WKB analysis via topological recursion for (confluent) hypergeometric differential equations

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Introduction

$$y^2 - \left(\frac{x^2}{4} - E\right) = 0 \quad \xrightarrow[\text{([EO1], [CEO])}]{\text{Topological recursion}} \quad \{W_{g,n}\}_{g \geqq 0, n \geqq 1}$$

Quantization ([EO1], [DM], [BE]) $\downarrow$

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left\{\frac{x^2}{4} - E - (\nu - \frac{1}{2})\hbar\right\}\right] \psi(x, \hbar) = 0$$

How can we study properties of solutions in terms of $W_{g,n}$?

Purpose

To report the result we recently obtain, i.e., there exist close relationships between Voros coefficients in the exact WKB analysis and free energies in the matrix model.

Outline of this talk

1. Exact WKB analysis and Voros coefficients
2. Topological recursion
3. The expressions of WKB solutions in terms of the topological recursion
4. The relationship between Voros coefficients and free energies
5. Other equations

Exact WKB analysis

Consider a differential equation

$$\left(-\hbar^2 \frac{d^2}{dx^2} + \frac{x^2}{4} - E\right)\psi = 0,$$

and its WKB solutions

$$\begin{aligned}\psi_\pm(x,\hbar) &= \exp\left(\int^x \sum_{j \geqq -1} \hbar^j S_j(x)\,dx\right) \\ &= \frac{1}{\sqrt{S_{\mathrm{odd}}(x,\hbar)}} \exp\left(\pm \int_{2\sqrt{E}}^x S_{\mathrm{odd}}(x,\hbar)\,dx\right),\end{aligned} \tag{1}$$

where $S = \sum \hbar^j S_j$ is a solution of

$$S^2 + \frac{dS}{dx} = \hbar^{-2}\left(\frac{x^2}{4} - E\right), \tag{2}$$

and S_{odd} is its odd degree part with respect to $\hbar$.

Voros coefficient (for the Weber equation)

Then, the Voros coefficient is defined by

$$V = \int_\gamma \left(S_{\mathrm{odd}}(x,\hbar) - \hbar^{-1} S_{-1}(x) \right) dx,$$

where γ is a path from a singular point to a singular point, so that the following holds:

$$\begin{aligned}
\psi_\pm(x,\hbar) &= \frac{1}{\sqrt{S_{\mathrm{odd}}(x,\hbar)}} \exp\left(\pm \int_{2\sqrt{E}}^{x} S_{\mathrm{odd}}(x,\hbar)\, dx \right) \\
&= e^V \frac{1}{\sqrt{S_{\mathrm{odd}}(x,\hbar)}} \exp\left(\pm \hbar^{-1} \int_{2\sqrt{E}}^{x} S_{-1}(x)\, dx \right) \\
&\quad \times \exp\left(\pm \int_{\infty}^{x} \left(S_{\mathrm{odd}}(x,\hbar) - \hbar^{-1} S_{-1}(x) \right) dx \right).
\end{aligned}$$

A path of Voros coefficient

Stokes curves : $\Im \int_a^x \sqrt{x^2 - 4E}\, dx = 0 \quad (a = \pm 2\sqrt{E})$

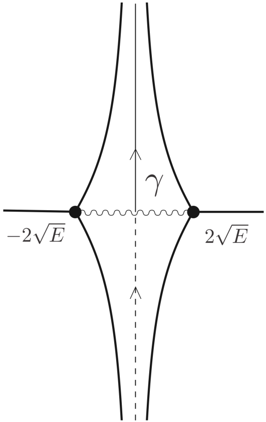

Figure: A path γ of Voros coefficient

Topological recursion (cf. [EO1]) I

Let us consider an algebraic curve

$$\mathcal{C} : P(x,y) = p_0(x)y^2 + p_2(x) = 0 \tag{3}$$

with

$$\begin{cases} p_0(x) = 1 \\ p_2(x) = -\dfrac{x^2}{4} + E. \end{cases} \tag{4}$$

