ENERGY LEVELS OF MEAN FIELD SPIN GLASS MODELS

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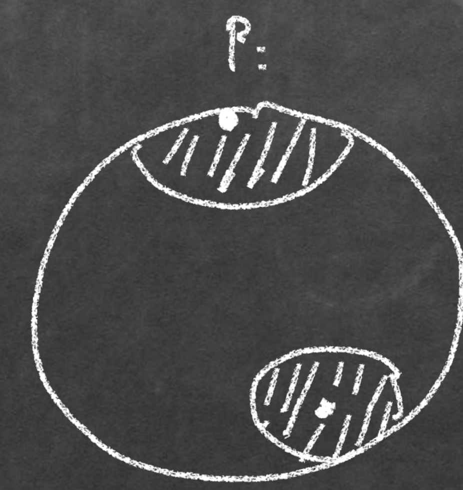
SPHERICAL PERCEPTRON

N dimensional sphere S^N

M IID UNIFORM POINTS ON S^N

$p_1, \dots, p_M$

K "RADIUS" OF THE BALL AROUND EACH p_i



$$B_i = \{ \sigma \in S^N : \langle \sigma, p_i \rangle \geq K \}$$

$$A = \bigcap_{i=1}^{M} S^N \setminus B_i$$

EQUIVALENTLY LOOK AT $\bigcup_{i=1}^{M} B_i$

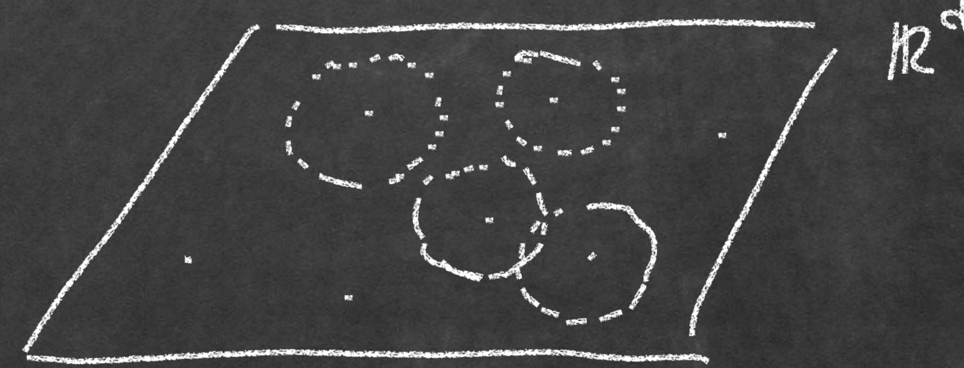
Q: UNDERSTAND THE TOPOLOGY OF THE RANDOM SET A.

TWO REGIMES

- K FIXED, $M = \alpha N$, $N \to \infty$
- N FIXED, $K = -1 + M^{-\beta}$, $M \to \infty$

TOPOLOGY AND GEOMETRY OF RANDOM SETS

① FINITE DIMENSIONS:



- CONTINUUM PERCOLATION

  (MEESTER, PENROSE, ROY, KESTEN, ...)

  - CONNECTIVITY QUESTIONS
  - #CONNECTED COMP., PERSISTENCE HOMOLOGY, ...

