Positive Geometry of the Wilsonahedra

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Context

Understanding Amplitudes of SYM $N=4$

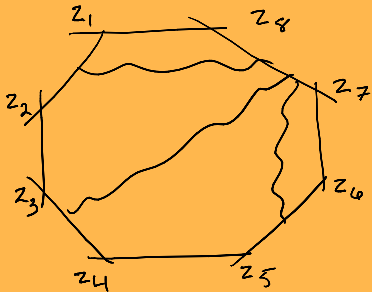
$$\boxed{\mathcal{A}_{n,k} \sim \mathbf{W}_{n,k}}$$

Context

- $$\mathbf{W}_{n,k} = \sum_{\substack{\text{admissible} \\ \text{Wilson Loop} \\ \text{Diagrams}}}$$

- MHV Diagrams
- Tree level
- Geometry

Wilson Loop Diagrams



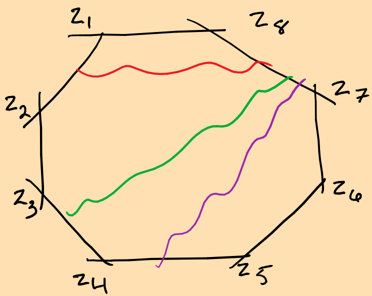
$$= W = ([n], \mathcal{P})$$

- $Z_i = (Z^{\mu}, Z^{t}) \in \mathbb{R}^4 \oplus \mathbb{R}^k$
- $\mathcal{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_n \end{pmatrix} \in M_+(n, 4+k)$
- planar

Dfn For $q \in \mathcal{P}$, $\boxed{V_q = \{i_p, i_p+1, i_q, i_{q+1}\}}$

$Q \subset \mathcal{P}$, $\boxed{V_Q = \bigcup_{q \in Q} V_q}$

$$\boxed{\text{Prop}(v) = \{\text{set of MHV Propagators dependent on } v\}}$$

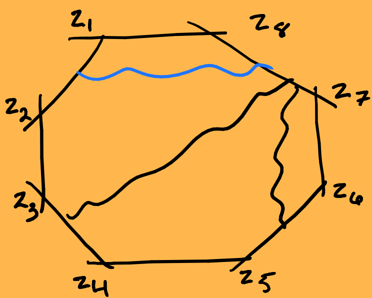


$$V_{\{g, r\}} = \{1, 2, 3, 4, 7, 8\}$$

$$\text{Prop}(Z_4) = \{g, p\}$$

$$\text{Prop}(Z_6) = \emptyset$$

NMHV propagators



$$Y_q = c_{q,*} Z_* + \sum_{v \in V_q} c_{q,v} Z_v$$

$$Y_b = c_{b,*} Z_* + c_{b,1} Z_1 + c_{b,2} Z_2 + c_{b,7} Z_7 + c_{b,8} Z_8$$

Feynman Integrals

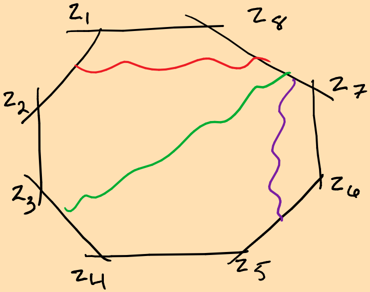
$$Y_q = c_{q,*} Z_* + \sum_{v \in V_q} c_{q,v} Z_v$$

$$I(W) = \int_{(\mathbb{RP}^4)^k} \prod_{q \in p} \frac{dc_{q,*}}{c_{q,*}} \prod_{v \in} \frac{d\hat{c}_{q,r}}{\hat{c}_{q,r}} \, \delta^4(Y_q^\mu) \left(\pi_{4+q}(Y_q)\right)^4$$

$\delta^4(Y_q^\mu)$ sets values for $c_{p,i}$:

$$\begin{cases} c_{q,*} = 1 \\ c_{q,j_q} = \dfrac{\langle Z^\mu_{i_q} Z^\mu_{i_q+1} Z^\mu_{*} Z^\mu_{j_q+1} \rangle}{\langle Z^\mu_{i_q} Z^\mu_{i_q+1} Z^\mu_{j_q} Z^\mu_{j_q+1} \rangle} \end{cases}$$

