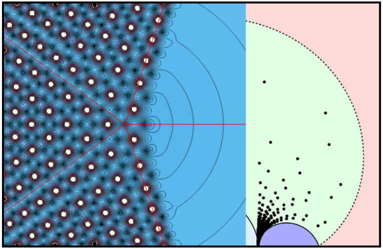
Physics and Mathematics of 2d Gravity: Stokes and Large N Anti-Stokes



Inês Aniceto

(Jagiellonian University, Krákow)

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This presentation is based in work in progress (to appear), from an on-going collaboration with Ricardo Schiappa and Marcel Vonk

It directly links to the results just presented by Marcel Vonk.

Goal: get a **full** understanding of the physics and mathematics encoded in resurgent asymptotic (trans)series.

Outline

1. Motivation
2. From local to global: gluing sectorial solutions in Painlevé I
    - Stokes transitions
    - Resonance
3. From local to global: off-criticality and the quartic matrix model
    - Phase diagram: evidence of different phases
    - Building one-parameter family of solutions
    - Analytic transseries summation: from Stokes to anti-Stokes
    - Analytic data: determining zeros of partition function
4. Conclusions

Next

1. Motivation

Painlevé I, 2d Gravity and Matrix models

- Matrix models:
  - NP description of string theory in simpler backgrounds: non-critical strings and Dijkgraaf-Vafa type topological strings[Dijkgraaf,Vafa '02]
  - Simper models for studying NP structure behind large N 't Hooft expansions
  - Can help us understand large-N duality
- 2d quantum gravity is obtained by taking a double scaling limit: large N and small coupling g_s[Douglas,Shenker '90][Brézin,Kazakov '90][Gross,Migdal '90]
- Free energy of 2d gravity related to the Painlevé I NLODE
  - $u(z) = -F''(z)$ where $z^{-5/4} \sim g_s$.
- Study Painlevé I: simpler model, but already showing major features from string theory
  - Asymptotic series with $(2g)!$ growth $\Rightarrow$ g_s^2 expansion

Global vs local

What we have seen so far (from Marcel Vonk's talk):

- Transseries as the most general formal solution of the Painlevé I equation
- Analysed sectorial solutions within Stokes wedges, with different asymptotic behaviour, generally dependent on two parameters
- Boundary data determines fully a particular sectorial solution
- Use analytic transseries summation to obtain analytic data: Lee-Yang zeros within these sectors

Our present goal:

- How to construct a **global** solution, i.e., how to "glue" the different sectors?
- How to generalise these results beyond Painlevé I, i.e. **off-criticality** in the context of matrix models?

Next

2. Global solutions of Painlevé I

Review: general solution for Painlevé I

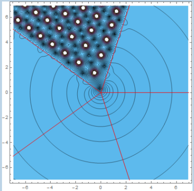
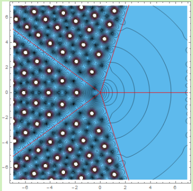
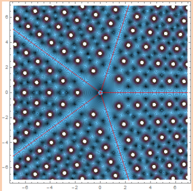
Use a **2-parameter transseries**: [Garoufalidis,Its,Kapaev,Mariño '10] [IA,Schiappa,Vonk '11]

$$u(x;\sigma_1,\sigma_2) = x^{-\frac{2}{5}} \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} L_{nm}(x;\sigma_1,\sigma_2)\,\sigma_1^n \sigma_2^m \,\mathrm{e}^{-\frac{(n-m)A}{x}}\, x^{\beta_{nm}}\,\Phi^{(n|m)}(x)$$

- Two instanton actions $A = \pm 8\sqrt{3}/5$: evidence of resonance, many sectors with same exponential grading
- $x = z^{-5/4} \sim g_s$ is open string coupling; σ_i are boundary data
- Asymptotic series: $\Phi^{(n|n)}(\mathrm{x})$ have a topological genus expansion (g_s^2), $\Phi^{(n|m)}$, $n \neq m$ have expansions in g_s: evidence of resonance
- Logarithmic sectors (from resonance) not independent, to be summed away:

$$L_{nm}(x;\sigma_1,\sigma_2) = \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\frac{2}{\sqrt{3}} (m-n)\,\sigma_1\sigma_2 \log x \right)^k$$

From local to global: Painlevé I

	0 parameter	1 parameter	2 parameter
Painlevé I	Tritronquée 	Tronquée 	General 
Quartic MM	?	?	?