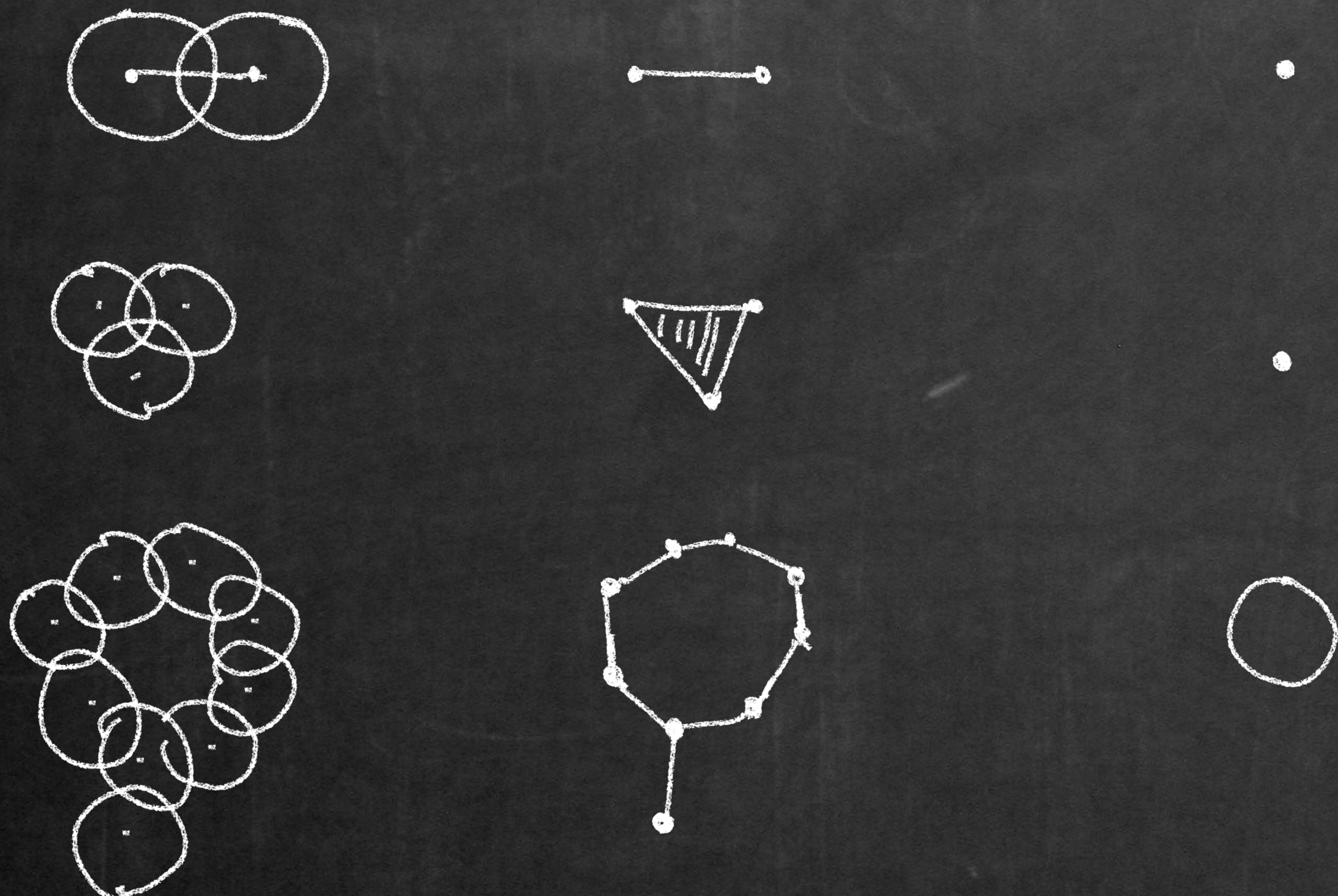
  # POINTS, volume $\to \infty$.

- NODAL DOMAINS OF RANDOM FUNCTIONS

  (NAZAROV, SODIN, SARNAK, WIGMAN, ...)

  - RANDOM EIGENFUNCTIONS OF Δ. $\lambda \to \infty$ SEMI CLASSICAL LIMIT.
  - ISOTROPIC GAUSSIAN R.F. $L\phi_\lambda = \lambda \phi_\lambda$

ČECH COMPLEX AND ITS EMPIRICAL MEASURE.



$$\mu = \frac{1}{N_c} \sum_{\substack{S:\\ \text{CONNECTED} \\ \text{COMP.}}} \delta_{[S]}$$

Q: DOES THE SEQUENCE μ_n CONVERGE?

THM: YES.

EX:

$$\mu = \frac{2}{3}\,\delta_{[pt]} + \frac{1}{3}\,\delta_{[S^1]}$$

② HIGH DIMENSIONAL LIMIT.