Explicitly

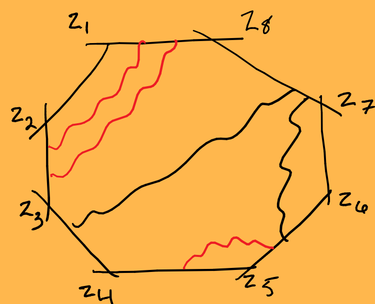


$$\to \left[\begin{array}{c|c} \begin{matrix}1\\1\\1\end{matrix} & \underbrace{\begin{matrix} c_{1,1} & c_{1,2} & 0 & 0 & 0 & 0 & c_{1,7} & c_{1,8} \\ 0 & 0 & c_{2,3} & c_{2,4} & 0 & 0 & c_{2,7} & c_{2,8} \\ 0 & 0 & 0 & 0 & c_{3,5} & c_{3,6} & c_{3,3} & c_{3,4} \end{matrix}}_{C(W)(\mathcal{Z}^{\mu})} \end{array}\right]$$

Admissible WLD

Dfn $([n], \mathcal{P})$ admissible if

$$\nexists\ Q \subset \mathcal{P} \text{ s.t. } |Q| + 3 > V_Q$$



Admissibility, Positivity and Geometry

<u>Thm</u> For $W = ([n], P)$ admissible;

