
slowest UV-mode

\textbf{This was used to analyze experiment.}

\textit{[Brown et al.; Science (2018)]}

\textit{concise summary in my subsection V.}
\textit{B of white paper [Sorensen et al.; arXiv:2301.13253]}

e.g., Mueller-Israel-Stewart theory, one may also think of this as Hydro+

\textbf{[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]}
Consider one conserved charge $n$ and relaxation time $\tau$:

$$\partial_t n + \nabla \cdot J = 0, \quad \tau \partial_t J + J + D \nabla n = 0$$

Fick’s law of diffusion (UV-regulated):

$$\tau \partial_t^2 n + \partial_t n - D \nabla^2 n = 0$$

Fourier transform to read off eigen-frequencies:

$$\omega = \frac{-i}{\tau} \sqrt{1 - \frac{4\pi D\omega^2}{k^2}}$$

Method

Dense Nuclear Matter Equation of State from Heavy-Ion Collisions

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V. Connections to other areas of nuclear physics

A. Applications of hadronic transport
1. Detector design
2. Space exploration, radiation therapy, and nuclear data

B. Hydrodynamics
1. Status
2. Range of applicability
3. Challenges and opportunities

E.g., Mueller-Israel-Stewart theory, one may also think of this as Hydro+

concise summary in my subsection V. B of white paper [Sorensen et al.; arXiv:2301.13253]
Method: Martin-Siggia-Rose

**nonlinear diffusion via effective action from exponentiated e.o.m.**

**Martin-Siggia-Rose formalism (MSR)**

write stochastic differential equations as a field theory formulated using path integrals

[Martin, Siggia, Rose; PRA (1973)]

Ideas:

\[ \langle \mathcal{O} \rangle \sim e^{e.o.m.} \]

Stochastic differential equation (e.o.m.):

\[ \partial_t x(t) = F(x(t), t) + \xi(x(t), t), \]

Noise correlation:

\[ \langle \xi(x, t)\xi(x', t') \rangle = G(x, t, x', t'). \]

Observables averaged over solutions of this stochastic differential equation may be written:

\[ \langle \mathcal{O}[x(t)] \rangle = \int \mathcal{D}[x, \tilde{x}] \mathcal{O}[x(t)] e^{-S[x, \tilde{x}]}, \]

\[ S[x, \tilde{x}] = \int_t i\tilde{x}(t) [\partial_t x(t) - F(x(t), t)] + \frac{1}{2} \int_{t,t'} G(x(t), t, x(t'), t')\tilde{x}(t)\tilde{x}(t'). \]
Results: Spectrum of eigen-frequencies

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

➡ larger range of applicability than SK without UV mode
Method

Self-interactions enter through dependence of diffusion coefficient on charge fluctuations:

\[ D(n) = D + \lambda_D n + \frac{\lambda_D'}{2} n^2 \]

\text{Note: corrections to } \tau(n) = \tau + \lambda_{\tau,1} n + \lambda_{\tau,2} n^2 + \ldots \text{contribute to higher order only}
Method

Self-interactions enter through dependence of diffusion coefficient on charge fluctuations:

\[ D(n) = D + \lambda_D n + \frac{\lambda_D'}{2} n^2 \]

Leading to the nonlinear equation of motion:

\[ \tau \partial_t^2 n + \partial_t n - \nabla^2 \left( D n + \frac{\lambda_D}{2} n^2 + \frac{\lambda_D'}{6} n^3 \right) = 0 \]

Note: corrections to \( \tau(n) = \tau + \lambda_{\tau,1} n + \lambda_{\tau,2} n^2 + \ldots \) contribute to higher order only
Method

Self-interactions enter through dependence of diffusion coefficient on charge fluctuations:

\[ D(n) = D + \lambda_D n + \frac{\lambda'_D}{2} n^2 \]

Leading to the nonlinear equation of motion:

\[ \tau \partial_t^2 n + \partial_t n - \nabla^2 \left(D n + \frac{\lambda_D}{2} n^2 + \frac{\lambda'_D}{6} n^3 \right) = 0 \]