TOP / GEO OF RANDOM FUNCTIONS $H_N : \Sigma_N \to \mathbb{R}$ $N \to \infty$

SUBLEVEL SETS: FOR $E \in \mathbb{R}$ $\Sigma_N = S^N$ OR $\{\pm 1\}^N$

$$L(E) = \{ \sigma \in \Sigma_N : H_N(\sigma) \leq E \}.$$

QUESTIONS: (a) # CONNECTED COMPONENTS

(b) BETTI NUMBERS

(c) VOLUME OF $L(E)$

TRY TO RECONSTRUCT M_N BY SCANNING $L(E)$ AS E CHANGES

MIXED p-SPIN GLASS MODEL

- H_N CENTERED GAUSSIANS
- $M_N = S^{N-1}(\sqrt{N})$

$$\mathbb{E} H_N(\sigma) H_N(\tau) = N \xi\left(\tfrac{1}{N}\langle \sigma, \tau\rangle\right)$$

ξ INDEPENDENT OF N.

$$\xi(t) = \sum_{p \geq 1} \beta_p^2 t^p$$

: SCHOENBERG '38, '42

CLASSIFY ALL POSITIVE DEFINITE FUNCTIONS ON S^{N-1}

$$\mathcal{F}_N \supseteq \mathcal{F}_{N+1} \supseteq \dots \supseteq \bigcap_{N \geq 1} \mathcal{F}_N := \mathcal{F}$$

$p = 1$ EXTERNAL FIELD $(\beta_1 = 0)$

$\xi(t) = t^2$ SPHERICAL SHERRINGTON-KIRK

$\xi(t) = t^p$, $p \geq 3$ PURE p-SPIN MODEL

If $\beta_p \neq 0 \ \forall p \geq 2$ GENERIC.

$$H_N(\sigma) = \sum_{p \geq 2} \beta_p H_{N,p}(\sigma)$$

$$H_{N,p} = \frac{1}{N^{\frac{p-1}{2}}} \sum_{i_1,\dots,i_p} g_{i_1,\dots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}, \quad \sigma = (\sigma_1,\dots,\sigma_N).$$

$g_{i_1,\dots,i_p} \sim N(0,1)$ IID

$$\boxed{\frac{\min_\sigma H(\sigma)}{N} \longrightarrow GS \in \mathbb{R}}$$

HOW TO UNDERSTAND THE TOPOLOGY OF SUBLEVEL SETS?

$$L(E) = \{ \sigma \in \Sigma_N \mid H_N(\sigma) \le NE \} \qquad \text{AS } N \to \infty$$

1st METHOD: INVESTIGATE # AND LOCATION OF CRITICAL POINTS? CRITICAL VALUES.

CHANGE IN TOPOLOGY REQUIRES CROSSING A CRITICAL VALUE.

$L(E)$:

MATH:

- A., BEN AROUS, ČERNY, FYODOROV, SUBAG, ZEITOUNI, ...

FAR FROM COMPLETE / RELIES ON KAC-RICE'S FORMULA, RMT.

COUNTING CRITICAL POINTS / VALUES. WHAT IS KNOWN?

let $Crt_k(E) = \#$ of critical points of index k in $L(E)$

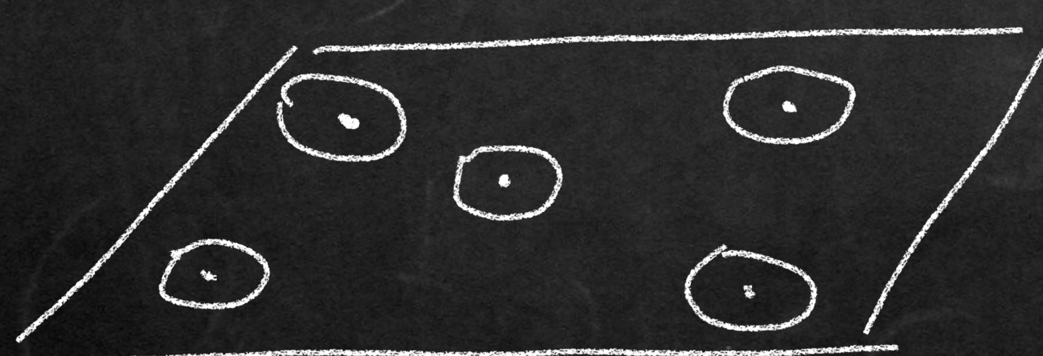
For any model:

- $\mathbb{E}Crt_k(\infty) \approx e^{N\Sigma_k}$, $\Sigma_k > 0$ (total number)
- $\mathbb{E}Crt_k(E) \approx e^{N\Sigma_k(E)}$, $\Sigma_k(E)$ explicit.
- $\mathbb{E}\chi(L(E)) \approx e^{N\Theta(E)}$, $\Theta(E)$ an "Airy-type" function
  oscillatory, ± values

For the pure p spin (homogeneous polynomials of deg-p) and small pertubations

- exponentially many local minima (not in average)
- near the ground state: no saddles
- diverging barriers btw minima

$$L(E) \simeq \bigcup_{i=1}^{e^{N\Sigma_1(E)}} B(\rho_i, r_i) \quad \text{disjoint}$$



S^N

N

2^{nd} METHOD: THERMODYNAMICAL APPROACH.

$\beta \geq 0$

$$G_N(A) = \frac{\int_A e^{\beta H_N(\sigma)}\, d\sigma}{Z_N}$$

$$Z_N = \int_{\Sigma^N} e^{\beta H_N}\, d\sigma$$

BASED ON TWO IDEAS.

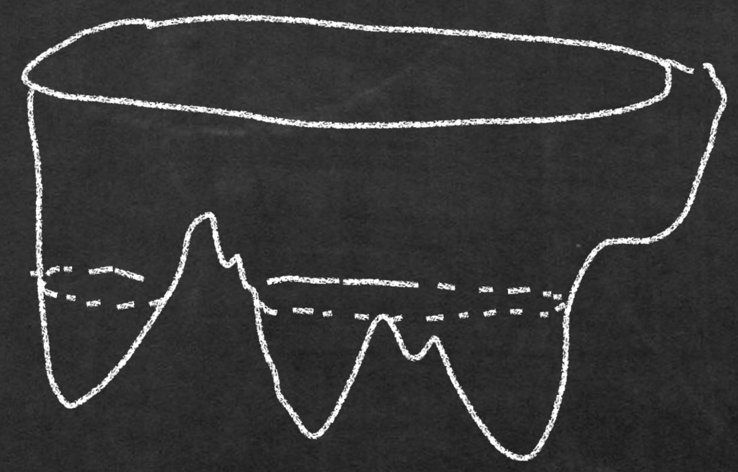
- EQUIVALENCE OF ENSEMBLES:

$$\beta \longleftrightarrow E$$

FOR EACH $\beta \geq 0$ $\exists$ $E(\beta) \in \mathbb{R}$,

$$G_N\left(\left\{\sigma : \left|\frac{H_N(\sigma)}{N} - E(\beta)\right| \leq \varepsilon\right\}\right) \to 1$$

If $\lim_{N\to\infty} \frac{1}{N} \log Z_N = F(\beta)$ and F is deterministic differentiable THEN $E(\beta) = F'(\beta)$.



- PHYSICISTS BEAUTIFUL PREDICTIONS FOR THE BEHAVIOR OF G_N AS $N\to\infty$.
- I WILL CONSIDER BOTH $\Sigma_N = S^{N-1}$ AND $\Sigma_N = \{\pm 1\}^N$

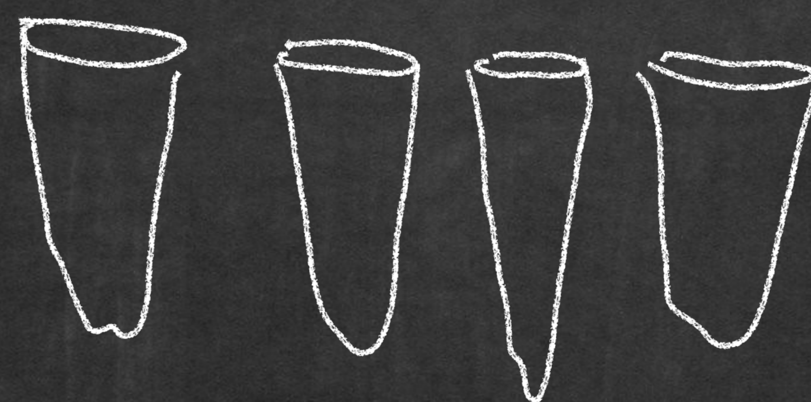
THE LIMITING STRUCTURE OF THE GIBBS MEASURE IS DESCRIBED BY A SINGLE ORDER PARAMETER : PARISI MEASURE.