Z_i^{μ} generic position $\Rightarrow$

$$C(W)(Z) \in Gr_+(|P|, n)$$

Invariants of WLDs

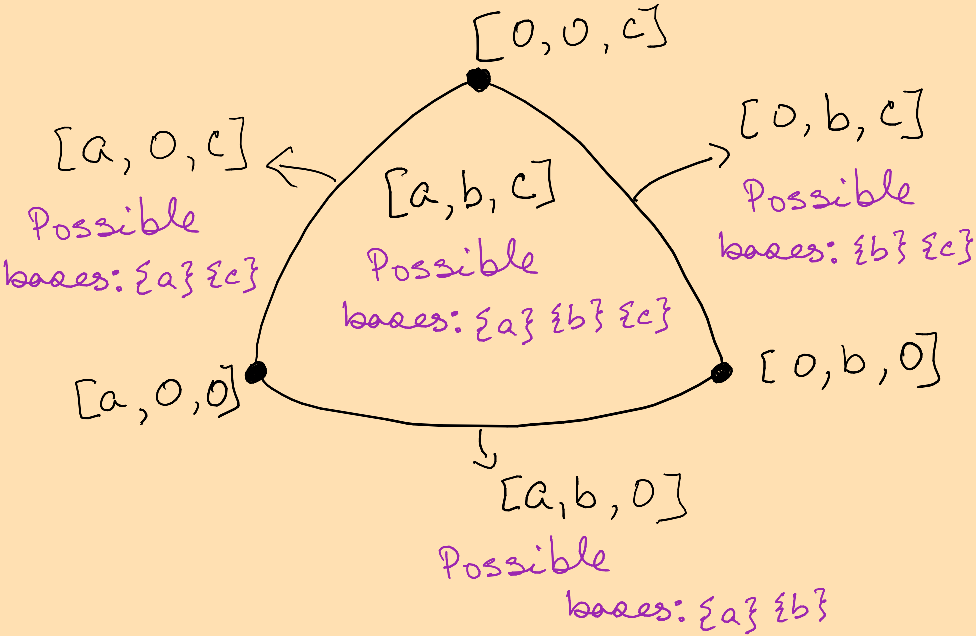
For all Z_i^μ in generic position

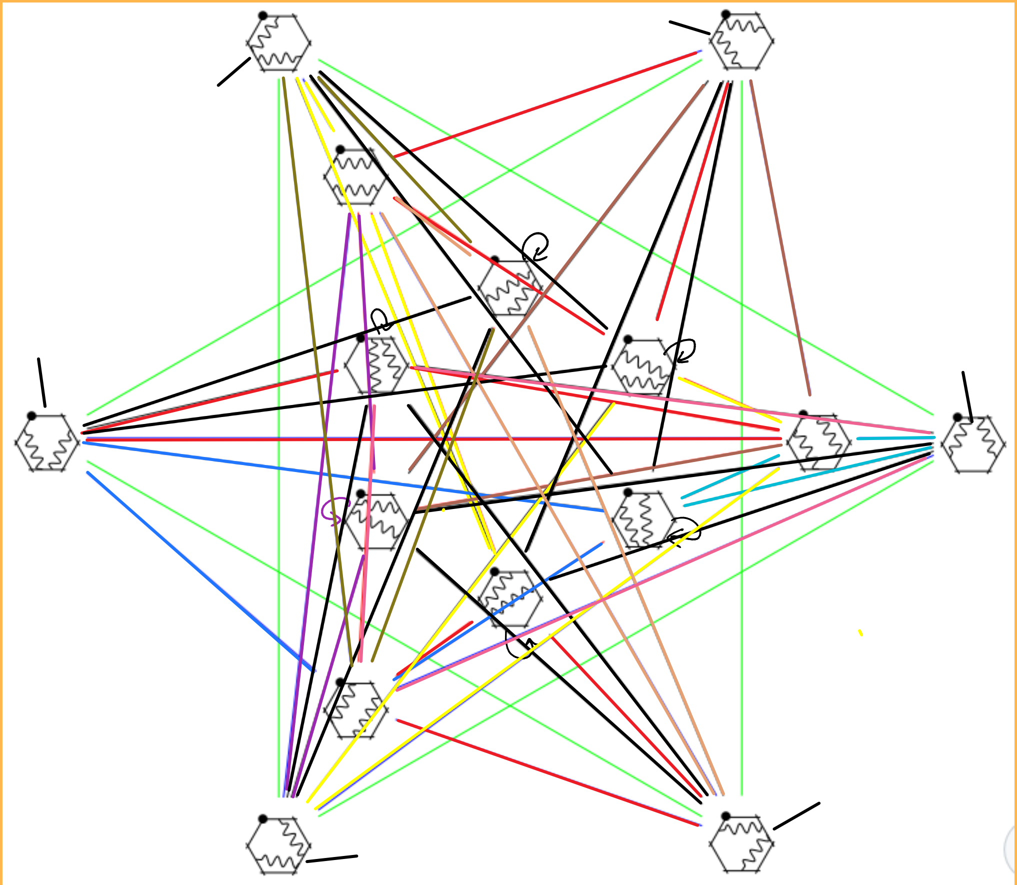
the matrices $C(W)(Z^\mu)$ have the

same independence data

$$\text{WLD} \longrightarrow \text{cells of } Gr_+(k,n)$$

Cells of $Gr_+(1,2)$





Missing Cells

- In $Gr_+(2,6)$, 6 6-dim cells, 6 6 dim cells missing.

$$\begin{array}{|c|c|c|c|}\hline 0&0&+&+\\\hline +&+&+&+\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline +&+&+&+\\\hline 0&0&+&+\\\hline\end{array}\quad
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\begin{array}{|c|c|c|c|}\hline +&+&+&+\\\hline +&+\\\cline{1-2}\end{array}$$

TABLE 3. 6-dimensional Le diagrams in $Gr(2,6)$ which are not associated to an admissible Wilson Loop diagram

- In $Gr_+(2,6)$, 6 6-dim cells, 6 5 dim cells missing.

$$\begin{array}{|c|c|c|c|}\hline 0&0&0&+\\\hline +&+&+&+\\\hline\end{array}\quad
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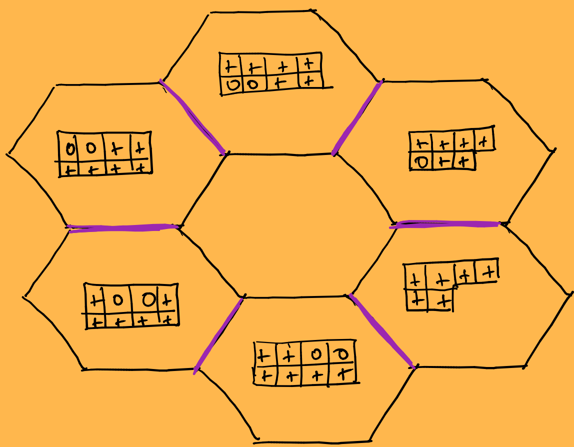
TABLE 4. 5-dim cells in $Gr(2,6)$ which do *not* appear as boundaries.

