Transserics

Transseries –viz. multi-instanton expansions, occur in singularly perturbed equations in mathematics and physics as combinations of formal power series and small exponentials. In normalized form, the simplest nontrivial transseries is

$$\sum_{k=0}^{\infty} e^{-kx}\Phi_k(x);\ \Phi_k = \sum_{j=0}^{\infty} \frac{c_{kj}}{x^j} \tag{1}$$

($x^{k\beta} \log x^j e^{-kx}$, more generally) where the c_{kj} grow factorially in j.

In physics Φ_0 is known as the perturbative series, $e^{-kx}\Phi_k$ are the nonperturbative terms and Φ_k are the fluctuations in the various instanton sectors.

In mathematics, Φ_0 is the asymptotic expansion and the Φ_k are the higher series in the transseries; e^{-x}, $1/x$ are called transmonomials; in math, the whole transseries is a **perturbative** expansion and the nonperturbative object is the generalized Borel sum of the transseries. General transseries involve iterated exponentials, but these very rarely show up in practice.

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Typically transseries are resurgent, in particular resummable.

Why are transseries & resurgence so universal?

Mathematically, transseries are provably closed under all operations (we) used in perturbative expansions [10]. While no similarly general theorem exists yet for summability, the proofs in the known types of problems almost fit one template.

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Closure $\Rightarrow$ universality. To remain in the space of convergent Taylor series ("viz." analytic functions) you must avoid divisions. If we allow for division we get (one-sided) Laurent expansions. But now we have to avoid integration, or allow for the log. If we allow for the log we need to avoid division+integration, since the asymptotic expansion of $\int(1/log)$ is factorially divergent.

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The miracle is that the proliferation of new objects stops here. Allowing also for $\exp = \log^{-1}$, nothing qualitatively new happens if we keep closing under operations (including infinite iterations involved in perturbation expansions). "In the limit" we get the transseries. Transseries are formally *sentences in a language where the words are* $1/x$, e^{-x}, $\log x$, *with an asymptotic rule of formation: terms are ordered decreasingly.*

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To go beyond transseries, one has to exit all operations used for solving ODEs, PDEs, integral equations, difference equations etc. (or regularity domain)

Transasymptotics [9] deals with the latter.

Resurgent transseries

Transseries arising in applications in analysis, mathematical physics and, at this stage conjecturally, numerically corroborated, in QCD, QFT, path integrals etc are **resurgent**. In particular this means that the series Φ_k are *generalized* Borel summable, and after generalized summation the transseries converges in the usual sense, giving a unique answer.

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A **resurgent function** (in "p plane", "Borel plane" or the "convolutive model") is the Borel transform ($\approx$ inverse Laplace transform) of a resurgent series. The Laplace transform of a resurgent function is also called resurgent (in the physical domain or "geometric model").

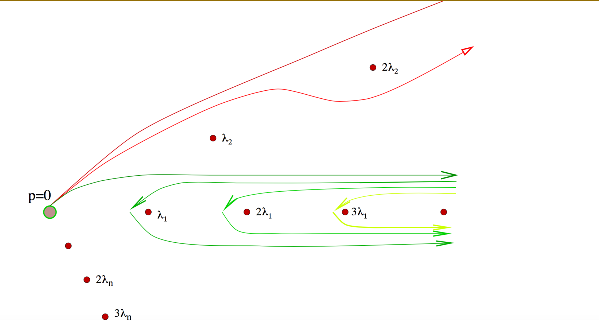

Figure: ◦ The red dots are singularities of a typical Borel transform of a resurgent series, $\mathcal{L}^{-1}\Phi_0$: poles or algebraic or logarithmic branch points. The Borel sum in direction α is the Laplace transform along a ray of angle $-\alpha$ [2].
◦ The branch jump $\Delta_k\mathcal{L}^{-1}\Phi_0$ at the k-th is $\sim \mathcal{L}^{-1}\Phi_k$, the Borel transform of the fluctuations in the k-th sector [2].
◦ Along singular lines there is a universal average of the analytic continuations (**medianization**) whose Laplace transform has all the good properties of the usual one (e.g., transforms convolution to product) [1,2]
◦ Resurgence $\Rightarrow$ linear relations among all singularities subject to **Alien calculus**. [1,2,13]. Example: $\Delta_j\Phi_k = \binom{k+j}{j} S^j \Phi^-_{k+j}$

Some (of many known) results

ODEs: resurgence of the transseries is known for systems of the type $y' = f(1/x, y)$, $y \in \mathbb{C}^n$, $x \to \infty$, f analytic at 0 in $(1/x, y)$ under a genericity condition (weaker than): the Jacobian $\frac{\partial f}{\partial(1/x,y)}\big|_{0,0}$ has nonresonant eigenvalues over $\mathbb{Q}$. The structure of singularities in Borel plane is known rigorously, as well as the resurgent relations among instantons, and medianization [2]. The resonant case is dealt with by **Écalle acceleration** (Braaksma-Ramis [4]) $\nexists$ similarly general results for resonant cases, but no conceptual difficulty is expected.

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PDEs Resurgence in t, of the propagator of time-periodic Schrödinger equations (ionization settings) [4] is well understood. Borel resummation of divergent expansions has been shown for fairly general systems of nonlinear evolution PDEs [5] $\partial_t f = \mathcal{E}(1/x, f)f + L$, $f \in \mathbb{C}^d$, $x \in \mathbb{C}^n$, $\mathcal{E}$ elliptic (N-S [8]).