Note: corrections to \( \tau(n) = \tau + \lambda_{\tau,1} n + \lambda_{\tau,2} n^2 + ... \) contribute to higher order only

Exponentiate stochastic version of this equation to obtain path integral, \([\text{Martin, Siggia, Rose; PRA (1973)}]\)

from which the effective action can be read:

\[ \mathcal{L} = iT \sigma \nabla n_a C \nabla n_a - n_a \left( \tau \partial_t^2 n + \partial_t n - D \nabla^2 n \right) \]

\[ + iT \chi \lambda_{\sigma} n \nabla n_a C \nabla n_a + \frac{\lambda_D}{2} \nabla^2 n_a n^2 + \frac{1}{2} iT \chi' \lambda_{\sigma} n^2 \nabla n_a C \nabla n_a + \frac{\lambda'_D}{6} \nabla^2 n_a n^3 \]

with conductivity \( \sigma(n) = \sigma + \chi \lambda_{\sigma} \delta n + \frac{1}{2} \chi' \lambda_{\sigma} \delta n^2 \) and \( C = \left( \frac{i \partial_t}{2T} \right) \coth \left( \frac{i \partial_t}{2T} \right) \)
Method

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Exponentiate stochastic version of this equation to obtain path integral, [Martin, Siggia, Rose; PRA (1973)] from which the effective action can be read:

\[
\mathcal{L} = i T \sigma \nabla n_a C \nabla n_a - n_a \left( \tau \partial_t^2 n + \partial_t n - D \nabla^2 n \right) \\
+ i T \chi \lambda_\sigma n \nabla n_a C \nabla n_a + \frac{\lambda_D}{2} \nabla^2 n_a n^2 + \frac{1}{2} i T \chi' \lambda_\sigma n^2 \nabla n_a C \nabla n_a + \frac{\lambda_D'}{6} \nabla^2 n_a n^3
\]

with conductivity \( \sigma(n) = \sigma + \chi \lambda_\sigma \delta n + \frac{1}{2} \chi' \lambda_\sigma \delta n^2 \) and \( C = \left( \frac{i \partial_t}{2T} \right) \coth \left( \frac{i \partial_t}{2T} \right) \)

Perform perturbation theory computation to one-loop order, like done in particle physics (e.g. QED).
Method

Perform perturbation theory computation to one-loop order, like done in particle physics (e.g. QED).

\[
G_{nn_a}(p) = G_{nn_a}^{(0)}(p) + G_{nn_a}^{(0)}(p)(-\Sigma(p))G_{nn_a}^{(0)}(p) = \frac{1}{\omega + i D_0 k^2 - i \tau \omega^2 + \Sigma(\omega, k)}
\]

\[
G_{nn_a}^{(0)} \Sigma(p) G_{nn_a}^{(0)} = \begin{array}{c}
\text{Diagram 1}
\end{array} + \begin{array}{c}
\text{Diagram 2}
\end{array}
\]

\[
G_{nn_a}^{(0)}(p)(-C(p))G_{nn_a}^{(0)}(p) = \begin{array}{c}
\text{Diagram 3}
\end{array} + \begin{array}{c}
\text{Diagram 4}
\end{array} + \begin{array}{c}
\text{Diagram 5} + c.c
\end{array}
\]
Results: charge correlation function

Charge correlator to one-loop order:

\[
G^R_{nm}(\omega, k) = \frac{i \left( \sigma + \delta \sigma(\omega, k) \right) k^2}{-i \tau \omega^2 + \omega + iDk^2 + \Sigma(\omega, k)}
\]

Branch point singularities:

\[
\tilde{\omega}_{11} \& \tilde{\omega}_{22} = -\frac{i}{\tau} \left( 1 \mp \sqrt{1 - Dk^2} \right), \quad \tilde{\omega}_{12} \& \tilde{\omega}_{21} = -\frac{i}{\tau} \pm |k| \sqrt{\frac{D}{\tau}}
\]

