Combinatorics of the (tree) amplituhedron

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Based on:

- joint work with Steven Karp arXiv:1608.08288
- joint work with Steven Karp and Yan Zhang (in preparation)

Outline:

- Review of (tree) amplituhedron $\mathcal{A}_{n,k,m}$
- Orthogonal description of $\mathcal{A}_{n,k,m}$
- Sign variation description of $\mathcal{A}_{n,k,m}$ when $m=1$
- Explicit description of BCFW cells $(m=4)$
- Conjectures on numerology of $\mathcal{A}_{n,k,m}$
- Disjointness of BCFW cells of $\mathcal{A}_{n,k,4}$ for $k=2$

The positive Grassmannian

<u>Def</u>: The Grassmannian $Gr_{k,n} = \{V \subset \mathbb{R}^n \mid \dim V = k\}$

Represent $V \in Gr_{k,n}$ by full rank $k \times n$ matrix $A = (A_1 | \dots | A_n)$

For $J \in \binom{[n]}{k}$, $\Delta_J(A) :=$ minor of A using columns J.
Plucker coordinate.

The <u>totally non-negative Grassmannian</u> is

$$Gr_{k,n}^{\geq 0} := \{A \in Gr_{k,n} \mid \Delta_J(A) \geq 0 \quad \forall\ J \in \binom{[n]}{k}\}$$

The <u>totally positive Grassmannian</u> is

$$Gr_{k,n}^{>0} := \{A \in Gr_{k,n} \mid \Delta_J(A) > 0 \quad \forall\ J \in \binom{[n]}{k}\}$$

Def: Given $\mathcal{M} \subseteq \binom{[n]}{k}$, set

$$S_{\mathcal{M}} := \{ A \in Gr_{kn}^{\geq 0} \mid \Delta_J(A) > 0 \text{ iff } J \in \mathcal{M} \}.$$

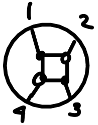
"positroid cell."

Thm (Postnikov): If $S_{\mathcal{M}} \neq \phi$ then $S_{\mathcal{M}}$ is open ball.

So have cell decomposition $Gr_{kn}^{\geq 0} = \coprod S_{\mathcal{M}}$

(Nonempty) cells in bijection with:

- decorated permutations
- (equivalence classes of) reduced plabic graphs, ie, on shell diagrams



- $\reflectbox{L}$-diagrams

+	+	0	+	0	+
0	0	0	0	+	+
+	+	+	+		
0	0	0	+		

<u>Def</u>: A <u>⅃-diagram</u> for $Gr_{kn}^{\geq 0}$ is a Young diagram $\leq k \times (n-k)$ filled with 0, + s.t. no

$$\begin{array}{ccc} & & + \\ & & \vdots \\ + & \cdots & 0 \end{array}$$

<u>Ex:</u>

0	+	+	0
0	0	0	+
+	+	+	

<u>Thm (Postnikov)</u>: Cells of $Gr_{kn}^{\geq 0}$ in bijection with ⅃-diagrams.

From ⅃-diagram can read off all points of the cell (as matrices or in terms of Plucker coord's)

Dim of cell = # of +'s.

The Amplituhedron

<u>Def:</u> (Arkani-Hamed, Trnka) Let Z be a $(k+m) \times n$ real matrix w/ maximal minors positive.

$\rightsquigarrow$ map $\tilde{Z} : Gr_{kn}^{\geq 0} \longrightarrow Gr_{k,k+m}$ defined by:

If A a $k \times n$ matrix in $Gr_{kn}^{\geq 0}$,

$$A \longmapsto A Z^t = \begin{array}{cc} & k+m \\ k & \left(\quad \right) \end{array}$$

The <u>(tree)</u> <u>amplituhedron</u> $\mathcal{A}_{n,k,m}$ is $\tilde{Z}(Gr_{kn}^{\geq 0}) \subset Gr_{k,k+m}$

Triangulating the amplituhedron

Recall $\mathcal{A}_{n,k,m}(Z) = \tilde{Z}(Gr_{kn}^{\geq 0})$ where

$\tilde{Z}: Gr_{kn}^{\geq 0} \longrightarrow Gr_{k,k+m}$. Image has full dimension km

For $m=4$:

<u>Conj (AH-T)</u>: The BCFW cells (which have dim $4k$)

in $Gr_{kn}^{\geq 0}$ give a "triangulation" of $\mathcal{A}_{n,k,4}$:

ie. their images are disjoint & cover a dense

subset of $\mathcal{A}_{n,k,4}$.