- Sectorial solutions in Painlevé I (from Marcel Vonk's talk):
  - Specified by boundary data σ_i
  - Analytic transseries summation $\Rightarrow$ analytical data: Lee-Yang zeros
- Global solutions: "glue" different sectors together
  - Stokes transitions and resonance

Stokes transitions

$$u(x,\boldsymbol{\sigma}) = \sum_{\mathbf{n}\in\mathbb{N}_0^2} \boldsymbol{\sigma}^{\mathbf{n}} \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}/x} \Phi_{\mathbf{n}}(x), \quad \mathbf{A} \equiv (A,-A),\ \boldsymbol{\sigma}^{\mathbf{n}} \equiv \sigma_1^n \sigma_2^m$$

- Specific boundary data $\boldsymbol{\sigma}$ fully defines a sectorial solution, valid on a Stokes wedge. Different $\boldsymbol{\sigma}$ lead to different solutions and asymptotics.
- Global solution will have different asymptotic properties in different sectors, or Stokes wedges.
- Thus **Stokes phenomena** occurring at Stokes lines translates to changes in the $\boldsymbol{\sigma}$: it propagates the boundary data from one wedge to another.
- These changes are encoded in the **Stokes transitions**, and depend generically on a infinite set of **Stokes data**.

Stokes transitions

$$u(x,\boldsymbol{\sigma}) = \sum_{\mathbf{n}\in\mathbb{N}_0^2} \boldsymbol{\sigma}^{\mathbf{n}} \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}/x} \Phi_{\mathbf{n}}(x), \quad \mathbf{A} \equiv (A,-A),\ \boldsymbol{\sigma}^{\mathbf{n}} \equiv \sigma_1^n \sigma_2^m$$

- Stokes lines in Painlevé I at $\arg x = 0, \pi$ (dictated by the instanton actions)
- How does Stokes transition at $\arg x = 0$ look like? Take $\sigma_2 = 0$

$$\sigma_1 \to \sigma_1 + S_1 \quad ; \qquad \sigma_2 = 0.$$

$S_1 = -\mathrm{i}\frac{3^{1/4}}{2\sqrt{\pi}}$ is the well known Stokes constant of Painlevé I

- A general Stokes transitions looks like:

$$\sigma_1 \to \mathbb{S}_1(\sigma_1,\sigma_2)\ ; \qquad \sigma_2 \to \mathbb{S}_2(\sigma_1,\sigma_2)$$

where $\mathbb{S}_i$ depend on the original boundary data and a collection of problem specific numbers: **Stokes data**

Stokes transitions for the full two-parameter solution are completely determined by the collection of Stokes data.

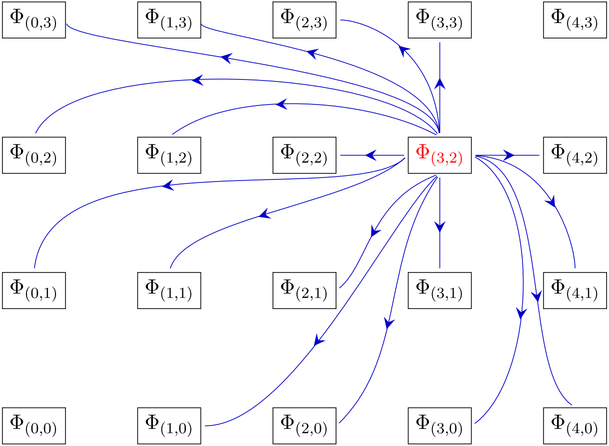
Stokes transitions

$$u(x, \boldsymbol{\sigma}) = \sum_{\mathbf{n} \in \mathbb{N}_0^2} \boldsymbol{\sigma}^{\mathbf{n}} \mathrm{e}^{-\mathbf{n} \cdot \mathbf{A}/x} \Phi_{\mathbf{n}}(x)$$

- Resurgent relations between sectors give rise to contributions to Stokes transitions
- These contributions are dictated by Stokes data
- Stokes data organises into vectors $\mathbf{S}_{\boldsymbol{\ell}}$, parametrized by vectors $\boldsymbol{\ell}$ in a $\mathbb{Z}^2$ (semi-)lattice
- Stokes data is fundamental to each problem, but usually only accessible numerically
- Are they all independent in Painlevé I? No! **Resonance**

$$\begin{tikzcd} \boxed{\Phi_{\boldsymbol{n}}} \arrow[d, "\sim \boldsymbol{S}_{\boldsymbol{\ell}}"] \\ \boxed{\Phi_{\boldsymbol{n}+\boldsymbol{\ell}}} \end{tikzcd}$$

Example: contributions from $\Phi_{(3,2)}$ to Stokes transitions

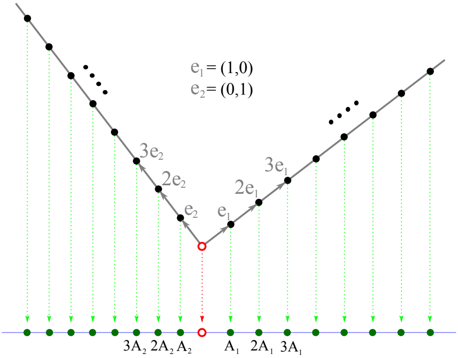


- All contributions: **fundamental** (weighed by $\mathbf{S}_\ell$), plus iterations