LANDSCAPE ROUGHLY 3 TYPES OF BEHAVIOR:

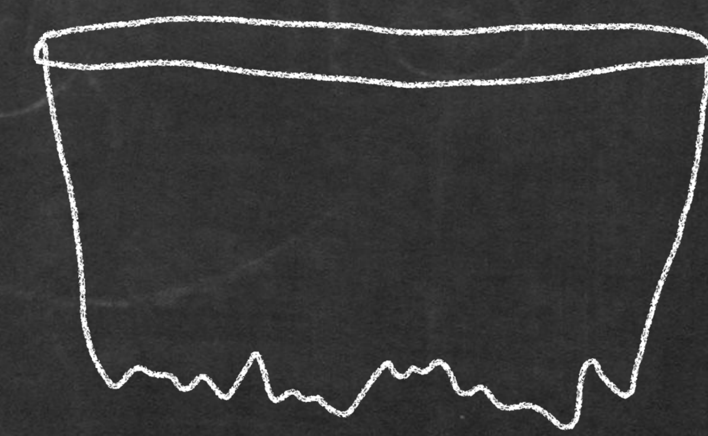
REPLICA SYMMETRIC

ZERO COMPLEXITY

(OR FINITE)
1 STEP REPLICA SYMMETRY BREAKING



FULL STEP REPLICA SYMMETRY BREAKING



EXPONETIALLY MANY LOCAL MINIMA

HOW TO DIFFERENTIATE THESE TWO?

PARISI MEASURE: PROB. MEASURE μ^*

SIZE OF ITS SUPPORT CLASSIFIES THE LANDSCAPE

PARISI '79: limiting FREE ENERGY VIA REPLICA METHOD

$$\lim_N \frac{1}{N} \log \int_{\Sigma_N} e^{\beta H_N}\, d\sigma = \lim_N \frac{1}{N} \log Z_N = \inf_{\mu\ \text{P.M.}} \mathcal{P}(\mu)$$

$$\xi(x) = x^2$$

How to "do" this?

STEP ①: COMPUTE MOMENTS OF Z_N: $n \geq 1$, $n \in \mathbb{N}$.

$$\mathbb{E} Z_N^n = \mathbb{E} \sum_{\{\sigma^a\}} \exp\Big(\beta \sum_{a=1}^n H(\sigma^a)\Big) = \mathbb{E} \sum_{\sigma^a} \prod_{i<j} \exp\Big(\beta \frac{g_{ij}}{\sqrt{N}} \sum_{a=1}^n \sigma_i^a \sigma_j^a\Big)$$

$$\mathbb{E} e^{\lambda X} = e^{\frac{1}{2}\Delta\lambda^2}$$

$$= \sum_{\{\sigma^a\}} \exp\Big(\frac{\beta^2}{4N} \sum_{1\le a,b\le n} \Big(\sum_{i=1}^N \sigma_i^a \sigma_i^b\Big)^2\Big)$$

INTRODUCE NEW VARIABLES THAT REPRESENT THE OVERLAP BETWEEN σ^a, σ^b.

$$1 = \int dQ_{ab}\, \delta\Big(Q_{ab} - \frac{1}{N}\sum_{i=1}^N \sigma_i^a \sigma_i^b\Big) = N \int dQ_{ab} \int \frac{d\lambda_{ab}}{2\pi}\, e^{-i\lambda_{ab}(NQ_{ab} - \sum \sigma_i^a \sigma_i^b)}$$

NOW:

$$\mathbb{E} Z_N^n = \int \prod_{a<b} dQ_{ab} \sum_{\{\sigma^a\}} \exp\left(\frac{N\beta^2 n}{4} + \frac{N\beta^2}{2} \sum_{a<b} Q_{ab}^2 \right) \delta\left(Q_{ab} - \frac{1}{N} \sum \sigma_i^a \sigma_i^b \right)$$

$$= \int \prod_{a<b} dQ_{ab}\, d\lambda_{ab}\; e^{N G(Q,\lambda)}$$

WHERE $G(Q,\lambda) = \frac{n\beta^2}{4} + \frac{\beta^2}{2} \sum_{a<b} Q_{ab}^2 \; i \sum_{a<b} \lambda_{ab} Q_{ab} + \log \left[\sum_{\{\sigma^a\}} e^{\sum_{a<b} i \lambda_{ab} \sigma_a \sigma_b} \right]$

depends on $n(n-1)$ VARIABLES, $1 \le a < b \le n$, $n \in \mathbb{N}$.