Open problem: Topology of $\mathcal{A}_{n,k,m}$?

Conj: Homeomorphic to closed ball.

Note: $Gr_{kn}^{\geq 0}$ $(= \mathcal{A}_{k+m,k,m})$

is conjecturally a closed ball.

Lots of evidence for this:

- it is combinatorially a ball, i.e. poset of cells shellable + Eulerian (W.)
- contractible w/ bdy homotopy equiv to sphere (Rietsch-W.)

Orthogonal point of view on $\mathcal{A}_{n,k,m}$

- $\mathcal{A}_{n,k,m} \in Gr_{k,k+m}$. For small m, prefer to work with $Gr_{m,k+m} \cong Gr_{k,k+m}$.

Theorem (Karp, W.): Let $Z \in \mathrm{Mat}^{>0}_{k+m,n}$.
Let $W \subset \mathbb{R}^n$ be rowspan(Z).
Let $\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in Gr^{\geq 0}_{k,n}\} \subseteq Gr_m(W)$.

Then $\mathcal{A}_{n,k,m}(Z)$ homeomorphic to $\mathcal{B}_{n,k,m}(Z)$.

Pf idea:

$$\underbrace{V^\perp \cap W}_{\substack{m\text{-dim'l} \\ \text{subspace} \\ \text{of } \mathbb{R}^n}} \xmapsto[\substack{\text{Take orthog} \\ \text{complement} \\ \text{in } \mathbb{R}^n}]{} V + W^\perp \xmapsto{Z:\mathbb{R}^n \to \mathbb{R}^{k+m}} \underbrace{Z(V)}_{\substack{k\text{-dim'l} \\ \text{subspace} \\ \text{of } \mathbb{R}^{k+m}}} \subseteq Gr_{k,k+m}$$

Note: $Z(W^\perp) = \{0\}$

Sign Variation

<u>Def:</u> For $v \in \mathbb{R}^n$, let $\text{var}(v) = \#$ times v changes sign, reading coordinates L to R

e.g. for $v = (4,-1,0,-2)$, $\text{var}(v) = 1$.

Let $\overline{\text{var}}(v) =$ max # sign changes after we choose a sign for each 0 coordinate.

e.g. $\overline{\text{var}}(4,-1,0,-2) = 3$.

<u>Theorem (Gantmacher-Krein, 1950):</u> Let $V \in Gr_{k,n}(\mathbb{R})$.

(i) $V \in Gr_{kn}^{\geq 0} \iff \text{var}(x) \leq k-1 \quad \forall$ vectors $x \in V$

$\iff \overline{\text{var}}(w) \geq k \quad \forall$ vectors $w \in V^{\perp}$

(ii) $V \in Gr_{kn}^{>0} \iff \overline{\text{var}}(x) \leq k-1 \quad \forall$ vectors $x \in V \setminus \{0\}$

$\iff \text{var}(w) \geq k \quad \forall$ vectors $w \in V^{\perp}$

Simple Description of Amplituhedron

Theorem (Karp, W.): For $W \in Gr_{k+m,n}^{>0}$, we have:

automatic from GK

① $\mathcal{B}_{n,k,m}(W) \subseteq \{X \in Gr_m(W) \mid k \le \overline{\text{var}}(x) \le k+m-1 \ \forall x \in X\} \subseteq Gr_m(W)$

Moreover, when $m=1$,

② $\mathcal{B}_{n,k,1}(W) = \{x \in \mathbb{P}(W) \mid \overline{\text{var}}(x) = k\}$

Open: In ①, is the $\subseteq$ an $=$? True for $m=1$, $k+m=n$.

