# QCD on a small circle

Aleksey Cherman
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work done with

Kyle Aitken (Univ. of Washington),

Erich Poppitz (Univ. of Toronto)

Larry Yaffe (Univ. of Washington )

1707.08971

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# YM on a small circle

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# Resurgence for QFT

Belief: QFT observables = resurgent transseries in some couplings

$$\mathcal{O}(\lambda) \simeq \sum_{n} p_n \lambda^n + \sum_{c} e^{-\frac{S_c}{\lambda}} \sum_{k} p_{k,c} \lambda^k + \cdots$$

Lots of evidence in special cases:

| | |
|---|---|
| Integrals with saddles | Stokes, Dingle, Berry, Howls … |
| matrix models | Marino, Schiappa, Weiss … |
| topological strings | Aniceto, Hatsuda, Marino, Schiappa, Vonk, … |
| QM (d=1 QFT) | Basar, Dunne, Kawai, Misumi, Nitta, Sakai, Takei, Sulejmanpasic, Unsal, Zinn-Justin … |
| some SUSY theories | Aniceto, Dorigoni, Hatsuda, Honda, Russo, Schiappa, … |

Realistic QFTs with asymptotic freedom like QCD or YM?

# Basic resurgence questions for QCD

(I) What is a **useful** expansion parameter λ?

$$\mathcal{O}(\lambda) \simeq \sum_n p_n \lambda^n + \sum_c e^{-\frac{S_c}{\lambda}} \sum_k p_{k,c} \lambda^k + \cdots$$

# Basic resurgence questions for QCD

(I) What is a **useful** expansion parameter $\lambda$?

(2) What kind of transseries should we expect?

In simpler cases studied so far, transmonomials restricted to

$$\{\lambda,\ 1/\lambda,\ \log\lambda,\ e^{-1/\lambda}\}$$

Turns out in QCD we also need

$$e^{-e^{+1/\lambda}}$$

as well, at the least!

Aitken, AC, Poppitz, Yaffe,  1707.08971

anticipated from an OPE perspective by M. Shifman, hep-ph/9405246

# Outline

YM coupling $\lambda$ changes with energy scale. Most interesting observables are "low-energy" ones.

Relevant coupling isn't small.

Big difference from QM and special QFT/string examples.

Challenge: construct a useful semiclassical expansion for QCD + its cousins.

useful = first few orders already give good guide to expected behavior.

1. Explain currently best-understood approach, giving its motivation and why we believe it works.

2. Highlight physical and mathematical lessons.

# Semiclassics requirements for QCD

Need coupling to be small at long distance

Keep as many features of QCD as many possible:

1.  asymptotic freedom

2.  quark confinement

3.  chiral symmetry breaking

4.  Lorentz invariance

5.  No microscopic scalars

Turns out we can **almost** have it all.
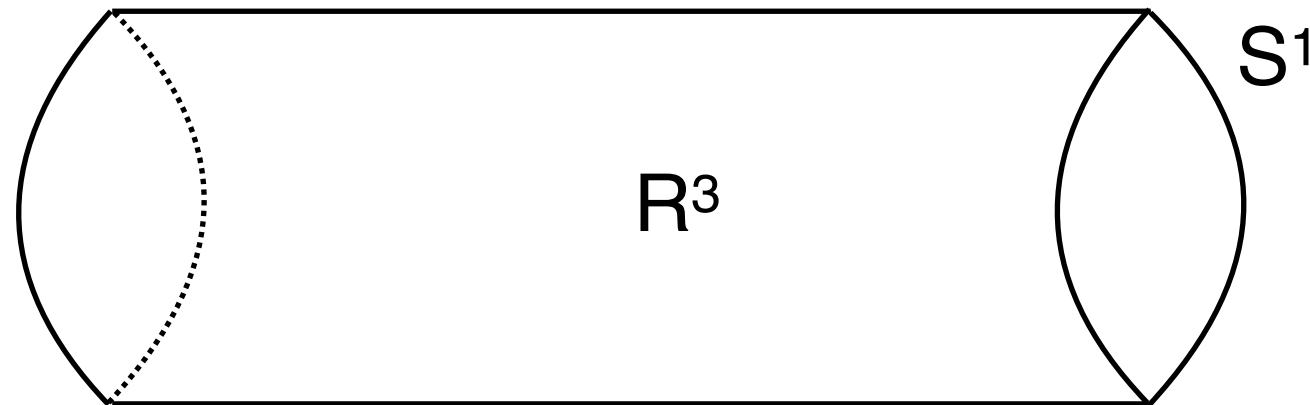
Lorentz-invariant route to weak coupling:

Scalar fields $\Rightarrow$ Higgs mechanism $\Rightarrow$ weak coupling ?