Non-analyticities:

\[
\Sigma_d(\omega, k) = \alpha_d(\omega, k) (\tau D)^{2-d} \frac{T \chi}{D^2} k^2 \left[ f_{1d}(\omega, k) \lambda_D^2 + f_{2d}(\omega, k) \lambda_D \lambda_\sigma \right]
\]

\[
\alpha_1(\omega, k) = \frac{1}{16} \left( \frac{(1 - i\tau \omega)^2(Dk^2 \tau - i\omega \tau(2 - i\tau \omega))}{Dk^2 \tau + (1 - i\tau \omega)^2} \right)^{-1/2}, \quad (d = 1)
\]

\[
\alpha_2(\omega, k) = -\frac{1}{64\pi} \log \left( \frac{(1 - i\tau \omega)^2(Dk^2 \tau - i\omega \tau(2 - i\tau \omega))}{Dk^2 \tau + (1 - i\tau \omega)^2} \right), \quad (d = 2)
\]

\[
\alpha_3(\omega, k) = -\frac{1}{128\pi} \left( \frac{(1 - i\tau \omega)^2(Dk^2 \tau - i\omega \tau(2 - i\tau \omega))}{Dk^2 \tau + (1 - i\tau \omega)^2} \right)^{1/2}, \quad (d = 3)
\]
Results: conductivity correction & current correlator

\[
\frac{\delta \sigma_d(\omega, \mathbf{k})}{\sigma} = \frac{-\alpha_d(\omega, \mathbf{k}) (\tau D)^{\frac{2-d}{2}}}{D^2} \frac{2T\chi}{k^2} \left[ f_{3d}(\omega, \mathbf{k}) \lambda_D^2 + f_{4d}(\omega, \mathbf{k}) \lambda_D \lambda_\sigma \right]
\]

equivalent to result from [Chen-Lin, Delacrétaz, Hartnoll; PRL (2022)]

\[
\frac{G_{JJ}(t, q)}{12 q^2 \lambda_{eff} D T \chi} = \frac{1}{(t/\tau)^{1/2}}
\]

⇒ long time tail: power law,
⇒ UV mode inconsequential for conductivity correction
Reduction of susceptibility & conductivity peaks

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

\[ G_{nn}(q, w) \]

\[ \chi \]

\[ \frac{\sigma(q, w)}{\sigma_0} \]
Dynamic susceptibility

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

dynamic susceptibility $G_{nn}^R/\chi$
Method: Effective formalism for hydrodynamic fluctuations

Supplemental Material of [Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

Goal is to compute correlator:

\[
\langle \varepsilon(t, x)\varepsilon(t', x') \cdots \rangle_\beta \equiv \text{Tr} \left( \rho_\beta \varepsilon(t, x)\varepsilon(t', x') \cdots \right) \quad \rho_\beta = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}}
\]

Generating functional:

\[
Z[A_1^1, A_2^1] \equiv \text{Tr} \left( U[A_1] \rho_\beta U[A_2]^\dagger \right) \quad Z[A_1^1, A_2^1] = \int D\psi_1 D\psi_2 e^{iS[\psi_1, A_1] - iS[\psi_2, A_2]}
\]

Constraints on effective action:

\[
Z[A_\mu, A_\mu] = 1, \quad Z[A_1^1, A_2^2] = Z^*[A_2^2, A_1^1], \quad Z[A_1^1, A_2^2] = Z[A_1^1 + \partial_\mu \lambda^1, A_2^2 + \partial_\mu \lambda^2], \quad Z[A_1^1, A_2^2] = Z[A_1^1(-t, x_{PT}), A_2^2(-t - i\beta, x_{PT})]
\]

Local effective action \( I \):

\[
Z[A_1^1, A_2^1] = \int D\varphi_1 D\varphi_2 e^{i[I[B_1^1, B_2^2]]}
\]

Auxiliary fields:

\[
\varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2), \quad \varphi_a = \varphi_1 - \varphi_2
\]