To parametrize this curve, we use

$$\begin{cases} x = x(z) = \sqrt{E}(z + \dfrac{1}{z}) \\ y = y(z) = \dfrac{\sqrt{E}}{2}(z - \dfrac{1}{z}) \end{cases} \tag{5}$$

with $z \in \mathbb{P}^1$. Then,

$$dx(z) = \sqrt{E}(1 - \frac{1}{z^2})dz = \frac{\sqrt{E}(z+1)(z-1)}{z^2}dz.$$

Topological recursion (cf. [EO1]) II

We first define

$$W_{0,1}(z) = y(z)\frac{dx}{dz}(z)dz, \quad W_{0,2}(z_1, z_2) = B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

For $g \geqq 0$, $n \geqq 0$ and $2g - 2 + n \geqq 0$, we construct meromorphic differentials $W_{g,n}(z_1, \ldots, z_n)$ on Σ^n by the following recursive formulas.

$$W_{g,n+1}(z_0, z_1, \ldots, z_n) = \sum_{a\,:\ \text{branch point}} \operatorname*{Res}_{z=a} \frac{\left(\frac{1}{z_0 - z}\right) dz_0}{(y(z) - y(\bar{z}))\, dx(z)}$$

$$\times \left\{ W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1,2,\ldots,n\}}}^{\prime} W_{g_1,1+|I|}(z, z_I) W_{g_2,1+|J|}(\bar{z}, z_J) \right\}.$$

- branch points are zeros of $dx(z)$ (assume that all branch points are simple);
- $\bar{z}$ is a local conjugate point of z near a branch point (i.e. $x(\bar{z}) = x(z)$).

Topological recursion (cf. [EO1]) III

In the case of (3), branch points are $z = \pm 1$, $\bar{z}$ is given by $1/z$, and $W_{0,1}$, $W_{0,2}$, $W_{1,1}$ and $W_{0,3}$ can be explicitly calculated as follows:

$$W_{0,1}(z) = \frac{E(z^2-1)^2}{2z^3}dz,$$

$$W_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2},$$

$$W_{1,1}(z) = \frac{1}{32E}\left\{\frac{z^2-4z+1}{(z-1)^4} - \frac{z^2+4z+1}{(z+1)^4}\right\}dz,$$

$$W_{0,3}(z_1,z_2,z_3) = \frac{1}{2E}\left\{\frac{1}{(z_1+1)^2(z_2+1)^2(z_3+1)^2} - \frac{1}{(z_1-1)^2(z_2-1)^2(z_3-1)^2}\right\}dz_1\,dz_2\,dz_3.$$

Theorem 1 (for the Weber equation ; cf. [BE])

We define

$$\psi(x,\hbar) = \exp\left[\hbar^{-1}\int^z W_{0,1}(z) + \frac{1}{2!}\int_D\int_D \frac{dz_1\,dz_2}{(z_1z_2-1)^2} + \sum_{m=1}^{\infty}\hbar^m\left\{\sum_{\substack{2g+n-2=m\\ g\geq 0,\, n\geq 1}}\frac{1}{n!}\int_D\cdots\int_D W_{g,n}(z_1,\ldots,z_n)\right\}\right]\Bigg|_{z=z(x)},$$

where $z = z(x)$ is an inverse function of $x = x(z)$ and

$$\int_D = \nu\int_0^z + (1-\nu)\int_\infty^z .$$

Here, ν is a parameter. Then, $\psi(x,\hbar)$ is a solution of

$$\left[\hbar^2\frac{d^2}{dx^2} - \left\{\frac{x^2}{4} - E - \left(\nu - \frac{1}{2}\right)\hbar\right\}\right]\psi(x,\hbar) = 0 \qquad \text{(W)}$$

which has a WKB-type expansion.