Questions: ① Can we triangulate $\mathcal{A}_{n,k,1}$?
② Is it a ball?

$m=1$ Amplituhedron

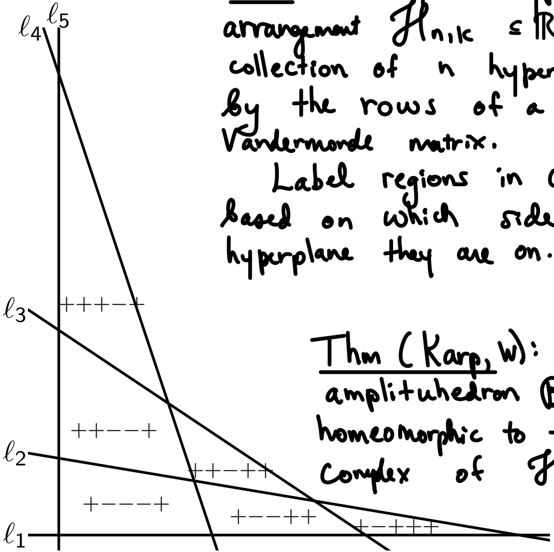
<u>Def:</u> Let $\overline{\text{Sign}}_{n,k,1} \subseteq \{0,+,-\}^n$ be the set of sign vectors σ s.t. $\overline{\text{var}}(\sigma)=k$.
Let $\text{Sign}_{n,k,1}$ " " s.t. $\text{var}(\sigma)=k$.

E.g. for $n=5$, $k=2$,
$\text{Sign}_{5,2,1} = \{+++-+,\ ++--+,\ ++-++,\ +---+,\ +--++,\ +-+++\}$.

Let $\mathcal{B}_\sigma(W) := \{x \in \mathcal{B}_{n,k,1}(W) \mid \text{sign}(x) = \sigma\}$

So $\mathcal{B}_{n,k,1}(W) = \coprod_{\sigma \in \text{Sign}_{n,k,1}} \mathcal{B}_\sigma(W)$.

The cyclic hyperplane arrangement and $\mathcal{A}_{5,2,1}$



Def: The cyclic hyperplane arrangement $\mathcal{H}_{n,k} \subseteq \mathbb{R}^k$ is the collection of n hyperplanes given by the rows of a $n \times (k+1)$ Vandermonde matrix.

Label regions in complement based on which side of each hyperplane they are on.

Thm (Karp, W): The amplituhedron $\mathcal{B}_{n,k,1}(W)$ is homeomorphic to the bounded complex of $\mathcal{H}_{n,k}$.

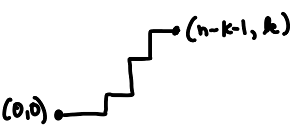
Combining our result with result of Dong,

Cor: The $m=1$ amplituhedron $\mathcal{B}_{n,k,1}$ (or $\mathcal{A}_{n,k,1}$) is homeomorphic to a closed ball.

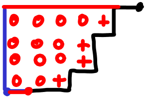
Moreover, have complete cell decomp of $\mathcal{B}_{n,k,1}$

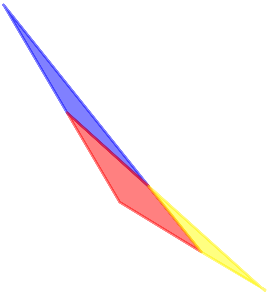
$$\mathcal{B}_{n,k,1}(W) = \coprod_{\sigma \in \mathrm{Sign}_{n,k,1}} \mathcal{B}_\sigma(W),$$ and we can describe cells in terms of decorated perms, or ⅃-diagrams, etc.