Most general isotropic quadratic action:

\[
\beta L_2 = c_1 \partial_r \varphi_a + \kappa \varphi_r \partial^2 \varphi_a + i T \kappa (\nabla \varphi_a)^2 + \cdots
\]

\[
\mathcal{L} = iT^2 \kappa (\nabla \varphi_a)^2 - \varphi_a (\varepsilon - D \nabla^2 \varepsilon) + \nabla^2 \varphi_a \left( \frac{\lambda}{2} \varepsilon^2 + \frac{\lambda'}{3} \varepsilon^3 \right) + iT^2 (\nabla \varphi_a)^2 (\tilde{\lambda} \varepsilon + \tilde{\lambda}' \varepsilon^2) + \cdots
\]

\[
\varepsilon = c T \dot{\varphi}_a
\]

quartic action (constraints imposed)
Results: Effective formalism for hydrodynamic fluctuations

Energy correlator:

\[ G^{R}_{\tau \tau}(\omega, k) = \frac{i[\kappa + \delta \kappa(\omega, k)]Tk^2}{\omega + iDk^2 + \Sigma(\omega, k)} \]

\[ \delta \kappa(\omega, k) = \delta \kappa + \kappa_*(\omega, k), \]

\[ \Sigma(\omega, k) = i\delta Dk^2 + \Sigma_*(\omega, k). \]

Analytic corrections to transport:

\[ \frac{\delta \kappa}{\kappa} = \frac{f_d}{c \ell_{\text{th}}^d} \lambda_\kappa, \quad \frac{\delta D}{D} = \frac{f_d}{c \ell_{\text{th}}^d} \lambda_D \]

Nonanalytic corrections:

\[ \kappa_*(\omega, k) = f_\kappa(\omega, k)\alpha_d(\omega, k), \]

\[ \Sigma_*(\omega, k) = k^2 f_\Sigma(\omega, k)\alpha_d(\omega, k), \]

\[ f_\kappa(\omega, k) = \frac{cT^2}{D^2} k^2 \lambda \overline{\lambda}, \]

\[ f_\Sigma(\omega, k) = \frac{cT^2}{D^2} [\omega \lambda(\lambda + \overline{\lambda}) + iDk^2 \lambda \overline{\lambda}] \]

\[ \mathcal{L} = iT^2 \kappa(\nabla \varphi_a)^2 - \varphi_a(\dot{\varphi} - D \nabla^2 \varphi) \]

\[ + \nabla^2 \varphi_a \left( \frac{\lambda}{2} \epsilon^2 + \frac{\lambda'}{3} \epsilon^3 \right) + \text{i}cT^2(\nabla \varphi_a)^2(\overline{\lambda} \epsilon + \overline{\lambda'} \epsilon^2) \]

\[ + \cdots, \]

\[ \alpha_1(\omega, k) = \frac{1}{4} \left( k^2 - \frac{2i\omega}{D} \right)^{-1/2}, \quad (d = 1) \]

\[ \alpha_2(\omega, k) = -\frac{1}{16\pi} \log \left( k^2 - \frac{2i\omega}{D} \right), \quad (d = 2) \]

\[ \alpha_3(\omega, k) = -\frac{1}{32\pi} \left( k^2 - \frac{2i\omega}{D} \right)^{1/2}, \quad (d = 3) \]

Nonanlalticities in energy correlator introduce branch point half-way to splitted diffusion pole
**Discussion**

**Summary**

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

- UV mode *does not affect renormalization of* diffusion coefficient $D$ and conductivity $\sigma$
- bound on local thermalization time $\tau$ is *protected* from renormalization
- charge current relaxes with *power law*
- proposed *measurement in ultracold atoms*
- relevant for QCD near critical point & Hubbard

**Outlook**

- extracting the running of $\tau$ from data will be an important check for the theory constructed in this work
- repeat our computation in Schwinger-Keldysh (generally equivalent?)