A path of Voros coefficient

Stokes curves : $\Im \int_a^x \sqrt{x^2 - 4E}\, dx = 0 \quad (a = \pm 2\sqrt{E})$

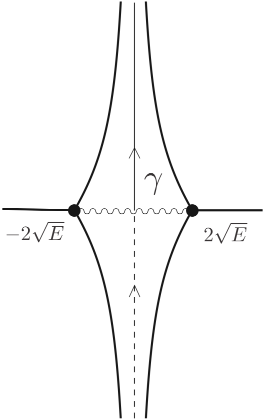

Figure: Weber equation (x-plane)

$$x = \sqrt{E}\left(z + \frac{1}{z}\right)$$

$$\longleftrightarrow$$

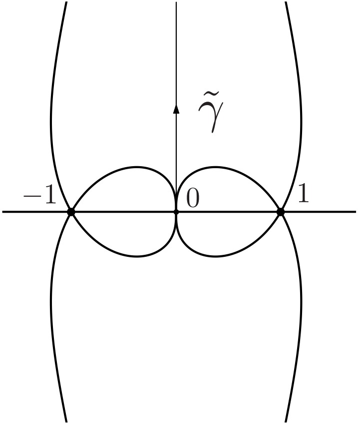

Figure: Weber equation (z-plane)

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Free energy (cf. [CEO])

We define $F_g = W_{g,0}$, called free energies, by the following ([EO1], [CEO]):

$$F_0 = -\frac{1}{2} \sum_{\alpha\,:\,\text{pole of } ydx} \operatorname*{Res}_{z=\alpha} V_\alpha(z) y(z)\, dx(z) - \frac{1}{2} \sum_{\alpha\,:\,\text{pole of } ydx} t_\alpha \mu_\alpha,$$

$$F_1 = -\frac{1}{2} \log(\tau_B(\{x(a)\})) - \frac{1}{24} \log\left(\prod_{a\,:\,\text{branch point}} y'(a) \right),$$

$$F_g = \frac{1}{2-2g} \sum_{a\,:\,\text{branch point}} \operatorname*{Res}_{z=a} \Phi(z) W_{g,1}(z) \quad (g \geqq 2).$$

- t_α is a residue of $y(z)\, dx(z)$ at $z = \alpha$;
- τ_B is the Kokotov-Korotkhon's Bergman τ-function;
- $\Phi(z)$ is any function satisfying $\frac{d\Phi}{dz} = y(z)\, dx(z)$.

$$\left(\text{Weber}: \quad F_0(E) = -\frac{3}{4} E^2 + \frac{1}{2} E^2 \log E, \quad F_1(E) = -\frac{1}{12} \log E. \right)$$

Theorem 2 (for the Weber equation)

Let $F_g(E)$ be free energies for the spectral curve corresponding to the Weber equation and

$$F(E,\hbar) = \sum_{g=0}^{\infty} F_g(E)\hbar^{2g-2}$$

be the generating function of $F_g(E)$. Then, we obtain

$$\begin{aligned} V(E,\nu,\hbar) &= F\left(E+\nu\hbar,\hbar\right) - F\left(E+(\nu-1)\hbar,\hbar\right) \\ &\quad - \frac{\partial F_0}{\partial E}\hbar^{-1} - \frac{2\nu-1}{2}\frac{\partial^2 F_0}{\partial E^2}, \end{aligned} \tag{6}$$

where $V(E,\nu,\hbar)$ is the Voros coefficient for the Weber equation.

Concrete form of free energies (Weber equation)

Lemma 3 (Weber equation)

For the Weber equation, the following relation holds:

$$V(E,0,\hbar) = V(E,1,\hbar). \tag{7}$$

From Lemma 3, $F(E,\hbar)$ satisfies the following difference equation:

$$F(E+\hbar,\hbar) - 2F(E,\hbar) + F(E-\hbar,\hbar) = \frac{\partial^2 F_0}{\partial E^2}. \tag{8}$$

We solve this equation to obtain the concrete form of free energies.

Concrete form of free energies (Weber equation)

$$F_g(E) = \frac{B_{2g}}{2g(2g-2)} E^{2-2g} \quad (g \geqq 2), \tag{9}$$

where B_{2g} designates the $2g$-th Bernoulli number.