In particular: $\mathcal{A}_{n,k,1}$ has $\binom{n-k-1}{k}$ cells, in bijection with lattice paths

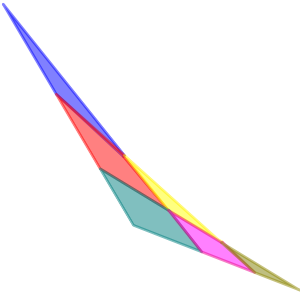


Corresponding ⅃-diagram

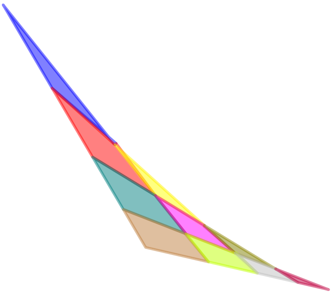




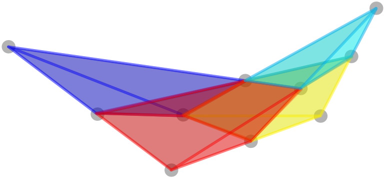
$\mathcal{A}_{4,2,1}$



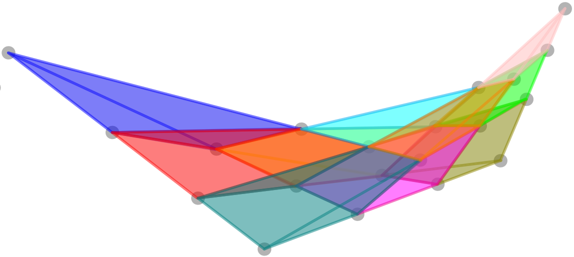
$\mathcal{A}_{5,2,1}$



$\mathcal{A}_{6,2,1}$



$\mathcal{A}_{5,3,1}$



$\mathcal{A}_{6,3,1}$

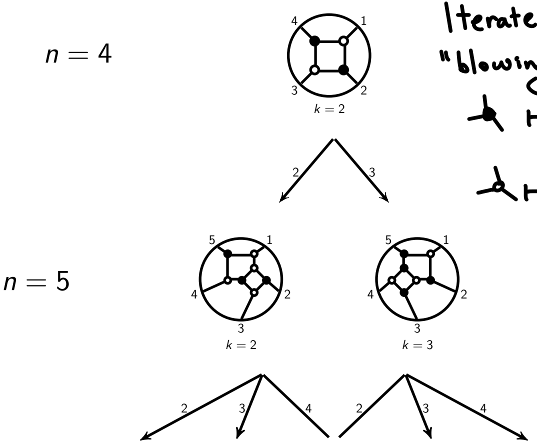
Next: $m=4$

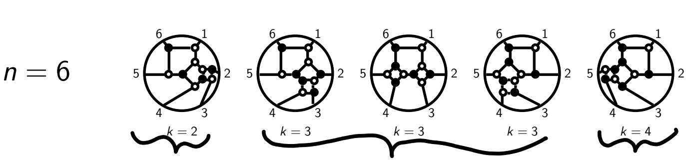
Recall

<u>Conj (AH–T)</u>: The BCFW cells (which have dim $4k$) in $Gr_{kn}^{\geq 0}$ give a "triangulation" of $\mathcal{A}_{n,k,4}$: ie. ie. their images are disjoint & cover a dense subset of $\mathcal{A}_{n,k,4}$.

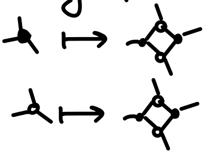
What are the BCFW cells? Obtained from recurrence.

(Many ways to do recurrence but we use canonical one)





Iterate recurrence by "blowing up" bdy vertex:



$k \mapsto k+1$
$n \mapsto n+1$

$k \mapsto k$
$n \mapsto n+1$

N.B. There are other ways to do the BCFW recursion but we always use this one.

After applying a "shift by 2," these graphs give us the BCFW cells that should triangulate the amplituhedron.