*for stability and causality, see [Mullins, Hippert, Noronha; arXiv:2306.08635]*
APPENDIX
Complex momentum spectra
Modified dispersion relation (in case $d=1$, convergent)

\[ m = 1, 3 : \quad w_m = \sum_{n=2} a_n (-1)^n \left[ \frac{m-1}{2} \right] (q^2)^{n/2} = a_2 q^2 \pm a_3 (q^2)^{3/2} + \cdots, \]

\[ m = 2, 4 : \quad w_m = -i + \sum_{n=2} c_n (-1)^n \left[ \frac{m}{2} \right] (q^2)^{n/2} = -i + c_2 q^2 \mp c_3 (q^2)^{3/2} + \cdots. \]
Loop calculations

\[ G_{nn_a}(p) = G_{nn_a}^{(0)}(p) + G_{nn_a}^{(0)}(-\Sigma(p)) G_{nn_a}^{(0)}(p) = \frac{1}{\omega + iD_0 k^2 - i\tau \omega^2 + \Sigma(\omega, k)} \]

\[ \Sigma(p) = \lambda_D^2 k^2 \int_{p'} k^2 G_{nn_a}^{(0)}(p') G_{nn_a}^{(0)}(p' + p) \]

\[ - i \frac{1}{2} \chi T \lambda_D \lambda_a k^2 \int_{p'} \left( k^2 + k \cdot k' \right) \left[ Q(\omega') + Q(\omega + \omega') \right] G_{nn_a}^{(0)}(p') G_{nn_a}^{(0)}(p + p') \]

\[ f_1(\omega, k) = \frac{\omega(1 - i\tau \omega)(Dk^2 \tau - \tau^2 \omega^2 - 3i\tau \omega + 2)^2}{(Dk^2 \tau + (1 - i\tau \omega)^2)^2} \left[ 1 + \frac{1}{(T\tau)^2} Q_1^{(2)}(\omega) + \frac{1}{(T\tau)^4} Q_1^{(4)}(\omega) \right], \]

\[ Q_1^{(2)} = \frac{(Dk^2 \tau - i\tau \omega(1 - i\tau \omega))^2}{48(Dk^2 \tau + (1 - i\tau \omega)^2)^2}, \]

\[ Q_1^{(4)} = \frac{(Dk^2 \tau - i\tau \omega(1 - i\tau \omega))^2 (3(Dk^2 \tau)^2 - 2(i\tau \omega)(Dk^2 \tau)(3 - i\tau \omega) - (i\tau \omega)^2(1 - i\tau \omega)^2)}{11520(Dk^2 \tau + (1 - i\tau \omega)^2)^2} \]

\[ f_2(\omega, k) = \frac{2i(Dk^2 - i\omega - i\tau \omega^2)(1 - i\tau \omega)(Dk^2 \tau - \tau^2 \omega^2 - 3i\tau \omega + 2)}{(Dk^2 \tau + (1 - i\tau \omega)^2)^2} \left[ 1 + \frac{1}{(T\tau)^2} Q_2^{(2)}(\omega) + \frac{1}{(T\tau)^4} Q_2^{(4)}(\omega) \right], \]

\[ Q_2^{(2)} = \frac{(Dk^2 \tau)^2 - 2(i\tau \omega)(Dk^2 \tau) - (i\tau \omega)^2(1 - i\tau \omega)^2}{48(Dk^2 \tau + (1 - i\tau \omega)^2)^2}, \]

\[ Q_2^{(4)} = \frac{1}{11520(Dk^2 \tau + (1 - i\tau \omega)^2)^2} \left[ (Dk^2 \tau)^4 - 4(Dk^2 \tau)^3(i\tau \omega)(1 + i\tau \omega), \right. \]