Note: [HZ], [CEO].

Concrete form of Voros coefficient (Weber equation)

The relation

$$V(E,\nu,\hbar) = F\left(E+\nu\hbar,\hbar\right) - F\left(E+(\nu-1)\hbar,\hbar\right) - \frac{\partial F_0}{\partial E}\hbar^{-1} - \frac{2\nu-1}{2}\frac{\partial^2 F_0}{\partial E^2} \quad (6)$$

and (9) give the concrete form of Voros coefficient $V(E,\nu,\hbar)$:

Concrete form of Voros coefficient (Weber equation)

$$V(E,\nu,\hbar) = \sum_{n=2}^{\infty} \frac{B_n(1-\nu)}{n(n-1)} \left(\frac{\hbar}{E}\right)^{n-1}, \quad (10)$$

where $B_n(X)$ designates the n-th Bernoulli polynomial.

Note: [T].

Proof of Theorem 2 I

Variation formula (cf. [EO2])

We assume that there exist an integration path γ on Σ and an analytic function $\Lambda(z')$ defined near γ satisfying

$$d\Omega(z) = \int_{z' \in \gamma} B(z, z')\Lambda(z'),$$

where $d\Omega$ denotes the first variation of ydx defined by

$$y(z; \lambda + \epsilon)dx(z; \lambda + \epsilon) = y(z; \lambda)dx(z; \lambda) + \epsilon d\Omega + O(\epsilon^2).$$

(Here λ is a parameter.) Then, the following relation holds:

$$\frac{\partial W_{g,n}}{\partial \lambda} = \int_{z_{n+1} \in \gamma} W_{g,n+1}(z_1, \ldots, z_n, z_{n+1})\Lambda(z_{n+1}). \tag{11}$$

From this formula, the following relation holds:

$$\frac{\partial W_{g,n}}{\partial E} = \int_0^\infty W_{g,n+1}(z_1, \ldots, z_n, z_{n+1}). \tag{12}$$

Proof of Theorem 2 II

Recall that $\int_D = \nu \int_0^z + (1-\nu)\int_\infty^z$, then we find

$$V(E,\nu,\hbar) = \sum_{m=1}^\infty \hbar^m \int_0^\infty \left\{ \sum_{2g+n-2=m} \frac{1}{n!}\frac{d}{dz}\int_D \cdots \int_D W_{g,n}(z_1,\ldots,z_n) \right\} dz$$

$$= \sum_{m=1}^\infty \hbar^m \sum_{2g+n-2=m} \frac{\nu^n-(\nu-1)^n}{n!}\int_0^\infty \cdots \int_0^\infty W_{g,n}(z_1,\ldots,z_n).$$

On the other hand, (12) gives

$$\frac{\partial^n F_g}{\partial E^n} = \int_0^\infty \cdots \int_0^\infty W_{g,n}(z_1,\ldots,z_n). \tag{13}$$

From these two formulas we obtain

$$V(E,\nu,\hbar) = \sum_{m=1}^\infty \hbar^m \sum_{2g+n-2=m} \frac{\nu^n-(\nu-1)^n}{n!}\frac{\partial^n F_g}{\partial E^n}$$

$$= \sum_{n=1}^\infty \frac{\nu^n-(\nu-1)^n}{n!}\hbar^n \frac{\partial^n F(E,\hbar)}{\partial E^n} - \frac{\partial F_0}{\partial E}\hbar^{-1} - \frac{2\nu-1}{2}\frac{\partial^2 F_0}{\partial E^2}.$$

