An overview of renormalization Hopf algebras

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The idea of combinatorial Hopf algebras

A **product** takes two things and puts them together into one thing.

A **coproduct** takes one thing and takes it apart into pairs of things.

Embody these compatibly on some combinatorial objects and you have a combinatorial Hopf algebra.

classic egs

words { product - concatenation
coproduct - deshuffle

{ product - shuffle
coproduct - deconcatenate

symmetric functions.

Lots of details including the antipode.

Hopf algebras of Feynman graphs

Say our things are Feynman diagrams. 1PI

A good product is disjoint union.

A good coproduct takes a diagram apart into divergent subgraphs and cographs.

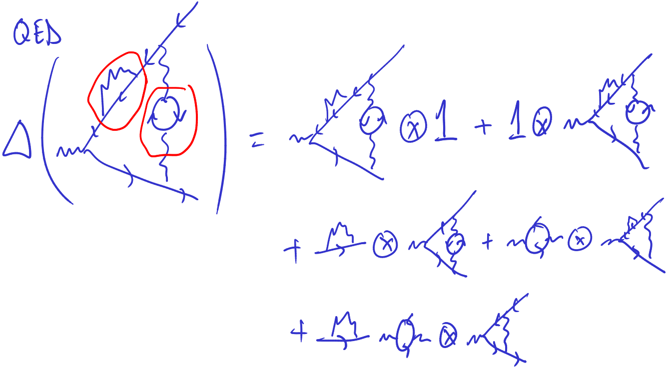
$$\Delta(G) = \sum_{\gamma \subseteq G} \gamma \otimes G/\gamma$$

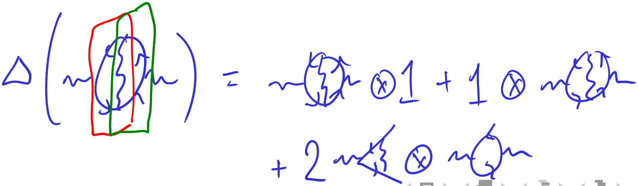
connected components 1PI

γ divergent

Examples

eg QED





The antipode

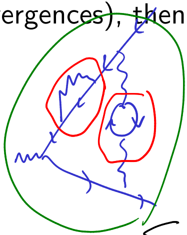
The antipode is given recursively.

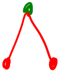
$$S(G) = -G - \sum_{\emptyset \subsetneq \gamma \subsetneq G} S(\gamma)\, G/\gamma$$

The tree-like case: Connes-Kreimer

When the subdivergences have a tree-like structure (no overlapping divergences), then the coproduct exists at the level of the tree.

Eg





$S=\emptyset$ $\quad$ $S=\{\text{root}\}$ $\quad$ $S=\{\text{leaf}\}$

$\Delta(\Lambda) = 1 \otimes \Lambda + \Lambda \otimes 1 + 2 \bullet \otimes \,!$

$S=$ both leaves

$+ \bullet\bullet \otimes \bullet$

$$\Delta(t) = \sum_{\substack{S \subseteq V(t) \\ S \text{ antichain}}} \left(\prod_{v\in S} t_v\right) \otimes \left(t \setminus \prod_{v \in S} t_v\right)$$

where t_v is the subtree rooted at v

(no element of S is an ancestor of another)

This gives the **Connes-Kreimer** Hopf algebra. It is **universal** and can take care of overlapping by sums.

What about IR?

Some infrared divergences can also be captured Hopf algebraically (arXiv:1512.06409, see "motic")

The rest of the IR story is also very combinatorial, but the exact structure is not so clear.

Renormalization by renormalization Hopf algebra

The Hopf algebra encodes the structure of BPHZ renormalization.

Essentially S is renormalization, but it needs to be twisted with the Feynman rules themselves and a regularization map.

Let

- ϕ be the unrenormalized Feynman rules.
- R be a regularization map.