Works in EW part of SM + many SUSY theories, but **not in QCD**.

# Adiabatic compactification

Unsal, Yaffe, Shifman, … 2008-onward

Break 4D Lorentz, but as little as possible!



$S^1$

$R^3$

If circle size L is small, get weak coupling by asymptotic freedom

NB: still have Lorentz invariance in $R^3$ directions.
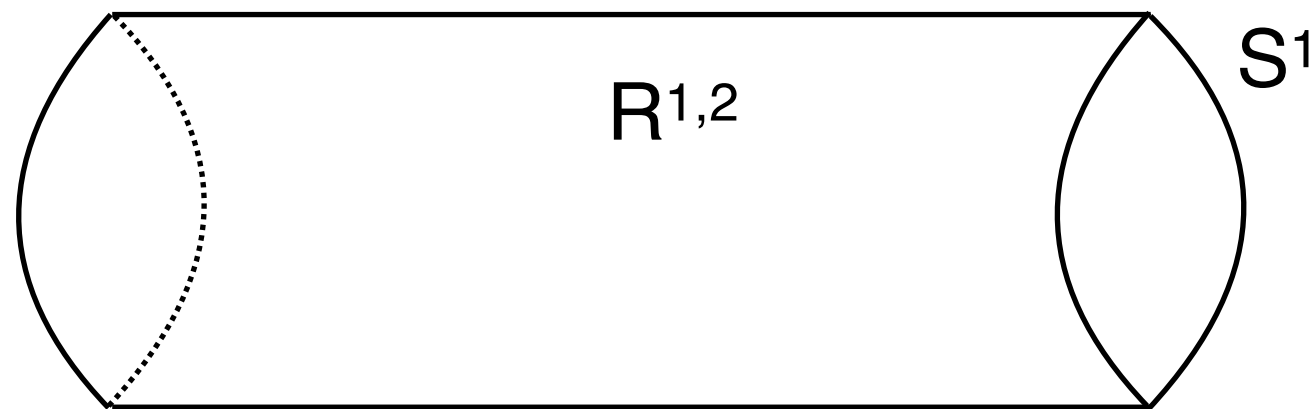
$3 > 2 \implies$ symmetries can break spontaneously.

Large L:  QCD has a diverse pattern of symmetry realizations.

Need d ≥ 3 non-compact directions to avoid Coleman-Mermin-Wagner thm.

Want **same** pattern at small L for our expansion to be useful.

# Self-Higgsing



When YM compactified on S[1],  tr(Polyakov loop) is an observable

$$\Omega = \mathcal{P}e^{i \oint A_3} = \text{diag}\left(e^{i\phi_1}, \ldots, e^{i\phi_N}\right)$$

Eigenvalues = classical moduli space

Non-coincident eigenvalues $\Rightarrow$
"broken" gauge group SU(N) $\rightarrow$ U(1)$^{N-1}$ in long-distance 3D EFT

"A$_3$"  acts like (compact) adjoint Higgs field!

But we don't get to choose eigenvalues:  theory picks own vacuum

# Adiabatic compactification

Unsal, Yaffe, Shifman, ... , 2008-onward

Confined phase at large L $\iff$ $< \text{tr } \Omega > \approx 0$ ; related to "center symmetry".

But at small L in pure YM dynamics force $A_4 = 0 \iff < \text{tr } \Omega > \neq 0$

Idea: add something that leaves large L theory
the "same", but makes small L limit smooth