Stokes transitions, Borel singularities and resonance

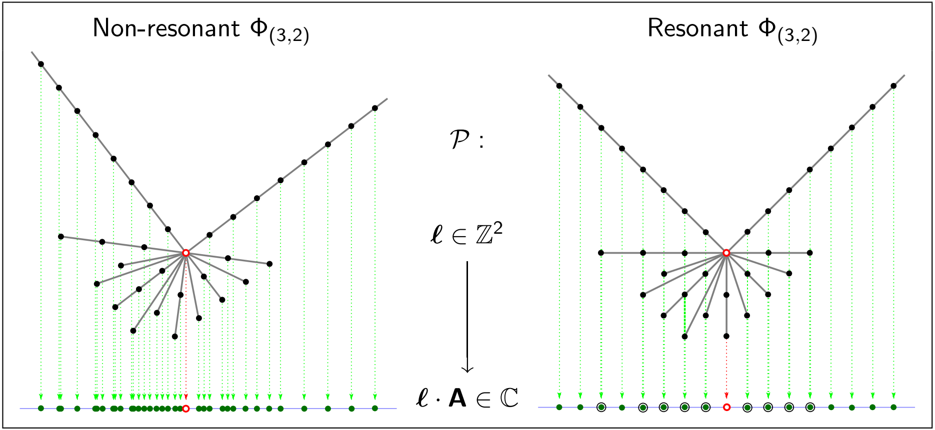
- Borel plane: singularities in Stokes directions are given by $\boldsymbol{\ell} \cdot \mathbf{A} \in \mathbb{C}$
- Contributions to Stokes transitions will collapse on different Stokes directions via the projection into Borel plane $\mathcal{P} : \boldsymbol{\ell} \to \boldsymbol{\ell} \cdot \mathbf{A}$

$$\mathcal{P}: \quad \begin{array}{ccc} \mathbb{Z}^2 & & \boldsymbol{\ell} \\ \downarrow & & \downarrow \\ \text{Borel plane } \mathbb{C} & & \boldsymbol{\ell} \cdot \mathbf{A} \end{array}$$



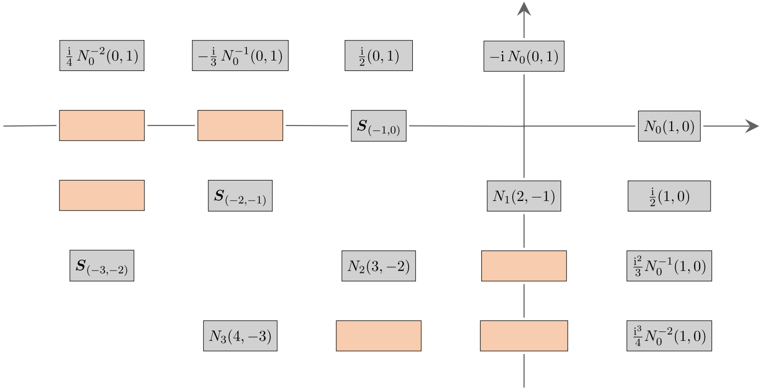
- Generally $\mathbf{A} = (A_1, A_2)$: each lattice point falls on a unique singularity
- Resonance: $\mathbf{A} = (A, -A)$: different lattice points go into same singularity

Resonance



- Non-resonant case: different lattice points $\boldsymbol{\ell}$ lead to different $\boldsymbol{\ell} \cdot \mathbf{A}$
- Resonant case: lattice points $\boldsymbol{\ell}' : \boldsymbol{\ell}' - \boldsymbol{\ell} \in \ker \mathcal{P}$ contribute to same $\boldsymbol{\ell} \cdot \mathbf{A}$
- This results in **many geometric relations constraining Stokes data**. We have a residual set of independent numbers.