Theorem 4 (cf. [BE]) I

For $P(x,y) = p_0(x)y^2 + p_2(x) = 0$, we define

$$\psi(x,\hbar) = \exp\left[\hbar^{-1}\int^z W_{0,1}(z) + \frac{1}{2!}\int_D\int_D \frac{dz_1\,dz_2}{(z_1z_2-1)^2} + \sum_{m=1}^{\infty}\hbar^m \left\{ \sum_{\substack{2g+n-2=m\\ g\geq 0,\, n\geq 1}} \frac{1}{n!}\int_D \cdots \int_D W_{g,n}(z_1,\ldots,z_n)\right\}\right]\Bigg|_{z=z(x)}, \tag{14}$$

where $z = z(x)$ is an inverse function of $x = x(z)$ and

$$\int_D = \sum_i \left(\nu_{\beta_i}\int_{\beta_i}^z\right) + \sum_j \left(\nu_{\gamma_j}\int_{\gamma_j}^z\right).$$

Here, β_i is a simple pole of $x(z)$, γ_j is a zero of $p_0(x(z))$, and ν_{β_i} and ν_{γ_j} are parameters satisfying

$$\sum_i \nu_{\beta_i} + \sum_j \nu_{\gamma_j} = 1.$$

Theorem 4 (cf. [BE]) II

Then, $\psi(x,\hbar)$ is a WKB solution of

$$\left[\hbar^2 p_0(x)\frac{d^2}{dx^2} + \hbar^2 Q(x)\frac{d}{dx} + \{p_2(x) + \hbar R(x)\}\right]\psi(x,\hbar) = 0, \tag{15}$$

where

$$Q(x) = \frac{1}{2}\frac{dp_0(x)}{dx} - \sum_j \frac{\nu_{\gamma_j} p_0(x)}{x - x(\gamma_j)}$$

$$R(x) = \left[\frac{y(z)p_0(x(z))}{\frac{dx}{dz}(z)}\left\{\sum_i \left\{\nu_{\beta_i}\left(\frac{1}{z-\beta_i} - \frac{1}{z-\overline{\beta_i}}\right)\right\}\right.\right.$$
$$\left.\left.\left.+\sum_j \left\{\nu_{\gamma_j}\left(\frac{1}{z-\gamma_j} - \frac{1}{z-\overline{\gamma_j}}\right)\right\}\right\}\right]\right|_{z=z(x)}.$$

Note: β_i is a simple pole of $x(z)$, and γ_j is a zero of $p_0(x(z))$.

Kummer equation

Let us consider the following algebraic curve

$$P(x,y) = 4x^2y^2 - \left(x^2 + 4t_0x + 4{t_1}^2\right) = 0 \tag{16}$$

In this case, we choose

$$\begin{cases} x = x(z) = \sqrt{{t_0}^2 - {t_1}^2}(z + \dfrac{1}{z}) - 2t_0 = \dfrac{\sqrt{{t_0}^2 - {t_1}^2}(z - s_1)(z - s_2)}{z} \\ y = y(z) = \dfrac{z^2 - 1}{2(z - s_1)(z - s_2)}, \end{cases} \tag{17}$$

where $z \in \mathbb{P}^1$, $s_1 = \sqrt{t_0 + t_1}/\sqrt{t_0 - t_1}$ and $s_2 = \sqrt{t_0 - t_1}/\sqrt{t_0 + t_1}$.
From Theorem 4, the differential equation corresponding to (16) is

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left\{R_0(x) + R_1(x)\hbar + R_2(x)\hbar^2\right\}\right]\psi(x,\hbar) = 0, \tag{K}$$

$$R_0(x) = \frac{x^2 + 4t_0x + 4{t_1}^2}{4x^2}, \quad R_1(x) = \frac{(\nu_0 - \nu_3)x + 2(\nu_1 - \nu_2)t_1}{2x^2},$$

$$R_2(x) = \frac{(\nu_1 + \nu_2 + 1)(\nu_1 + \nu_2 - 1)}{4x^2}.$$

Theorem 5 (for the Kummer equation)