Options:
(a) add a Dirac adjoint fermion with *periodic* BCs, $m \lesssim 1/NL$
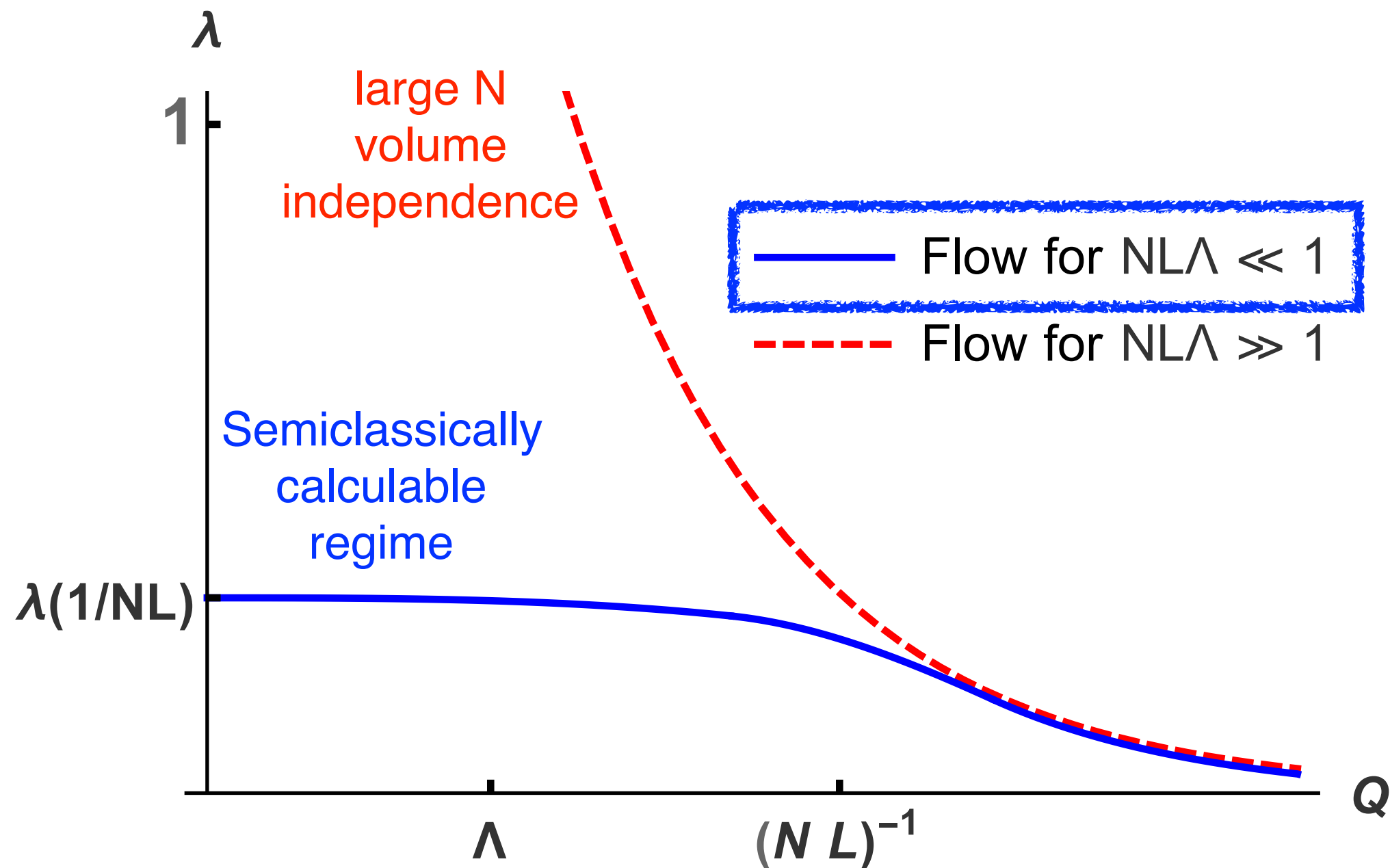(b) add appropriate double-trace deformation

Resulting theories remains center-symmetric at small L

Non-coincident eigenvalues of $\Omega \Rightarrow$ breaking $SU(N) \to U(1)^{N-1}$

Adjoint Higgs mechanism driven by VEV of "$A_4$" !

W-boson mass scale is $m_W = 2\pi/NL$

# Coupling flow with adiabatic compactification

When NLΛ << 1 physics is weakly-coupled at all scales!

# Small L limit in perturbation theory

At long distances $\ell >>$ N L $\sim$ 1/m$_W$

$$SU(N) \to U(1)^{N-1}$$

due to the center-symmetric background holonomy.

N$^2$ - N W-bosons with masses m$_W \sim$ 1/NL; ignore for now.

The light fields are N - 1 "Cartan gluons"

$$F_{\mu\nu}^j = \frac{1}{N} \sum_{p=0}^{N-1} e^{2\pi ijp/N} \,\mathrm{tr}\left(\Omega^p F_{\mu\nu}\right)$$

(added fictitious p = 0 mode for notational simplicity; it decouples exactly.)

Small-L physics easiest to describe using 3D Abelian duality

# Small L limit in perturbation theory

N - 1 Cartan gluons are classically gapless.

$$F_{\mu\nu}^i = g^2/(2\pi L)\epsilon_{\mu\nu\alpha}\partial^\alpha \sigma^i$$

$$S_\sigma = \int d^3x \, \frac{g^2}{8\pi^2 L}(\partial_\mu \vec{\sigma})^2.$$

$\sigma^i$ shift symmetry $\iff$ conservation of magnetic charge.

But there are no magnetic monopoles in perturbation theory.