\[ - 2(Dk^2 \tau)^2(i\tau \omega)^2(1 - 10i\tau \omega - 5\tau^2 \omega^2) + 4(Dk^2 \tau)(i\tau \omega)^3(1 - i\tau \omega)(3 - i\tau \omega) + (i\tau \omega)^4(1 - i\tau \omega)^4 \]
\begin{align*}
\lim_{Dk^2 \ll \omega} f_{1d}(\omega, k) &= -\omega(2 - 3i\tau\omega + \tau^2\omega^2) \left[ 1 + \frac{\omega^2}{48T^2} - \frac{\omega^4}{11520T^4} \right] \\
\lim_{Dk^2 \ll \omega} f_{2d}(\omega, k) &= 2\omega(1 - i\tau\omega) \left[ 1 + \frac{\omega^2}{48T^2} - \frac{\omega^4}{11520T^4} \right].
\end{align*}

\[(\omega + iDk^2 - i\tau\omega^2) \left(1 - \frac{\delta\sigma(\omega, k)}{\sigma}\right) + \Sigma(\omega, k) = 0\]

\[C(p) = 2T\chi Dk^2 Q(\omega) + 2\chi T\lambda_\sigma Q(\omega) \int_{p'} G_{nn}^{(0)}(p') + \frac{1}{2} \lambda_D k^4 \int_{p'} G_{nn}^{(0)}(p')G_{nn}^{(0)}(p-p') + i\chi T\lambda_\sigma \lambda_D k^2 \int_{p'} k \cdot k' G_{n_n}^{(0)}(p')G_{nn}^{(0)}(p+p')(Q(\omega) + Q(\omega')).\]

\[G_{nn}(\omega, k) = \frac{C(\omega, k)}{\omega^2 + D^2k^4 + 2\omega\Re\Sigma(\omega, k) + 2(Dk^2 - \tau\omega^2)\Im\Sigma(\omega, k)}\]

\[C(p) = 2T\chi Dk^2 Q(\omega) \left[ 1 + \frac{\Re\delta\sigma(p)}{\sigma} + \frac{Dk^2 - \tau\omega^2\Im\delta\sigma(p)}{\sigma} + \frac{\Re\Sigma(p)}{\omega} \right]\]

\[G_{nn}(p)(-C(p))G_{n_n}^{(0)}(p) = \]

\[+ \quad + \quad + \quad \left[ + \text{c.c.} \right]\]
Singular points of plane curves

Puiseux theorem:

Any equation $f(x, y) = 0$, where $f$ is a polynomial with $f(O) = 0$ or more generally $f \in \mathbb{C}[[x, y]]$ with zero constant term, admits at least one solution in formal power series of the form

$$x = t^n, \quad y = \sum_{1}^{\infty} a_r t^r$$

(some $n \in \mathbb{N}$).

Thus, $y$ can be expressed as power series in fractional powers of $x$.

Example: hydrodynamics

$$x = k, \quad y = \omega, \quad f(x, y) = \mathcal{P}(\omega, k)$$

$$\mathcal{P} \phi = 0 \Rightarrow \mathcal{P} = \omega + iDk^2 + \mathcal{O} = 0$$

⇒ There exists convergent hydrodynamic expansion. Critical points limit the radius of convergence in complex $k$. [https://xkcd.com/2605/]
Hydrodynamic expansion of dispersion relations around far-from-equilibrium state

- asymptotic expansion: coefficients $\sim n$!
- attractors [Heller, Spalinski; PRL (2015)]
  [Heller et al; PRL (2021)]
- resurgence
- far-from-equilibrium holography [Kurkela et al; PRL (2019)]
  [Janik, Jankowski, Soltanpanahi; PRL (2017)]
- far-from-equilibrium fluid dynamics [Romatschke; PRL (2017)]

Pressure anisotropy in $N=4$ SYM:

$$\mathcal{A} = \frac{8C_\eta}{w} + \frac{16C_\eta C_\tau}{3w^2} + \ldots = \sum_{n>0} \frac{a_n^{(0)}}{w^n} + \left(\sigma \frac{C_\eta}{w^2} e^{-\frac{3}{2C_\tau}w}\right) \sum_{n\geq 0} \frac{a_n^{(1)}}{w^n} + \ldots$$