Let $F_g(t_0, t_1)$ be free energies for the spectral curve corresponding to the Kummer equation and

$$F(t_0, t_1, \hbar) = \sum_{g=0}^{\infty} F_g(t_0, t_1)\hbar^{2g-2}$$

be the generating function of $F_g(t_0, t_1)$. Then, we obtain

$$\begin{aligned} V^{(0)}(t_0, t_1, \nu, \hbar) &= F\left(t_0 + A\hbar, t_1 + (B + \frac{1}{2})\hbar, \hbar\right) - F\left(t_0 + A\hbar, t_1 + (B - \frac{1}{2})\hbar, \hbar\right) \\ &\quad - \frac{\partial F_0}{\partial t_1}\hbar^{-1} - A\frac{\partial^2 F_0}{\partial t_0 \partial t_1} - B\frac{\partial^2 F_0}{\partial {t_1}^2}, \end{aligned} \tag{18}$$

$$\begin{aligned} V^{(\infty)}(t_0, t_1, \nu, \hbar) &= F\left(t_0 + (A - \frac{1}{2})\hbar, t_1 + B\hbar, \hbar\right) - F\left(t_0 + (A + \frac{1}{2})\hbar, t_1 + B\hbar, \hbar\right) \\ &\quad + \frac{\partial F_0}{\partial t_0}\hbar^{-1} + B\frac{\partial^2 F_0}{\partial t_0 \partial t_1} + A\frac{\partial^2 F_0}{\partial {t_0}^2}, \end{aligned} \tag{19}$$

where $V^{(0)}(t_0, t_1, \nu, \hbar)$ and $V^{(\infty)}(t_0, t_1, \nu, \hbar)$ are Voros coefficients for the Kummer quation and $A = (\nu_3 - \nu_0)/2$, $B = (\nu_2 - \nu_1)/2$ and $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$.

Concrete form of F_g (Kummer equation)

Therefore, we obtain the concrete form of $F_g(t_0, t_1)$.

Concrete form of $F_g(t_0, t_1)$ (Kummer equation)

$$F_g(t_0, t_1) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{(t_0 - t_1)^{2g-2}} + \frac{1}{(t_0 + t_1)^{2g-2}} - \frac{1}{(2t_1)^{2g-2}} \right\} \quad (g \geqq 2).$$

Theorem 6 (for the Gauss hypergeometric equation)

Let us consider the following algebraic curve

$$P(x,y) = x^2(1-x)^2y^2 - \{{t_0}^2x^2 - ({t_0}^2 + {t_1}^2 - {t_2}^2)x + {t_1}^2\} = 0 \qquad (20)$$

Then, the corresponding equation is the Gauss hypergeometric differential equation. Let $F_g(t_0, t_1, t_2)$ be free energies for the spectral curve corresponding to (20) and

$$F(t_0, t_1, t_2, \hbar) = \sum_g F_g(t_0, t_1, t_2)\hbar^{2g-2}$$

be the generating function of $F_g(t_0, t_1, t_2)$. Then, we obtain

$$\begin{aligned} V^{(0)}(t_0, t_1, t_2, \nu, \hbar) = {} & F(t_0 + A\hbar, t_1 + (B + \frac{1}{2})\hbar, t_2 + C\hbar, \hbar) \\ & - F(t_0 + A\hbar, t_1 + (B - \frac{1}{2})\hbar, t_2 + C\hbar, \hbar) \\ & - \frac{1}{2}\frac{\partial F_0}{\partial t_1}\hbar^{-1} - \frac{A}{2}\frac{\partial^2 F_0}{\partial t_0 \partial t_1} - \frac{B}{2}\frac{\partial^2 F_0}{\partial {t_1}^2} - \frac{C}{2}\frac{\partial^2 F_0}{\partial t_1 \partial t_2}, \end{aligned} \qquad (21)$$

where $V^{(0)}(t_0, t_1, t_2, \nu, \hbar)$ is Voros coefficient for the Gauss hypergeometric differential equation and $A = (\nu_5 - \nu_0)/2$, $B = (\nu_3 - \nu_1)/2$, $C = (\nu_4 - \nu_2)/2$ and $\nu = (\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$. Other Voros coefficients can be expressed similarly.

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