$\sigma^i$ are massless to all orders in perturbation theory.

# Finite-action field configurations

Since SU(N) → U(1)$^{N-1}$, 4D BPST instanton breaks up
into N 'monopole-instantons' with action $S_I/N = 8\pi^2/\lambda$

N - 1 have Q = 1/N, magnetic charges +1,-1 under nearest-neighbor U(1)'s
Nth one is 'Kaluza-Klein' monopole.

Lee, Yi; Kraan,
van Baal; 1998

't Hooft
amplitude

$$\mathcal{M}_i \sim e^{-8\pi^2/\lambda} e^{i(\sigma_i - \sigma_{i+1})} \quad \lambda = g_{\mathrm{YM}}^2 N$$

$$\text{core size} \sim NL$$

$$\text{typical separation} \sim NLe^{+8\pi^2/(3\lambda)}$$

λ ≪ 1 when NLΛ ≪ 1, dilute gas approximation justified at small L.

Contrast with usual IR disasters with instantons in YM!

# Weak coupling confinement

Unsal,
Yaffe, Shifman, Poppitz,
Sulejmanpasic,
…

$$V(\sigma) \sim -m_W^3 e^{-8\pi^2/\lambda} \sum_{i=1}^{N} \cos(\sigma_i - \sigma_{i+1}) + \cdots$$

So dual photons pick a gap:

$$m_p \sim m_W e^{-4\pi^2/\lambda} |\sin(\pi p/N)|, \ \text{p = 1, ... , N - 1.}$$

Concrete realization of old Mandelstam, 't Hooft, Polyakov dreams: mass gap driven by `condensation' of magnetic monopoles.

Different in details:  here the monopoles aren't particles, and nothing is condensing.

String tension also calculable, and is finite.
Behaves just as expected from YM.

Poppitz,  Erfan S. T.,
1708.08821

Anber, Pellizzani,
1710.06509

(Far closer to real YM than Seiberg-Witten theory with N > 2.)

# Quantum number question

Are the σ's glueballs?

Not quite.  Recall mass formula and gauge-invariant representation

$$m_p \sim m_W e^{-4\pi^2/\lambda} |\sin(\pi p/N)|$$

$$F_{\mu\nu}^j = \frac{1}{N} \sum_{p=0}^{N-1} e^{2\pi ijp/N} \operatorname{tr}\left(\Omega^p F_{\mu\nu}\right)$$

'p' = charge under center symmetry, so σ's are center symmetry eigenstates:
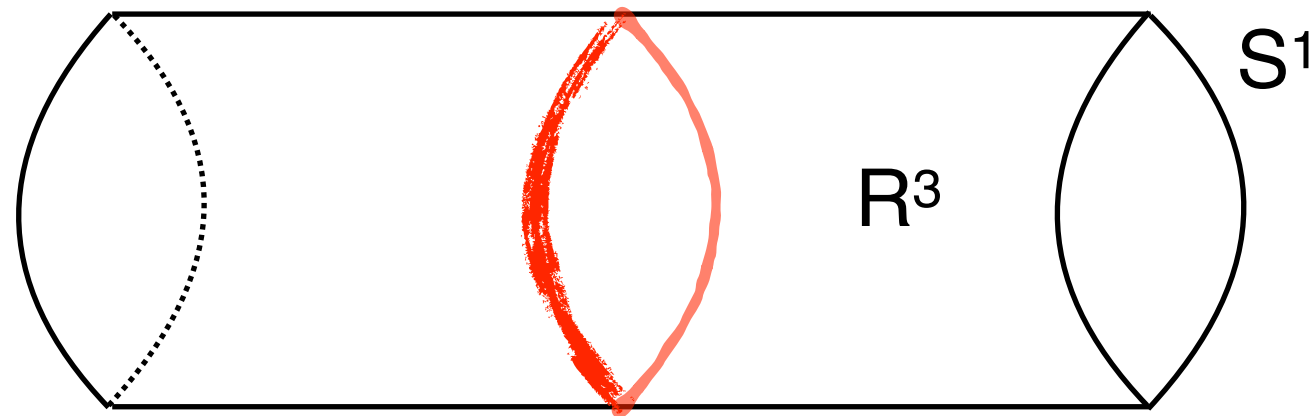
$$\mathcal{S} : \sigma_p \to e^{2\pi ip/N} \sigma_p$$

p = 1, … , N-1;  so no neutral modes.

Oops.  All states in light sector center-charged!?

# Large L extrapolation

center p-charged states = p-winding string states.



At small fixed L, their energy is small.

But as L → ∞ , energy ~ L * winding * (string tension) → ∞

What are the lightest states that have the right quantum numbers to extrapolate to finite energy states at large L?