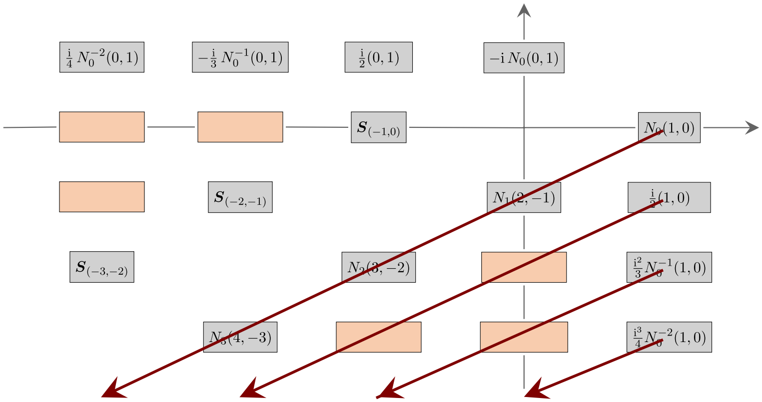
The Stokes vectors of Painlevé I



Residual unconstrained data: $N_0, N_1, \cdots, N_k, \cdots$

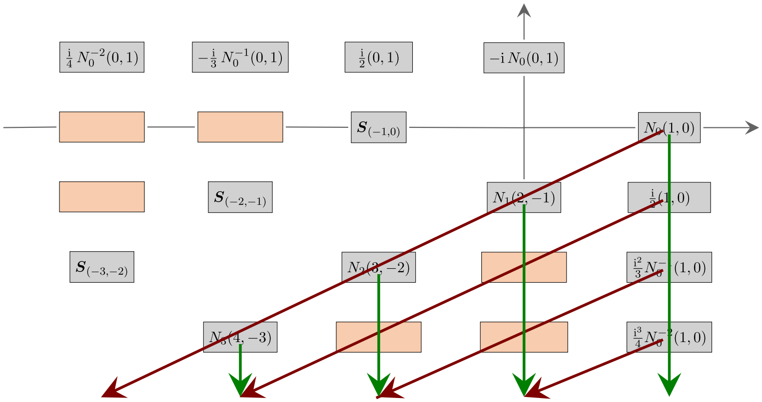
E.g. $\mathbf{S}_{(-2,-1)} = \left(\frac{\mathrm{i}}{8} N_0^{-3} - \frac{1}{2} N_0^{-2} N_1 - \mathrm{i} N_2\right)(-2,3)$

The Stokes vectors of Painlevé I



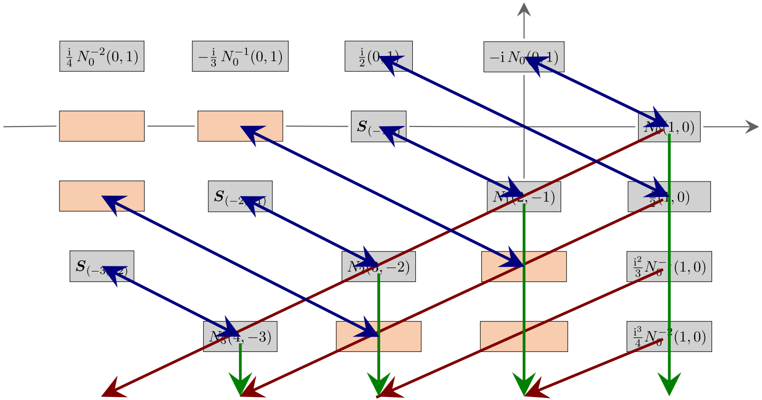
Resonance

The Stokes vectors of Painlevé I



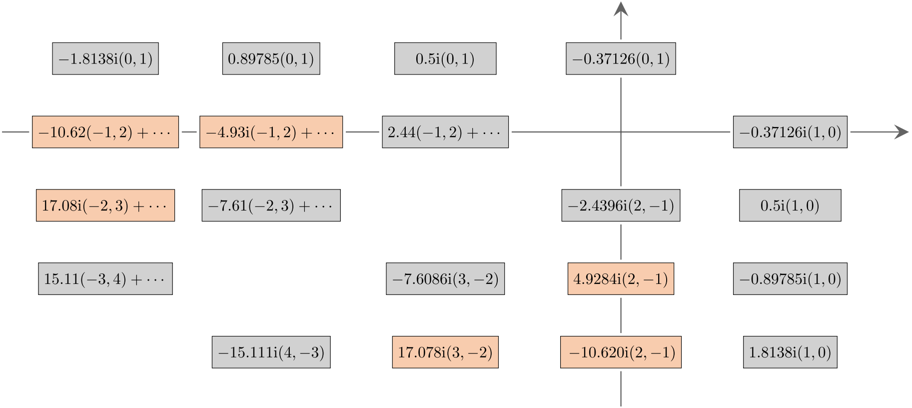
Resonance **Pattern**

The Stokes vectors of Painlevé I



Resonance **Pattern** **Conjugation+shift**

The Stokes vectors of Painlevé I



The geometric relations translate into knowledge of all Stokes data!
Starting data N_k, $k \in \mathbb{N}_0$

$\Longrightarrow$ construct global solutions of Painlevé I

Next

3. Off-criticality: large N asymptotic analysis of quartic matrix model

From local to global: off-criticality

- Analyse the large N asymptotic expansions of matrix model observables using transseries, resurgence analysis and summation
- Start with the local formal *transseries* solutions for large N free energies of the quartic MM, determined in "Stokes regions": the 1-cut [Mariño '08][IA,Schiappa,Vonk '11] and 2-cut [Schiappa,Vaz '13] backgrounds
- In [Couso-Santamaría,Schiappa,Vaz '15]: used Borel-Padé-Écalle summation of transseries to obtain finite N results

We want to understand the large-N physics and corresponding string dualities.