Missing Cells

- Missing cells are exactly the orbit of C_6 (acting by rotation)



- Homology of subspace of $Gr_+(2,6)$ parametrized by WLD:

$$H_i = \mathbb{R} \quad \text{if } i = 0, 5$$

$$H_i = 0 \quad \text{else.}$$

$\mathcal{M}$-Hedron

$$Y = \begin{bmatrix} * & \Big| & C(W) \\ * & & \\ * & & \end{bmatrix} \begin{bmatrix} - Z_* - \\ - Z_1 - \\ \vdots \\ - Z_n - \end{bmatrix}$$

- $* \in \{1, -1, 0\}$
- $\mathcal{M}$-Hedron given by

$$\{ Y \mid \langle Y_1 \ldots Y_k Z_i Z_{i+1} Z_j Z_{j+1} \rangle \geq 0 \text{ for some } i, j \}$$

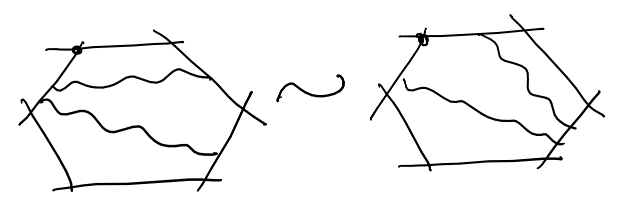
?! not every 3k face of $\mathcal{M}$-Hedron has a singularity

?!

A cabinet of curiosities

- Any subset $Q \subset \mathcal{P}$ s.t. $|Q| = V_Q + 3$
  $\Rightarrow C(W)$ totally positive subcell.

Eg)



2 different parametrizations for same cell.

- Need a linear combination of the two in order to cancel all boundaries
- In $Gr_+(2,6)$, all boundaries of a $C(W)$ matched by another $C(W')$

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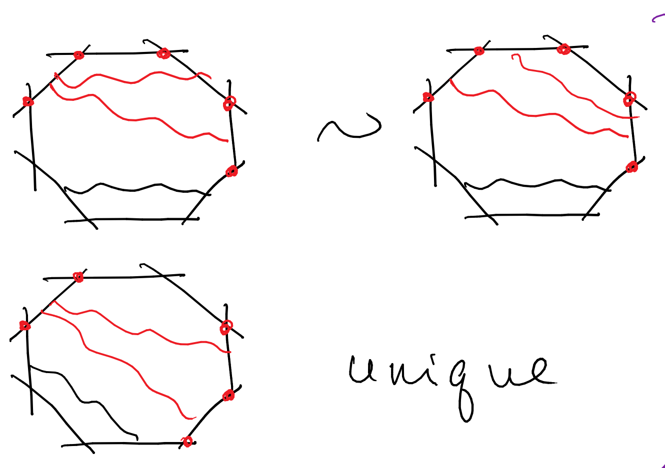