**Answer:  bound states of dual photons!**

# Light bound states

Set N = 2 to keep it simple.

$$S_{3D} = \int d^3x \left[ \tfrac{1}{2}(\partial_\mu \sigma)^2 + \tfrac{1}{2} m_\gamma^2 \sigma^2 - \tfrac{2}{3} \epsilon\, m_\gamma\, \sigma^4 \right.$$

$$\left. + \tfrac{16}{45} \epsilon^2 \sigma^6 - \tfrac{32}{315} \epsilon^3 m_\gamma^{-1} \sigma^8 + \cdots \right],$$

$m_\gamma \sim m_W\, e^{-4\pi/\lambda}$,     $\epsilon \sim e^{-4\pi/\lambda}$,  center symmetry: $\sigma \to -\sigma$

3D relativistic power counting: $\sigma^4$ is relevant, $\sigma^6$ marginal, $\sigma^8$ irrelevant.

[gap exists]+ [$\epsilon \ll 1$]  $\Rightarrow$ interactions negligible?

Indeed, they seem to be completely ignored in literature.

Polyakov 1977…

But they produce bound states!

# Non-relativistic effective field theory

Expect binding energy $\ll m_\gamma$, so take non-relativistic limit

$$\sigma = (2m_\gamma)^{-1/2} e^{-im_\gamma t} \Sigma + (\text{h.c.})$$

$$S_{\text{3D,NR}} = \int dt\, d^2x \left[ \Sigma^\dagger \left( i\partial_t + \frac{\nabla^2}{2m_\gamma} \right) \Sigma + \frac{\epsilon}{m_\gamma} (\Sigma^\dagger)^2 \Sigma^2 \right.$$
$$\left. - \frac{8\epsilon^2}{9m_\gamma^3} (\Sigma^\dagger)^3 \Sigma^3 + \cdots \right].$$

Non-relativistic power counting

$[t] = -2 \,,\ [x] = -1 \,,\ [\Sigma] = (d-1)/2 \,,\ [m] = 0$

# Non-relativistic effective field theory

Expect binding energy $\ll m_\gamma$, so take non-relativistic limit

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Non-relativistic power counting

$[t] = -2\ ,\ [x] = -1\ ,\ [\Sigma] = (d-1)/2\ ,\ [m] = 0$

Marginal interactions in QM

1d: none ,   2d: $\Sigma^6$,   3d: $\Sigma^4$,   4d: $\Sigma^{10/3}$,   5d: $\Sigma^3$, …

3d power counting: $|\Sigma|^6$ is irrelevant, **$|\Sigma|^4$ is marginal!**

# Non-relativistic effective field theory

$$S_{\text{3D,NR}} = \int dt \, d^2 x \left[ \Sigma^\dagger \left( i\partial_t + \frac{\nabla^2}{2m_\gamma} \right) \Sigma + \frac{\epsilon}{m_\gamma} \, (\Sigma^\dagger)^2 \, \Sigma^2 \right.$$

$$\left. - \frac{8\epsilon^2}{9m_\gamma^3} \, (\Sigma^\dagger)^3 \, \Sigma^3 + \cdots \right].$$

Coefficient of l$\Sigma$l$^4$ , $\epsilon$ , runs with energy scale:

$$\mu \frac{d \, \epsilon(\mu)}{d\mu} = -\frac{1}{\pi} \, \epsilon(\mu)^2$$

Follows from one-loop analysis

Or from Schroedinger problem with delta function interaction.

Asymptotic freedom, and hence "strong coupling" at long distances!

# Light bound states

Strong coupling in IR = disaster in relativistic QFT.  But
here it occurs in non-relativistic regime. This saves us!

Can sum all Feynman diagrams exactly in 2-particle sector,
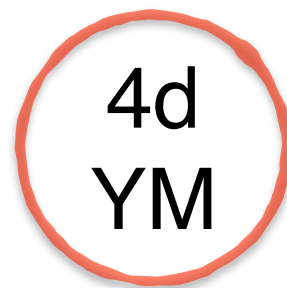$\Rightarrow$ solve for bound states!

In fact this is how Schrodinger equation is usually derived from QFT.