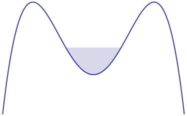
- Large-N structure:
  - construct full phase diagram of matrix model;
  - recover analytic data from analytic transseries summation;
  - connect different sectorial solutions with Stokes transitions

Quartic matrix model

Quartic model partition function ($N \times N$ matrix M)

$$\mathcal{Z}(N, g_s) \propto \int dM \exp\left(-\frac{1}{g_s}\mathrm{Tr}\, V(M)\right), \quad V(z) = \frac{1}{2}z^2 - \frac{1}{24}\lambda z^4$$

Local solutions in "Stokes regions": saddle point analysis around **1-cut solution**



Free energy has perturbative genus expansion at large N

$$F \equiv \log Z \simeq \sum_{g \geq 0} F_g(t)\, g_s^{2g-2}, \; t = g_s N$$

- Obey a NP **finite difference eq**: string equation
- **Resurgent properties**: 2 parameter transseries, instanton action and coefficients of transseries are functions!
- **Double-scaling limit**: recover the results for Painlevé I

Quartic matrix model

Quartic MM partition function:

$$\mathcal{Z}(N, g_s) \propto \int dM \, \exp\left(-\frac{1}{g_s}\mathrm{Tr}\, V(M)\right) , \quad V(z) = \frac{1}{2}z^2 - \frac{1}{24}\lambda z^4$$

- Introduce orthogonal polynomials $\{p_n(z)\}$, with normalisations h_n
- Define $r_n = \frac{h_n}{h_{n-1}}$, obeying the finite N relation

$$\mathcal{Z}_N \propto \prod_{n=1}^{N} r_n^{N-n}, \quad r_n = \frac{\mathcal{Z}_{n+1}\mathcal{Z}_{n-1}}{\mathcal{Z}_n^2}$$

- Recursion equations of the $\{p_n(z)\}$ lead to NL string equation for the r_n:

$$\mathcal{R}(t)\left(1 - \frac{\lambda}{6}(\mathcal{R}(t-g_s) + \mathcal{R}(t) + \mathcal{R}(t+g_s))\right) = t \, , \quad \mathcal{R}(n\, g_s) = r_n$$

  where and $\mathcal{R}(t)$ is directly related to the free energies

- Comparison with Painlevé I: $\mathcal{Z}_{\mathrm{MM}} \leftrightarrow \mathcal{Z}_{\mathrm{PI}}$; $\mathcal{R} \leftrightarrow u$

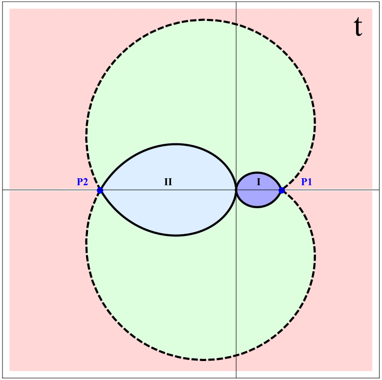
Quartic matrix model

$\mathcal{R}(t)$ has resurgent properties, with transseries solution:

$$\mathcal{R}(t,\sigma_1,\sigma_2) = \sum_{n,m\geq 0} \sigma_1^n \sigma_2^m \mathrm{e}^{-N\,(n-m)\,\frac{A(t)}{t}}\, t^{\beta_{nm}} R_{(n|m)}(t)$$

- $R_{(n|m)}(t)$ asymptotic expansions
- Instanton action $A(t)$ and coefficients $R_g^{(n|m)}(t)$ are functions.
- Large-N phase diagram (first studied in [Bertola '07,Bertola,Tovbis '11]): study the leading contributions to the exponentials, given by $\frac{A(t)}{t}$:
  - Stokes lines $\mathbb{Im}\,(A\,(t)\,/t) = 0$: instanton contributions maximally suppressed
  - Anti-Stokes lines $\mathbb{Re}\,(A\,(t)\,/t) = 0$: all contributions of same order

Phase Diagram



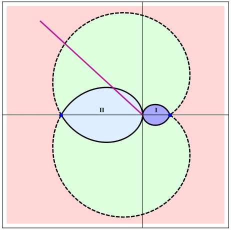
- light blue: Stokes regions, standard 't Hooft large N expansion
  - I: 1-cut solution is dominant
  - II: 2-cut sym solution dominant
- green: anti-Stokes region,dominated by 3-cuts solution, modular properties; no genus expansion [Bonnet,David,Eynard '00]
- light red: trivalent tree-like configuration dominant

- Re line in I and II: Stokes lines, exponentially suppressed saddles are maximally suppressed
- P1 (P2): DS point described by Painlevé I (II) equation

Evidence of different phases?
What local solutions are associated with each phase?
How to obtain analytic data? Global Solutions?