Navier–Stokes

[from Talk by Spalinski at QuarkMatter22]
Quantum chaos in (large) rotating AdS5 black holes

AdS5 Schwarzschild pole-skipping:

\[ w = i, \quad q = \pm \sqrt{\frac{3}{2}} i. \]

Apply transformation:

\[ q = \frac{a \nu + j}{\sqrt{1 - a^2}}, \quad w = \frac{a j + \nu}{\sqrt{1 - a^2}}. \]

Shifted pole-skipping points:

\[ \nu_{\text{scalar}} = \frac{i}{\sqrt{1 - a^2/L^2}} \left( 1 \mp \frac{\sqrt{3} a}{\sqrt{2} L} \right), \quad j_{\text{scalar}} = \frac{i}{\sqrt{1 - a^2/L^2}} \left( \pm \frac{\sqrt{3}}{\sqrt{2}} \frac{a}{L} \right) \]

\[ \lambda_L = 2\pi T \left( 1 - \sqrt{\frac{3}{2} \frac{|a|}{L}} \right) = 2\pi T \left( 1 - \frac{|v|}{v_B^{(0)}} \right) \]

quantum Lyapunov exponent

\[ v_B^{\pm} = \frac{\sqrt{\frac{2}{3} \mp \frac{a}{L}}}{1 \mp \sqrt{\frac{2}{3} \frac{a}{L}}} \]

butterfly velocity

Agrees with shock-wave computation and with near-horizon expansion method.

Pole-skipping points in rotating black holes in AdS4: [Blake, Davison; JHEP (2021)]

Pole-skipping points in rotating AdS5 black holes:

\[ \frac{a}{L} = \frac{3}{10}, \frac{6}{10}, \frac{9}{10}, \sqrt{\frac{2}{3}} \]

[Amano(Garbisio), Blake, Cartwright, Kaminski, Thompson; (2022)]
More topics in hydrodynamics

- **Spin hydrodynamics**
  
  [Hongo, Huang, Kaminski, Stephanov, Yee; JHEP (2021)]

- **Magnetohydrodynamics**
  
  [Ammon, Grieninger, Hernandez, Kaminski, Koirala, Leiber, Wu; JHEP (2021)]

- **Non-relativistic**
  
  [Kaminski, Moroz; PRB (2014)]
  [Garbiso, Kaminski; JHEP (2019)]
  [Davison, Grozdanov, Janiszewski, Kaminski; JHEP (2016)]

- **Far from equilibrium fluid dynamics**
  
  [Cartwright, Kaminski; JHEP (2019)]
  [Wondrak, Kaminski, Bleicher; PRB (2020)]
  [Cartwright, Kaminski, Knipfer; arXiv:2207.02875]
  [Cartwright, Kaminski, Schenke; PRC (2022)]

- **Quantum chaos**
  
  [Blake, Lee, Liu; JHEP (2018)]
  [Amano(Garbiso), Blake, Cartwright, Kaminski, Thompson; JHEP (2022)]

- **Convergence under rotation**
  
  [Cartwright, Garbiso-Amano; Kaminski, Noronha, Speranza; arXiv:2112.10781]

Thank you for listening!
Vision: Quantum fluids far from equilibrium

Hydrodynamics

- far from equilibrium
  
  [Romatschke; PRL (2018)]

  [Glorioso,Liu]
  [Haehl,Loganayagam,Rangamani]

- quantum chaos
  
  [Blake, Lee, Liu; JHEP (2018)]
  [Grozdanov et al. (2019)]

- convergence & stability
  
  [Kovtun; JHEP (2019)]
  [Grozdanov, Kovtun, Starinets, Tadic; PRL (2019)]
  [Withers; JHEP (2018)]
  [Heller, Janik, Witaszczyk; PRL (2013)]
  [Heller, Spalinski; PRL (2018)]

- most vortical fluid
  
  [Garbiso, Kaminski; JHEP (2019)]
  [Cartwright, Garbiso-Amano; Kaminski, Noronha, Speranza;arXiv:2112.10781]