But even though here asymptotic freedom is
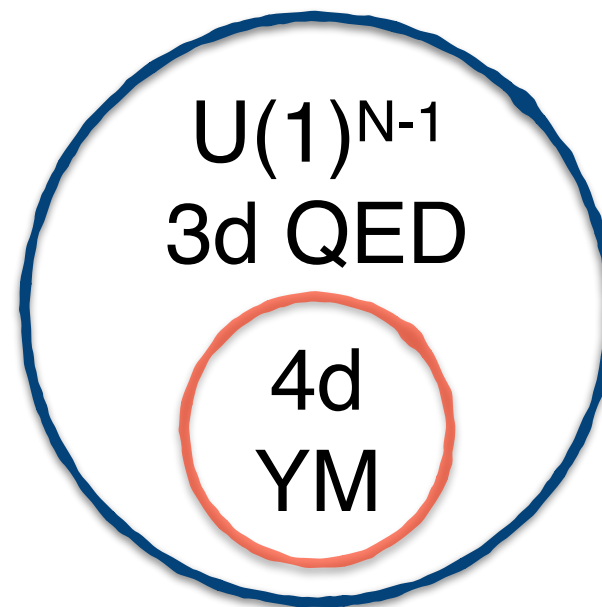harmless, it's quite funny that it appears!