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Integro-differential equations in hydrodynamics (Tanveer, [])

Some outstanding questions where resurgence was instrumental: Nonlinear stability of self-similar singularity formation in supercritical[1] Wave Maps and Yang Mills [8], the solution of the time-periodic Schrödinger equation in external fields which are $O(1)$ [4] proof of Dubrovin's conjecture (pole positions of special solutions of Painlevé P1) [9].

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Example of resurgent analysis: P1 [2] In normalized form, a modified Boutroux form, P1 reads

$$h'' - \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \quad (*)$$

When possible, instead of Borel transforming the asymptotic series we Borel transform the source of the series. The **Borel transform of (*)** is

$$H = (p^2 - 1)^{-1}\left(\frac{196}{1875}p^3 - \int_0^p sH(s)ds + \frac{1}{2}\int_0^p H(s)H(p-s)ds\right) \quad (**)$$

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We "see" that $p = \pm 1$ are singular points. Looking more carefully, both are $1/\sqrt{}$ branch points. If we iterate (**), convolution spreads these two singularities at all nonzero integers. Generated by convolution, the singularities are related to each-other. The above is mechanism is **typical** of any order ODEs.

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General (small) transseries in P1

The general solution decaying along $\mathbb{R}^+$ (the **tronquées**) depends on a constant C and has the transseries

$$h(C,x) = \sum_{k\geq 0} C^k h_k x^{-k/2} e^{-kx}$$

where h_k are (generalized) Borel sums of divergent series; the h_k satisfy linear nonhomogeneous second order ODEs. Across a Stokes line $C \to C + S$, where $S = i\sqrt{6/(5\pi)}$ is the Stokes constant [3].

[3] Calculated in closed form by isomonodromic deformations, recently by resurgence techniques; numerically, there are many methods.

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Resurgence. Let $H_k = \mathcal{L}^{-1} h_k$. The Borel plane jump at the j singularity of H_k is related to H_{k+j} through a formula independent of the ODE

$$(H_k^+ - H_k^-)_j = \binom{k+j}{j} S^j H_{k+j}^-$$

In particular, the whole structure of H_0 on the universal covering of $\mathbb{C} \setminus \mathbb{N}$ [4] is contained in H_k. Since it's all reduced to the first sheet, **endless continuation** also follows.

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Transasymptotics [5], [9] a sketch

We can view the transseries

$$h(x) = \sum_{j,k} c_{kj} C^k x^{-k/2} e^{-kx} x^{-j}$$

as a formal function of two variables $\xi = Cx^{-1/2}e^{-x}, \eta = 1/x$,

$$h(x) = F(\xi, \eta) = \sum_{k,j} c_{k,j} \xi^k \eta^j \quad (*)$$

When $\xi \ll \eta$ $(e^{-x} \ll 1/x)$, (*) was conveniently written in the standard "multiinstanton" form

$$\sum_{k \geq 0} h_k(\eta) \xi^k$$

[5] Instanton condensation!

However, when an antistokes line is approached (here $\pm i\mathbb{R}^+$, where the exponential becomes oscillatory), it is natural to write it in the form

$$(*)\ \sum_j F_j(\xi)\eta^k$$

Plugging (*) in P_1 and solving perturbatively in η we get

$$F_0(\xi) = \frac{\xi}{(\xi/12-1)^2}$$

and all F_k **are rational functions**. We see formation of singularities near antistokes lines, at the points $Ce^{-x}x^{-1/2} \approx 12$, infinitely many of them due to the periodicity of e^{-x}.

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More complex transasymptotic phenomena occur in PDEs [14].

A simple PDE example [15]

Simplest example: the heat equation, $f_t = f_{xx}$. Because the equation is parabolic, if we solve the initial value problem by a series expansion $f = \sum t^k f_k(x)$, $f_0 = f(0,x)$, the PDE implies $f_{k+1}(x) = f_k''(x)/k$, that is

$$f(x,t) = \sum_{k\geq 0} \frac{f_0^{(2k)}(x)}{k!} t^k$$

which diverges factorially even if f_0 is analytic (but not entire). Instead of Borel transforming the solution it is much better to Borel transform the equation, in $1/t$. This gives better analytic control, and more importantly we can allow non-analytic initial conditions. With $f(t,x) = t^{-1/2} g(1/t, x)$ and $\mathcal{L}^{-1}_{\frac{1}{t}} g(q) = q^{-1/2} G(x, 2q^{1/2})$, $2q^{1/2} = p$, the equation becomes

$$G_{pp} - G_{xx} = 0$$

the wave equation, for which power series solutions converge.

Cont, and more general PDEs

Using the elementary solution of the wave equation $G_1(x+p)+G_2(x-p)$ and the initial and boundary conditions, one gets, after returning to f by Laplace transform and changes of variables,

$$f(t,x) = t^{-1/2}\int_{-\infty}^{\infty} f(0,s)\exp(-(x-s)^2/(4t))ds$$

The point here, of course, is not to solve the heat equation in closed form. It is, rather, like in most applications of resurgence, to transform divergent series into convergent ones, more generally singular perturbations into regular perturbations. This approach allows $f(0,s)$ to be general, say in L^1 and also shows when resurgence is obtained: essentially iff $f(0,s)$ is analytic.

A conceptually similar approach applies to **very general systems of nonlinear PDEs** (Navier-Stokes included) [11,8], resulting in Laplace representations of actual solutions, proving (at least local) existence of solutions and the possibility to control solutions more globally.

Because of dependence on initial conditions, one studies resurgence of the **Green's function or of the unitary propagator**. Fairly well understood for time-periodic d-dim Schrödinger equations. In these models, the Borel sum of the series is insufficient; one needs the full transseries. [16]

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