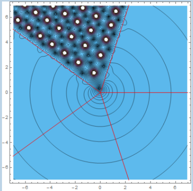
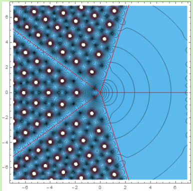
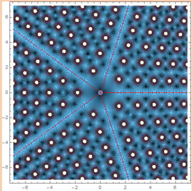
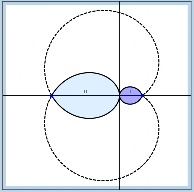
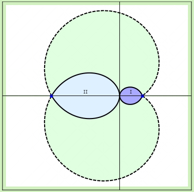
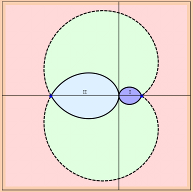
Eigenvalue distribution: numerical evidence

- Calculate and plot the position of zeros of $p_N(z)$: for large enough N it will reproduce correct density of eigenvalues
- keep $\arg t \in]\frac{\pi}{2}, \pi[$ fixed; changing $|t|$: eignevalues move in complex z-plane



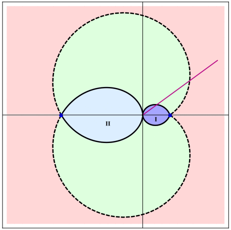
- Stokes region: dominant structure is two-cut
- Anti-Stokes region: distributions along 3 cuts of comparable length
- Trivalent phase: tree-like configurations
- Organization of eigenvalues is markedly different, but do these lead to different physical phases?

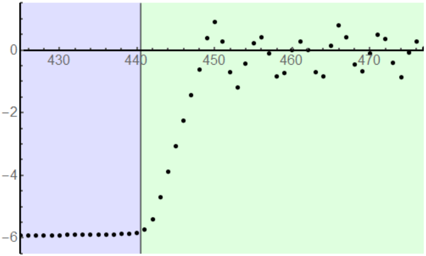
Sectorial solutions

	0 parameter	1 parameter	2 parameter
Painlevé I	Tritronquée 	Tronquée 	General 
Quartic MM	Stokes (1 and 2 cuts) 	Anti-Stokes 	Trivalent 

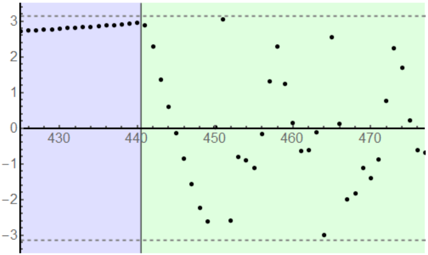
The anti-Stokes phase: numerical evidence

- Numerically calculate the recursion coefficients r_n with the boundary condition corresponding to 1-cut configuration
- Take $N = 1000$ $\arg t = \frac{\pi}{12}$ fixed, change $|t|$ from the 1-cut phase into anti-Stokes
- r: normalization factor (classical solution $g_s = 0$)





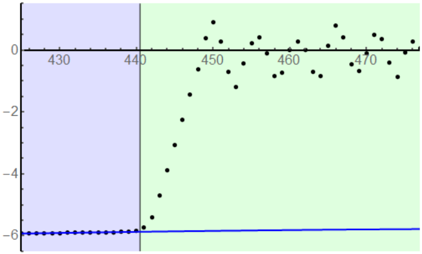
$(r_n - r)$: log of absolute value



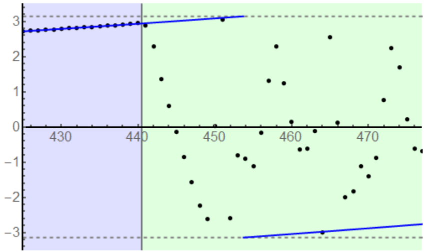
$(r_n - r)$: phase

The anti-Stokes phase: numerical evidence

- Perform optimal truncation to the one-parameter sectors of $\mathcal{R}(t, \sigma_1, 0)$:
  - perturbative $R_{(0,0)}(t)$
- Compare to the numerical results for the r_n



$(r_n - r)$ vs $(R_{pert} - r)$: log of absolute value

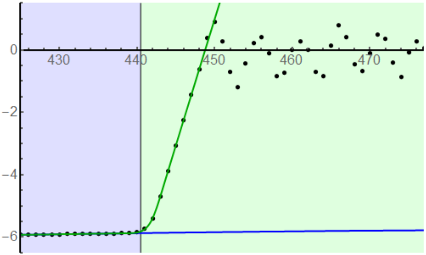


$(r_n - r)$ vs $(R_{pert} - r)$: phase

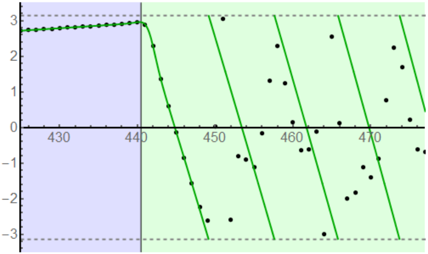
The summation of the perturbative expansion leads to very good results in the Stokes region, but it stops at the anti-Stokes boundary, where the instanton contributions become of the same order