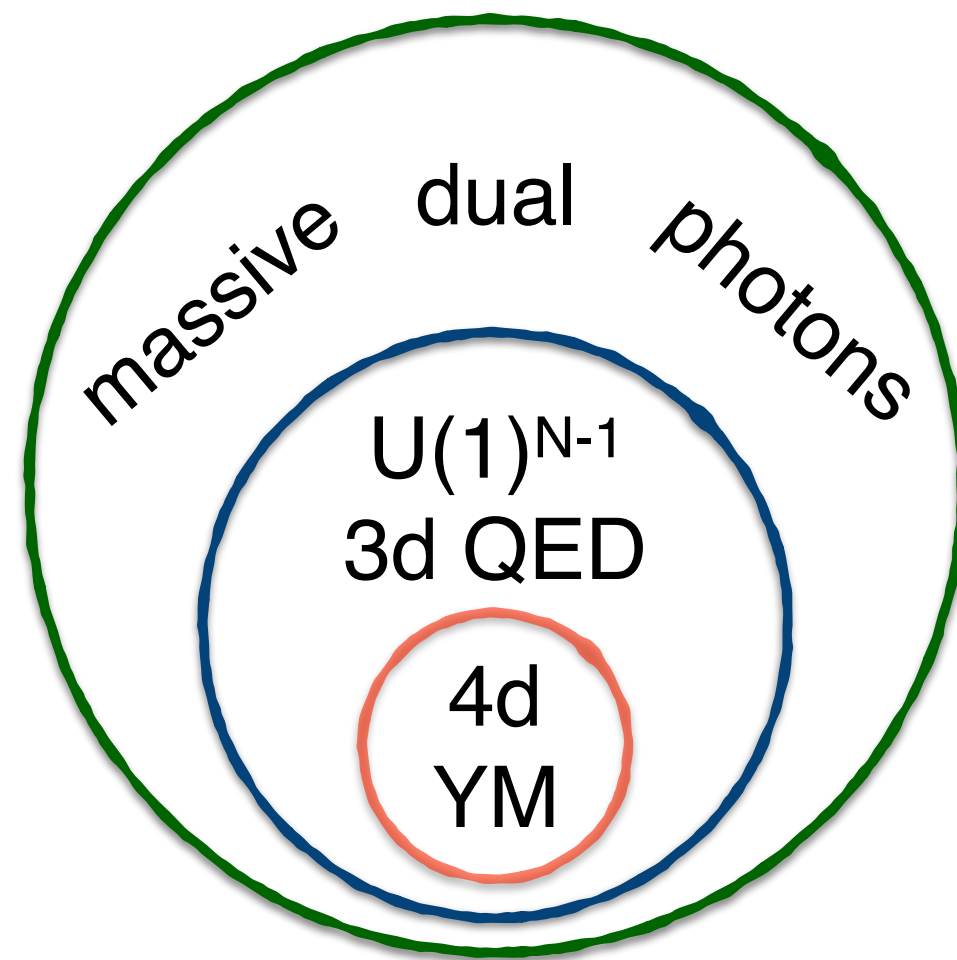
# Effective field theory onion
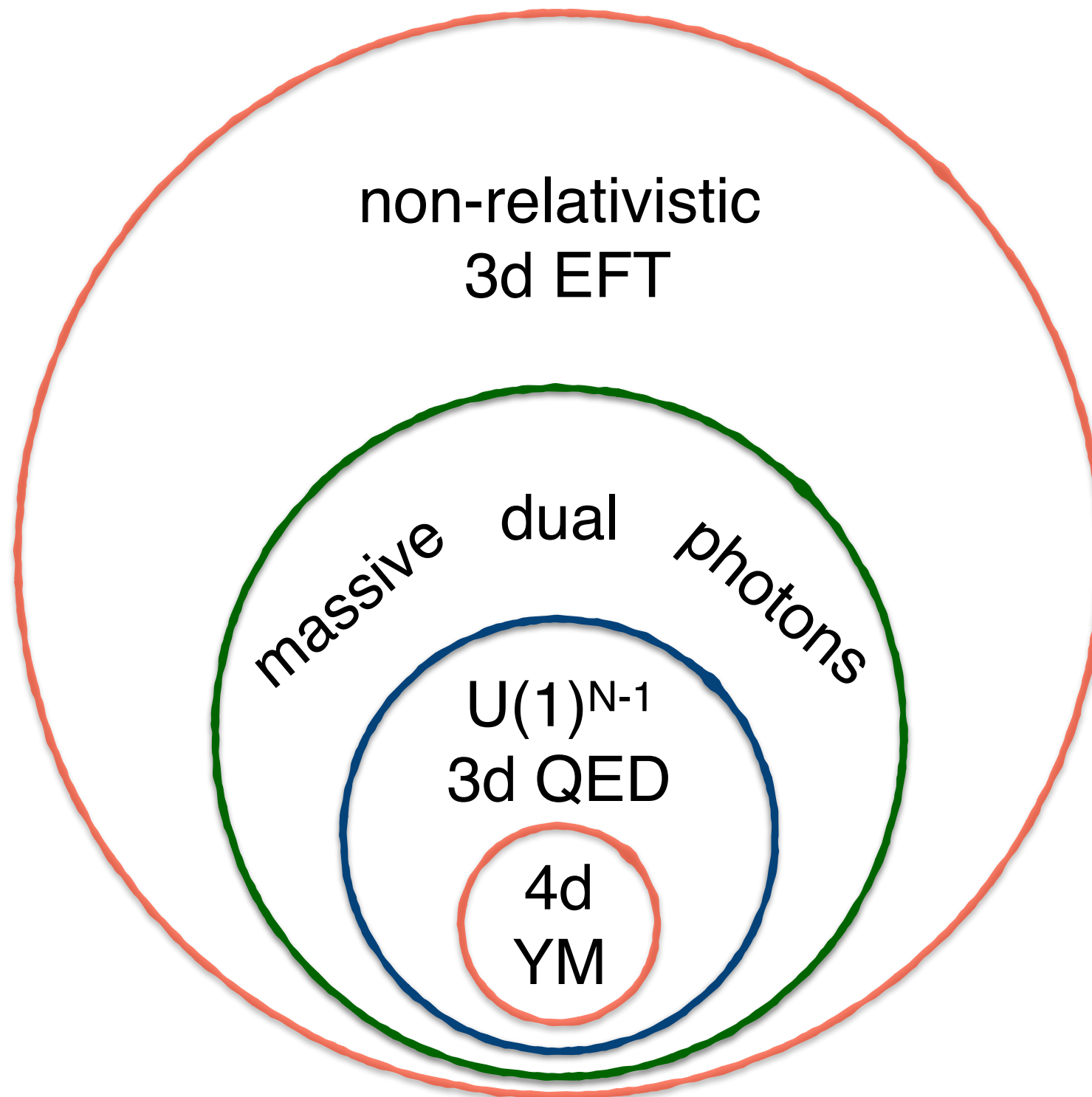
# Effective field theory onion