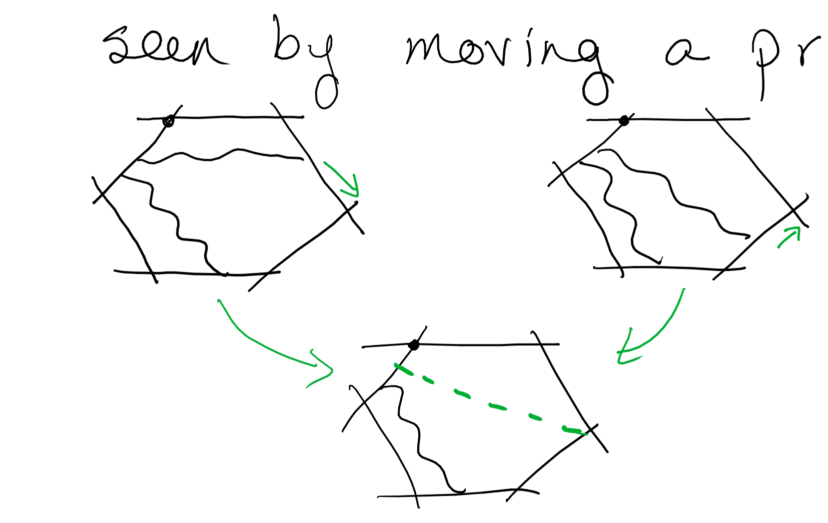
unique

speculative

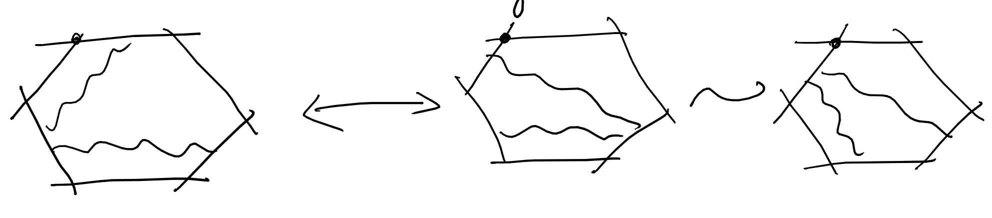
$\leftarrow$ need both

$\leftarrow$ all bdny accounted for

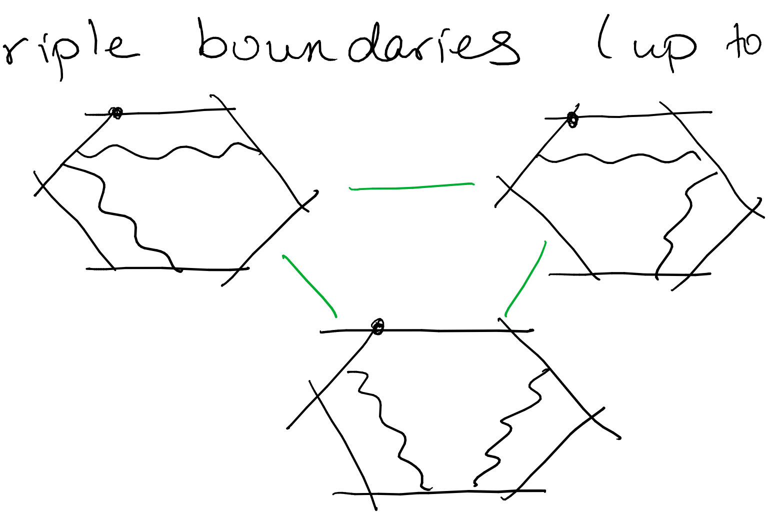
- Some cells sharing bdny can be seen by moving a propagator:

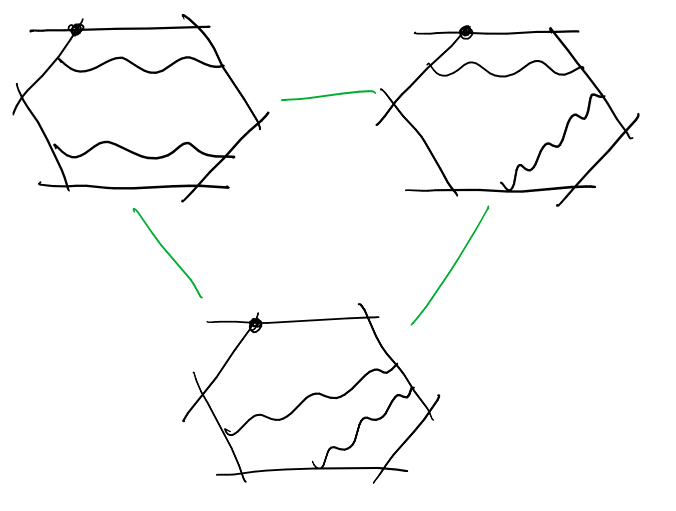


- But not always!