Define

$$S_R^{\phi}(G) = -R(\phi(G)) - \sum_{\emptyset \neq \gamma \subsetneq G} S_R^{\phi}(\gamma) R(\phi(G/\gamma))$$

Compare this to S itself: $S(G) = -G - \sum_{\emptyset \subsetneq \gamma \subsetneq G} S(\gamma)\, G/\gamma$

Renormalization continued

Let

- ϕ be the unrenormalized Feynman rules.
- R be a regularization map.
- $S_R^\phi(G) = -R(\phi(G)) - \sum_{\emptyset \neq \gamma \subsetneq G} S_R^\phi(\gamma) R(\phi(G/\gamma))$

S_R^ϕ is the **twisted antipode**. It gives the counterterms.

The **renormalized Feynman rules** are

$$\phi_R = m(S_R^\phi \otimes \phi)\Delta$$

Depending on your taste you can view this more concretely or more geometrically.

Importance of primitivity and 1-cocycles

The Hopf algebra is more than just a mathematical underpinning for BPHZ.

An element of a Hopf algebra is primitive if

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

Primitives generate everything. They are the physical skeletons. Insertion into a primitive is algebraically priviliged.

get Hochschild 1-cocycles

Physical identities and Hopf ideals

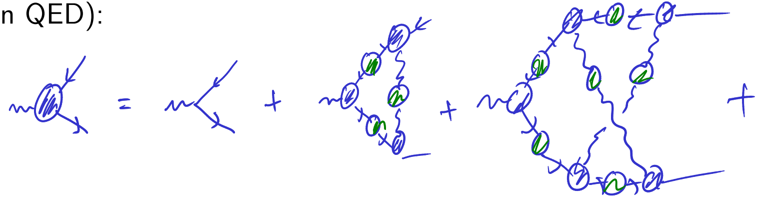
Physical identities become reasonable algebraic objects.

The Ward identities are a Hopf ideal.

The renormalization group equation becomes a combinatorial decomposition.

Dyson-Schwinger equations

Dyson-Schwinger equations are combinatorial specifications.
Eg (in QED):



etc.

Then apply Feynman rules.

Dyson-Schwinger equations again

Applying Feynman rules one gets an integral equation.
Eg:

$$G(x, L) = 1 - \frac{x}{q^2} \int d^4k \, \frac{k \cdot q}{k^2 G(x, \log \frac{k^2}{\mu^2}) (k+q)^2} - \cdots \Big|_{q^2 = \mu^2}$$

$L = \log \frac{q^2}{\mu^2}$

This the perturbative version of the usual physical Dyson-Schwinger equations.

Rewriting them into pseudo-differential form

In the example

- Expand out $G(x, L)$ under the integral.
- Use $\frac{d^k}{d\rho^k} y^\rho|_{\rho=0} = \log^k(y)$.
- Swap freely.

The series has the same shape but with ρ derivatives in place of powers of $\log q^2$. Get

in the eg

$$G(x, L) = 1 - xG\left(x, \frac{\partial}{\partial \rho}\right)^{-1} \left(e^{-L\rho} - 1\right) F(\rho)\Big|_{\rho=0}$$

where $F(\rho)$ is the Feynman integral for the primitive, regularized by ρ and $q^2 = 1$

This makes sense for formal power series, but ...

The P-equation

How to do better? One attempt: use the renormalization group equation and a geometric series approximation.

Put the extra stuff thrown away from the approximation into a series $P(x)$.

Get

$$\boxed{\gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x)}$$

The example above was $s = 2$. Can do systems similarly.

Resurgence view

Lutz Klaczynski in arxiv:1601.04140 looks at two cases where P is not mysterious using transseries.

He concludes that the obvious transseries Ansatz is not the right one for this problem, but why. . . .

Other people have looked too. Eg Marc Bellon and Pierre Clavier in a Wess Zumino version arxiv:1612.07813.

Chord diagram expansions

Another attempt, which I'm particularly excited about, is chord diagram expansions. In the running example get (arXiv:1210.5457, newer papers extend and enrich)

$$G(x, L) =$$

see other talks (sorry out of time)