4d
YM

# Effective field theory onion

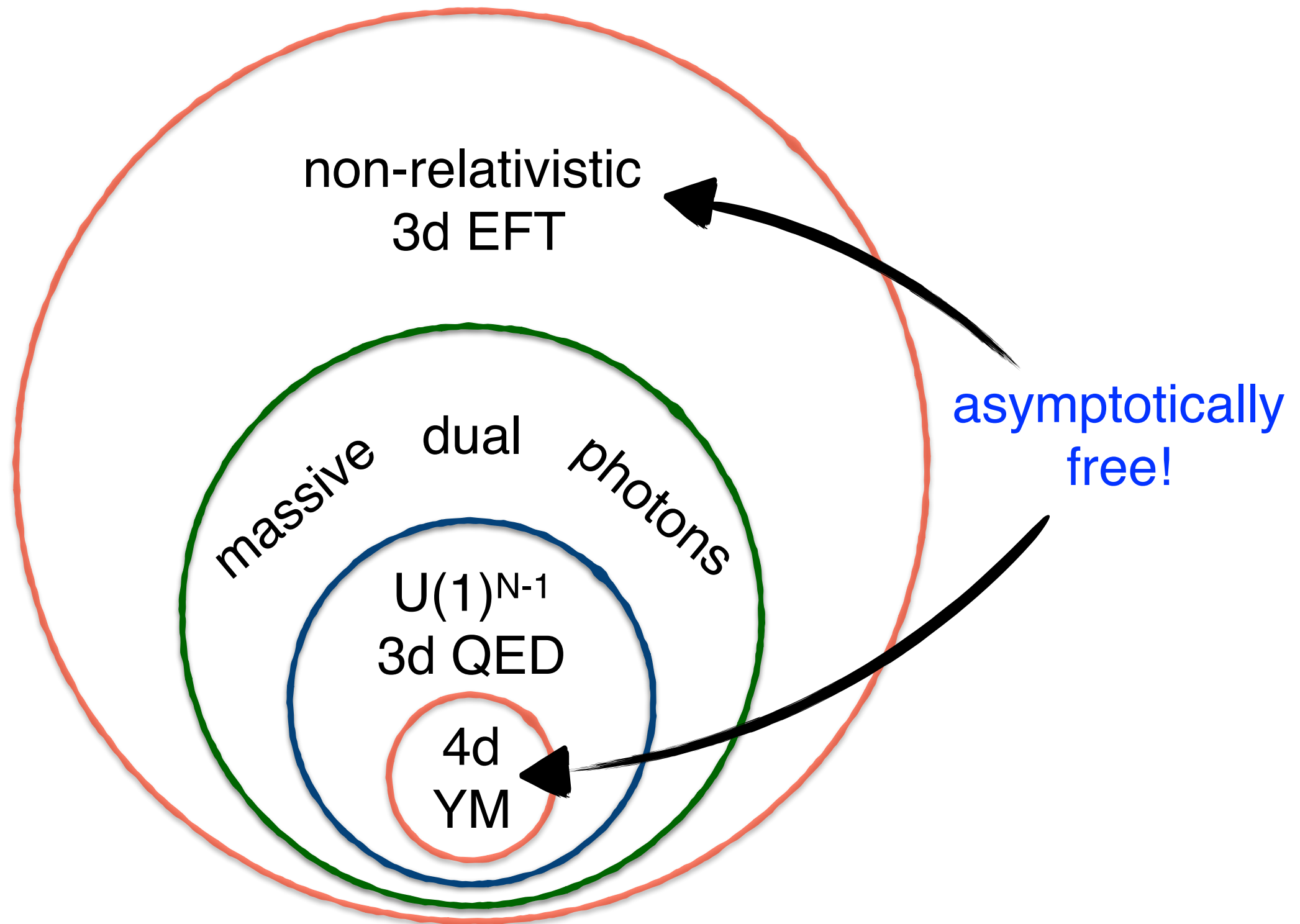# Effective field theory onion

# Effective field theory onion

# Effective field theory onion

non-relativistic
3d EFT

massive dual photons

$U(1)^{N-1}$
3d QED

4d
YM

asymptotically
free!

# Light bound states

Binding energies ~ (strong-coupling momentum)²/$m_\gamma$

$$\Lambda_{\mathrm{IR}} = \mu_{\mathrm{UV}} e^{-\frac{\pi}{\epsilon(\mu_{\mathrm{UV}})}} \ , \ \mu_{\mathrm{UV}} \sim m_\gamma \ , \ \epsilon(\mu_{\mathrm{UV}}) \sim e^{-\frac{4\pi^2}{\lambda}}$$

$$\textcolor{red}{\frac{\Delta E_2}{m_\gamma} \sim -\exp\left(-C\,\lambda^{5/2}\,e^{4\pi^2/\lambda}\right),}$$

When N = 2, two-photon bound state is center neutral!

# Light bound states

Analogous results for any N.
Always get some center-neutral bound states.

$$\text{mass} \sim e^{-4\pi/\lambda} \left( 1 - e^{-C\lambda^{5/2} e^{+4\pi/\lambda}} \right)$$

Earlier belief that dual photon mass scale
~ lightest states that extrapolate to large L was correct…

… but non-trivially so:

$$\log \left[ \langle \text{tr} \, F^2(x) \text{tr} \, F^2(0) \rangle \right] = \ldots + e^{-e^{+1/\lambda}} (\cdots) + \ldots$$

Should include iterated exponentials in transseries of YM!

# Conclusions

∃ small L limit of YM which is smoothly connected to
large L, according to all available evidence.

So we can do systematic semiclassics for low-energy observables.

Physical insights:

In YM, insights into confinement, θ dependence,
renormalons, Hagedorn behavior, large N limit, …

In QCD, insights into XSB, critical behavior, symmetries, …

Mathematical insight:

Direct computation revealing iterated-exponential
coupling dependence in confining gauge theories!