E.g. the 3 graphs for $n=6$, $k=3$ label the 3 BCFW cells triangulating $\mathcal{A}_{6,1,4}$.

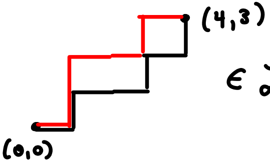
Rk (AH-B-C-G-P-T) The number of BCFW cells (which conjecturally triangulate $\mathcal{A}_{n,k,4}$) is $N_{k+1,n-3}$, where $N_{a,b} = \frac{1}{b}\binom{b}{a}\binom{b}{a-1}$ is the Narayana numbers.

Note: Description of BCFW cells is recursive. Would prefer more explicit description.

Note: $N_{a,b}$ counts various combinatorial objects.

Let $\mathcal{L}_{n,k,q}$ = set of all pairs of non-crossing lattice paths taking steps W and S from $(n-k-q, k)$ to $(0,0)$.

In particular, $N_{k+1,n-3} = |\mathcal{L}_{n,k,4}|$.



$\in \mathcal{L}_{11,3,4}$

Ex: If $k=1$, $n=6$,

$\mathcal{L}_{n,k,q}$ = pairs of paths from $(1,1)$ to $(0,0)$

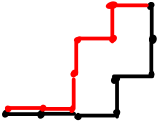
= 

Cardinality is 3! (same 3 from before)

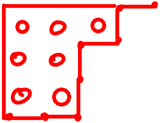
Theorem (Karp, W., Zhang): Explicit description of all BCFW cells for $\mathcal{A}_{n,k,4}$. Give bijection

$$\mathcal{L}_{n,k,4} \longrightarrow \text{⅃-diagrams of BCFW cells}$$

Ex:

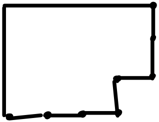
 $\in \mathcal{L}_{n,k,4}$. $\rightsquigarrow$

Step 0:



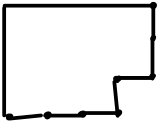
Y_u

and



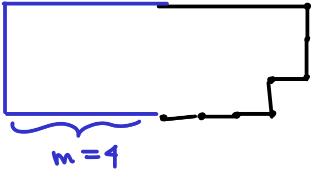
Y_L

Step 1: Use Y_L to get shape of ⅃-diagram

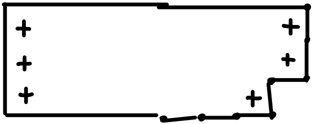


Y_L

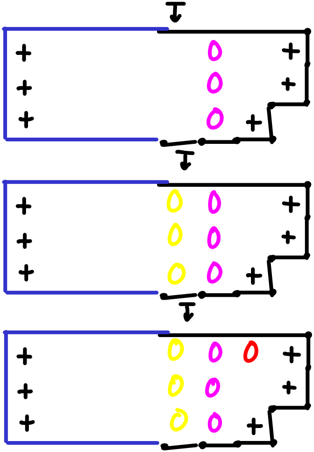
$\longmapsto$



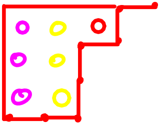
Step 2: Put + at L and R of each row



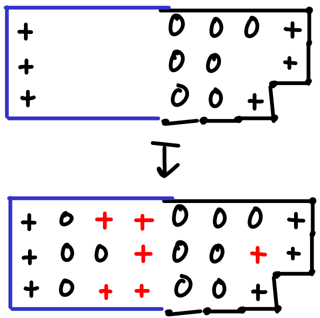
Step 3: Place columns of 0's in Y_u L to R into diagram.



Recall

$Y_u =$ 

Step 4: Place 2 more +'s in each row, justified to right



Step 5: Fix blocked 0's with J-move $\begin{matrix} + & + \\ + & 0 \end{matrix} \mapsto \begin{matrix} 0 & + \\ + & + \end{matrix}$

Alternatively, can read off perm from this diagram (simple algorithm).