The anti-Stokes phase: numerical evidence

- Perform optimal truncation to the one-parameter sectors of $\mathcal{R}(t, \sigma_1, 0)$:
  - perturbative $R_{(0,0)}(t)$ plus 1-instanton $R_{(1,0)}(t)$
- Compare to the numerical results for the r_n



(r_n−r) vs (R_{1-inst}−r): log of absolute value

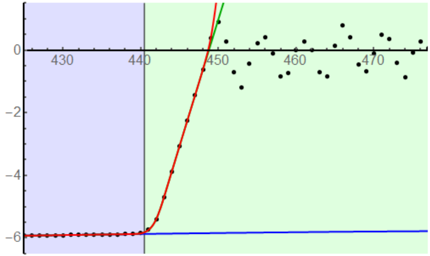


(r_n−r) vs (R_{1-inst}−r): phase

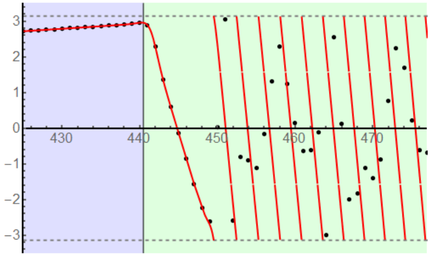
Adding the first instanton correction to the $\mathcal{R}(t)$ has corrected part of the first oscillation, still very close to the anti-Stokes boundary

The anti-Stokes phase: numerical evidence

- Perform optimal truncation to the one-parameter sectors of $\mathcal{R}(t, \sigma_1, 0)$:
  - perturbative $R_{(0,0)}(t)$ plus n-instantons $R_{(n,0)}(t)$, for $n = 1, 2, 3$
- Compare to the numerical results for the r_n



(r_n−r) vs ($R_{3-\text{inst}}$−r): log of absolute value



(r_n−r) vs ($R_{3-\text{inst}}$−r): phase

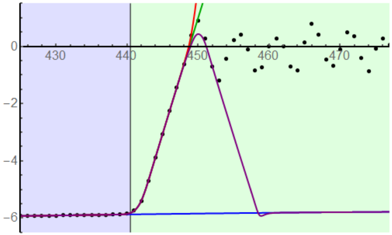
Adding the first three instanton correction to the $\mathcal{R}(t)$, has now produced a very small difference, we can reach one extra data point: all instanton contributions are of the same order and need to be included

The anti-Stokes phase

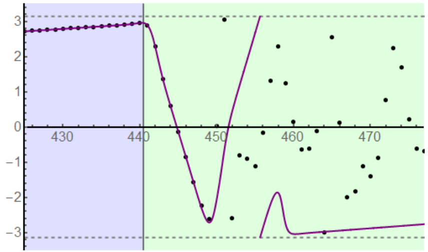
- Evidence that indeed the **different phases are physical**: they lead to different asymptotics of the $\mathcal{R}(t)$ in different regions
- Still, we cannot yet reach far into the anti-Stokes region
- Can we do better? Perform **analytic transseries summation** (as in Marcel Vonk's talk)

Linear analytic transseries summation

- Learn from the example of Painlevé I (Marcel Vonk's talk) and start by doing linear analytic transseries summation
- Sum the leading terms in g_s in the one-parameter transseries ($\sigma_2 = 0$)



($r_n - r$) vs ($R_{\text{lin.ATS},1} - r$): log of absolute value

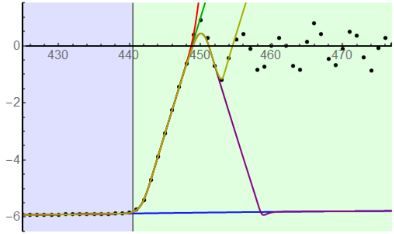


($r_n - r$) vs ($R_{\text{lin.ATS},1} - r$): phase

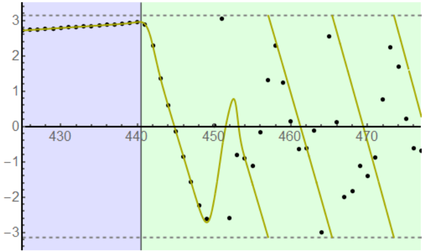
Leading g_s linear transseries summation for $\mathcal{R}(t)$ shows definite improvement, mostly for the absolute value results.