- Triple boundaries (up to rotation)





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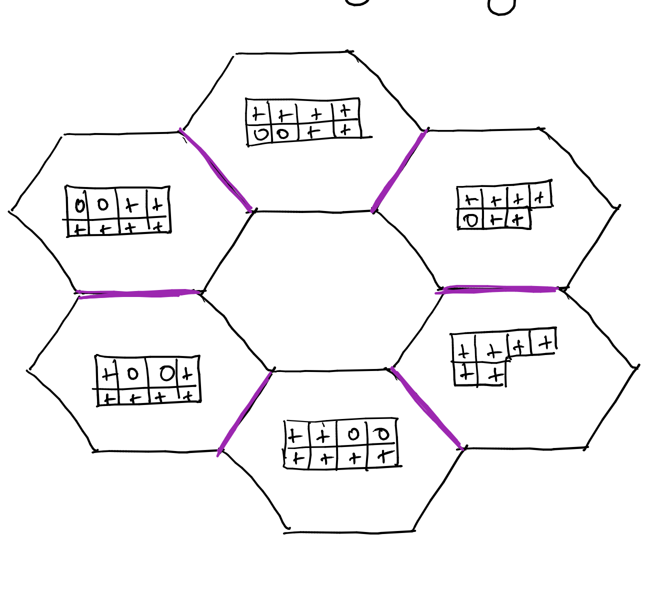


- Homology of subspace of $Gr_+(2,6)$ parametrized by WLD:

  $H_i = \mathbb{R}$ if $i = 0, 5$

  $H_i = 0$ else.

- Note! since we miss so much of $Gr_+(2,6)$ need to consider matrix $\left[\begin{smallmatrix}\vdots\\ \vdots\end{smallmatrix}\middle| C(W)\right] \in Gr(k, n+1)$

Grassmann Necklace

(1) Given a point in $Gr_+(k,n)$, write down representative matrix, M

(2) The Grassmann necklace is a string of n k-tuples $\mathcal{I} = \{I_1, \dots I_n\}$ such that I_a is the lexicographically minimal set of columns of M starting at a

WLD to GN

(0) Set $i = a$

(1) If i supports a propagator $\Rightarrow$

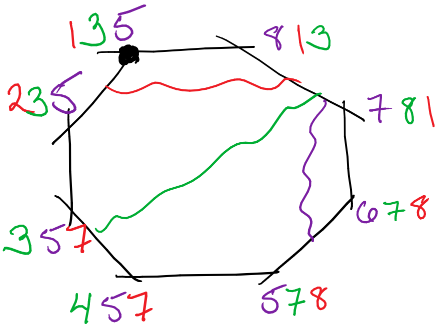
(a) $i \in I_a$

(b) remove clockwise most prop supported on i

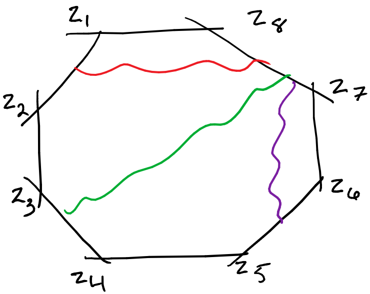
(2) $i \to i+1$

(3) If props remain, goto (1)

Eg]



$$\longrightarrow \begin{bmatrix} C_{1,1} & C_{1,2} & 0 & 0 & 0 & 0 & C_{1,3} & C_{1,4} \\ 0 & 0 & C_{2,1} & C_{2,2} & 0 & 0 & C_{2,3} & C_{2,4} \\ 0 & 0 & 0 & 0 & C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} \end{bmatrix}$$



$$\longrightarrow \left[\begin{array}{c|cccccccc} 1 & c_{1,1} & c_{1,2} & 0 & 0 & 0 & 0 & c_{1,7} & c_{1,8} \\ 1 & 0 & 0 & c_{2,3} & c_{2,4} & 0 & 0 & c_{2,7} & c_{2,8} \\ 1 & 0 & 0 & 0 & 0 & c_{3,5} & c_{3,6} & c_{3,3} & c_{3,4} \end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{C(W)(\tilde{Z})}$$

$$c_{3,3} = \hat{c}_{3,3} \; ; \; c_{3,4} = \hat{c}_{3,4}$$

$$c_{2,3} = c_{3,3}\,\hat{c}_{2,3} \; ; \; c_{2,4} = c_{3,4}\,\hat{c}_{2,3} + \hat{c}_{2,4}$$

$$c_{1,3} = c_{2,3}\,\hat{c}_{1,3} \; ; \; c_{1,4} = c_{2,4}\,\hat{c}_{1,3} + \hat{c}_{1,4}$$