Conjectures on numerology of $\mathcal{A}_{n,k,m}$

	# max'l cells in decomp. of $\mathcal{A}_{n,k,m}$	
$m=1$	$\binom{n-1}{k}$	Karp-W. (theorem)
$m=2$	$\binom{n-2}{k}$	Arkani-Hamed-Trnka-Thomas
$m=4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	Conj of AH-T
$k=1$	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n,m)$

Is there a formula which generalizes all of these?

Conjectures on numerology of $\mathcal{A}_{n,k,m}$

Let $N(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$

Note: $N(a,b,c)$ symmetric in a,b,c.

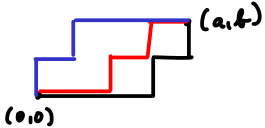
Conj: KWZ (90% confidence?) For even m, there is cell decomposition of $\mathcal{A}_{n,k,m}$ which has $N(k, n-m-k, \frac{m}{2})$ top-dim'l cells (of dim. km)

Conj: KWZ (70% confidence??) For odd m, " " $\mathcal{A}_{n,k,m}$ which has $N(k, n-m-k, \frac{m+1}{2})$ top-dim'l cells.

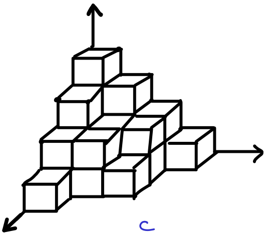
Rk: These conjectures generalize all previous results/ conjectures.

$N(a,b,c)$ counts:

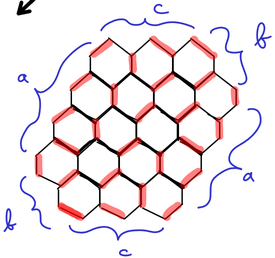
- collections of c noncrossing paths from (a,b) to $(0,0)$ taking steps W and S



- plane partitions $\leq a \times b \times c$ box



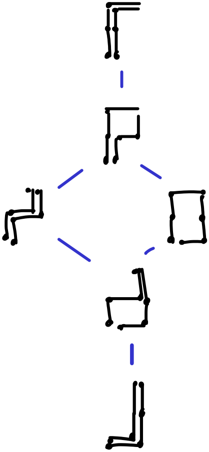
- Kekulé structures (perfect matchings) of a hexagon-shaped benzenoid w/ parameters a,b,c)

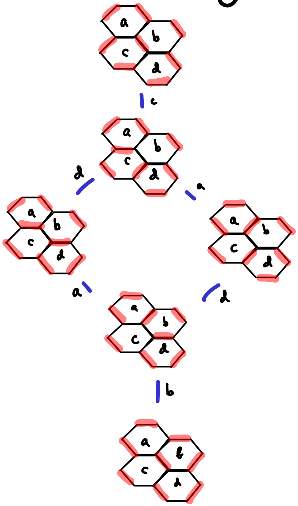


So conjecturally there's a triangulation of $A_{n,k,m}$ whose top-dim'l cells are in bijection with:

- collections of $\lfloor \frac{m+1}{2} \rfloor$ noncrossing paths from $(k, n-m-k)$ to $(0,0)$ taking steps W and S
- plane partitions $\leq k \times (n-m-k) \times \lfloor \frac{m+1}{2} \rfloor$ box
- Kekulé structures (perfect matchings) of a hexagon-shaped benzenoid w/ parameters $k, n-m-k, \lfloor \frac{m+1}{2} \rfloor$

Note: These combinatorial objects all have structure of distributive lattice
(= really nice kind of partially ordered set)





(Containment of plane partitions)

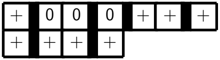
What does this mean for the amplituhedron?

Disjointness of BCFW cells in $\mathcal{A}_{n,k,4}$ for $k=2$

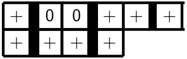
- Using $\reflectbox{L}$-diagram description of BCFW cells, can classify BCFW cells for $k=2$, and construct "domino basis" for each element of a BCFW cell.