Linear analytic transseries summation

- Learn from the example of Painlevé I (Marcel Vonk's talk) and start by doing linear analytic transseries summation
- Leading and subleading terms in g_s in the one-parameter transseries



$(r_n - r)$ vs $(R_{\text{lin.ATS},2} - r)$: log of absolute value



$(r_n - r)$ vs $(R_{\text{lin.ATS},2} - r)$: phase

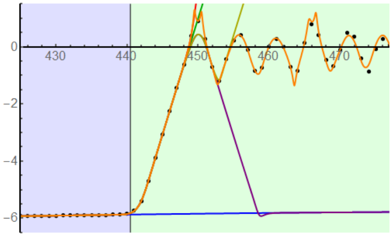
Leading and subleading g_s linear transseries summation for $\mathcal{R}(t)$ shows that we can go further into the anti-Stokes region, but slow improvement.

Can we do better?

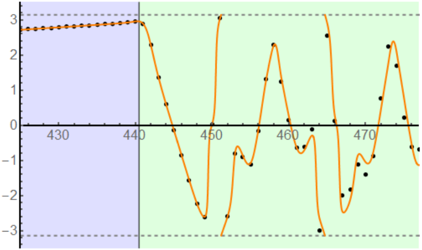
- Optimal truncation did not correctly recover the results in the anti-Stokes phase. The sum over instanton corrections is insufficient as all corrections are of the same order.
- Linear analytic transseries summation recovered results further into the anti-Stokes region, but subleading g_s corrections give slow improvement
- Can we do better? From Marcel Vonk's talk, we should perform **quadratic analytic transseries summation** or the partition function

Quadratic analytic transseries summation

- Perform quadratic analytic transseries summation for the one-parameter partition function
- Sum the leading terms in g_s for $\mathcal{Z}(t)$ ($n-$th instanton sector starts at order $g_s^{n^2}$)
- Determine the $\mathcal{R}(t)$ from these results



($r_n - r$) vs ($R_{\text{quad.ATS},1} - r$): log of absolute value



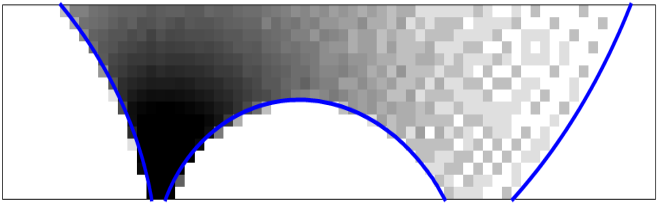
($r_n - r$) vs ($R_{\text{quad.ATS},1} - r$): phase

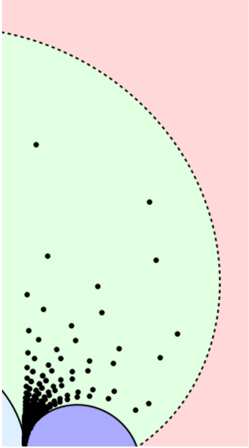
Leading g_s quadratic transseries summation for $\mathcal{Z}(t)$ follows the numerical results far into the anti-Stokes region!

Zeroes of the partition function

Use the quadratic analytic transseries summation to predict Lee-Yang zeros?

- **Left:** prediction of zeros of $\mathcal{Z}(t)$ obtained from quadratic analytic transseries summation with $N = 10$ eigenvalues
- **Down:** numerical calculation of zeros from direct calculation of the matrix integral ($N = 100$). The grayscale is proportional to number of zeros





Leading g_s quadratic transseries summation for $\mathcal{Z}(t)$ predicts analytic results deep into the anti-Stokes region!

Summary

Global solutions of Painlevé I

- "Glue" the sectorial solutions with Stokes transitions
- Use resonance to determine geometric relations constraining Stokes data

Asymptotic large-N analysis of quartic matrix model

- Study of different asymptotics across the complex 't Hooft coupling
- Summation of local solutions: optimal truncation, linear and quadratic analytic transseries summation
- Lee-Yang zeros prediction in the anti-Stokes region

Current work

Stokes phenomena in Painlevé I:

- Full geometric structure of Stokes data
- Determining the residual Stokes data analytically
- Modular properties of the analytic transseries summation

Stokes phenomena in quartic matrix model:

- Stokes transitions connecting local results from different regions
- Trivalent phase: full 2-parameter transseries solutions

Future applications

- Our analysis only needed the weak coupling expansions for the transseries.
- Using analytic transseries summation we can recover results everywhere in the 't Hooft complex plane, for arbitrarily large coupling.
- The methods we used are generic, and could be applied to more complex systems
  - e.g. HAE: the transseries for topological strings in particular backgrounds is already known[Couso-Santamaría *et al* '14,15,16], which can be matched to non-perturbative definitions of string theory [Couso-Santamaría,Mariño,Schiappa '16][Codesido,Mariño,Schiappa].

Thank you!