The 9 classes of BCFW cells for $k = 2$.

domino

Class 1. 

$$\begin{array}{|cc|cc|cc|cc|c} + & + & + & + & 0 & 0 & 0 & 0 & - \\ \hline 0 & 0 & 0 & 0 & + & + & + & + & + \end{array}$$ o

Class 2. 

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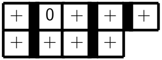
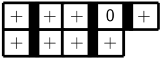
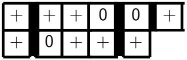
Class 3. 

$$\begin{array}{|cc|cc|cc|c} + & + & + & + & 0 & 0 & - \\ \hline 0 & 0 & + & + & + & + & + \end{array}$$ o

Class 4. 

$$\begin{array}{|cc|cc|cc|c} + & + & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & + & 0 \end{array}$$ d

Class 5. 

$$\begin{array}{|cc|cc|ccc|c} + & + & 0 & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & + & 0 & 0 \end{array}$$ d

The 9 classes of BCFW cells for $k = 2$.

Class 6. $\begin{array}{|c|cc|cc|c|c|}\hline + & + & + & 0 & 0 & 0 & + \\ \hline + & 0 & 0 & + & + & + & \\ \hline\end{array}$ $\quad \begin{array}{|cc|cc|cc|cc|c} + & + & 0 & 0 & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & + & 0 & 0 & 0 \end{array}$ d

Class 7. $\begin{array}{|c|cc|cc|c|}\hline + & + & + & 0 & 0 & + \\ \hline + & 0 & 0 & + & + & + \\ \hline\end{array}$ $\quad \begin{array}{|ccc|cc|cc|c} + & + & 0 & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & 0 & 0 & 0 \end{array}$ d

Class 8. $\begin{array}{|c|ccc|c|}\hline + & + & + & 0 & + \\ \hline + & 0 & + & + & + \\ \hline\end{array}$ $\quad \begin{array}{|ccc|ccc|c} + & + & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & 0 & 0 \end{array}$ d

Class 9. $\begin{array}{|c|cc|c|}\hline + & + & + & + \\ \hline + & + & + & + \\ \hline\end{array}$ $\quad \begin{array}{|ccc|cc|c} + & + & 0 & - & - & - \\ \hline + & + & + & + & + & 0 \end{array}$ d

Recall: $\mathcal{A}_{n,k,4} = \tilde{Z}(Gr_{kn}^{\geq 0})$, where

$\tilde{Z}: Gr_{kn}^{\geq 0} \to Gr_{k,k+4}$ defined by $A \mapsto {}_k\left(\overset{n}{}\right)_n\left(\overset{k+4}{}\right)$

Theorem (Karp-W.-Zhang): Let $k=2$. The images of two distinct BCFW cells in $\mathcal{A}_{n,k,4}$ are distinct.

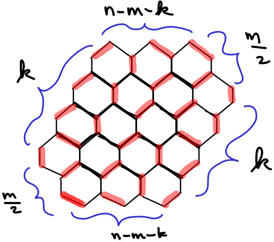
Idea: Suppose V_1 and $V_2 \in Gr_{kn}^{\geq 0}$ lie in 2 BCFW cells.

① $V_i \in Gr_{kn}^{\geq 0} \overset{\text{G.K.}}{\Rightarrow} \forall x \in V_i$, $\operatorname{var}(x) \leq k-1$. Variation small.

② If $\tilde{Z}(V_1) = \tilde{Z}(V_2)$, ie. $V_1 - V_2 \in \ker(Z)$, then

$Z \in Gr_{k+4,n}^{>0} \overset{\text{G.K.}}{\Rightarrow} \forall$ nonzero $y \in V_1 - V_2$, $\operatorname{var}(y) \geq k+4$. Variation big

To prove Thm, need to use ①, ② to get $\Rightarrow\Leftarrow$.



Thank you!

Part 1 of talk joint w/ Steven Karp
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Part 2 of talk joint w/ Steven Karp
+ Yan Zhang