ASYMPTOTIC ANALYSIS:

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_N^n = \sup_{Q,\lambda} \left\{ G(Q,\lambda) \right\}$$

Q $n \times n$ MATRIX

$$\left[Q_{ab} \right]$$

Q_{ab} OVERLAP BTW σ^a σ^b

- POSITIVE DEFINITE
- MAXIMIZER SHOULD HAVE REPLICA SYMMETRIES

STEP 2!

SWITCH LIMITS + CRAZY MAGIC

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\log Z_N = \lim_{N\to\infty}\frac{1}{N}\lim_{n\downarrow 0}\frac{1}{n}\log \mathbb{E}\, Z_N^n \quad \text{(L'HOPITAL)}$$

$$= \lim_{n\to 0}\left\{\lim_{N\to\infty}\frac{\log \mathbb{E}\, Z_N^n}{nN}\right\}$$

$$(**) \quad = \lim_{n\to 0}\frac{1}{n}\sup_{(Q,\lambda)}\left\{G_n(Q,\lambda)\right\}$$

$1 \leq n < \infty \longmapsto 0 \leq n \leq 1$

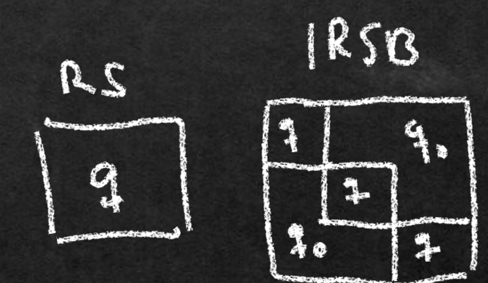
$\sup \longmapsto \inf$

$$(*) \quad = \inf_{Q}\left\{F(Q)\right\}$$

WHY? Q HAS $\frac{n(n-1)}{2}$ entries

ANSATZ ON THE STRUCTURE OF Q.

$Q_{ab} = q \quad \forall ab$ RS ANSATZ



$$(*) = \inf_{q\in[0,1]}\left(\log 2 + \frac{\beta^2}{2}(1-q)^2 + \mathbb{E}\log\cosh\left(z\beta\sqrt{2q}\right)\right)$$

$\boxed{\text{CORRECT}}$ IF $\beta < 1/\sqrt{2}$.

THEOREM: (GUERRA, TALAGRAND, PANCHENKO, CHEN)

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = \inf_{\mu} \left(\Phi_{\mu,\beta}(0,h) - \frac{1}{2}\int_0^1 \beta\, \mu[0,s] \cdot s\, \xi''(s)\, ds \right)$$

WHERE μ IS A PROBABILITY MEASURE ON $[0,1]$ AND $\Phi_{\mu,\beta}$ SOLVES:

$$\partial_t \Phi_{\mu,\beta} = -\frac{\xi''(t)}{2} \left(\partial_{xx} \Phi_{\mu,\beta} + \beta\, \mu[0,t] \left[\partial_x \Phi_{\mu,\beta} \right]^2 \right)$$

$$\Phi_{\mu,\beta}(1,x) = \begin{cases} \dfrac{\log\cosh \beta x}{\beta}, & \Sigma_N = \{\pm 1\}^N \\ x^2, & \Sigma_N = S^{N-1}. \end{cases}$$

REMARKS

- MINIMIZER IS UNIQUE
- THE ABOVE FORMULA SIMPLIFIES IN THE CASE OF $\Sigma_N = S^{N-1}$ CRISANTI SOMMERS

REMARKS:

RESCALE: $\beta\mu(0, 1-\underline{s}) \to \lambda(0,1-s)$

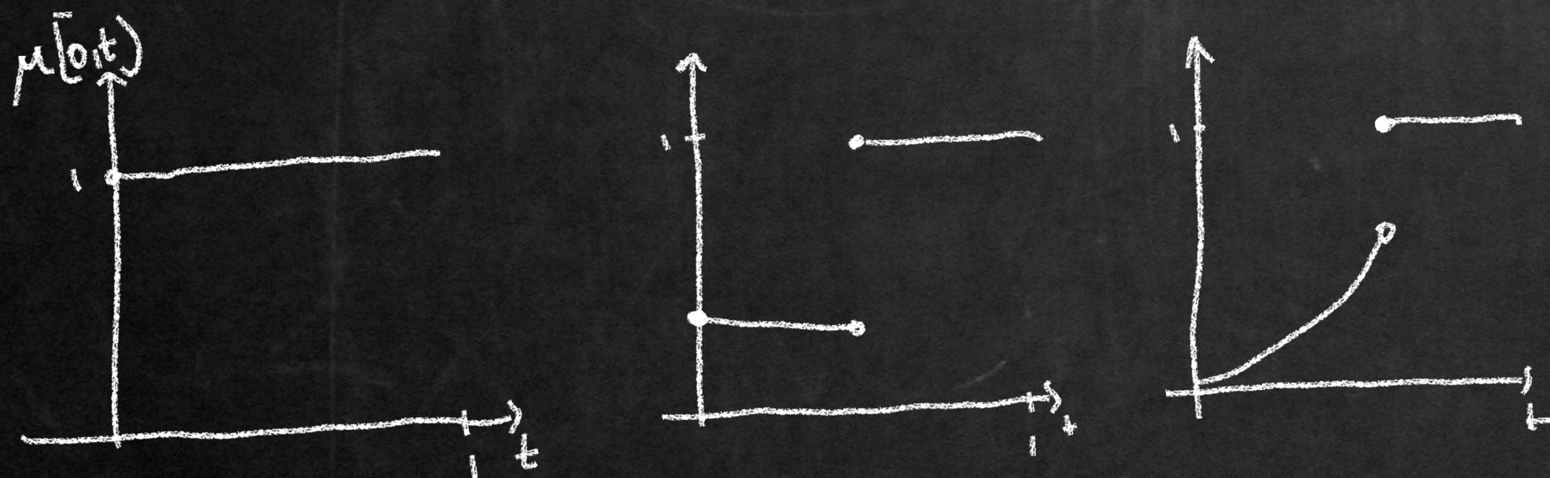
- EXTEND THE ABOVE FORMULA TO ZERO TEMPERATURE$^{\beta}$ (A.-CHEN)

$$\lim_{N\to\infty} \frac{\min H_N}{N} = \inf_{\lambda \text{ measure on } [0,1]} \left\{ \underline{\qquad\qquad} \right\}$$

- PARISI MEASURE IS DEFINED AS THE MINIMIZER OF THE ABOVE VAR. PRINCIPLES.

PREDICTIONS:

ON HYPERCUBE (PURE p-SPIN)



IN GENERAL: FOR THE HYPERCUBE:

- RS AT HIGH TEMPERATURE
- FRSB AT LOW ENOUGH TEMP.
- FRSB AT ZERO TEMPERATURE

THM (A., CHEN, ZENG) ANY MODEL ξ ON $\{\pm 1\}^N$ IS ∞-RSB AT ZERO TEMPERATURE, THAT IS, IF μ^* IS THE PARISI MEASURE THEN

$$\#\{\operatorname{supp} \mu^*\} = \infty.$$

PREVIOUS RESULTS ON $\{\pm 1\}^N$:

- AIZENMAN - LEBOWITZ - RUELLE '89, TONINELLI '02: IF $\beta > \beta_c$, $\#\{\operatorname{supp} \mu^*\} \geq 2$
  $\beta \leq \beta_c$ $\#\{\operatorname{supp} \mu^*\} = 1$

A.-CHEN: FOR β LARGE ENOUGH $\#\{\operatorname{supp} \mu^*\} \geq 3$.

ON THE SPHERE THE PREVIOUS RESULT DOES NOT HOLD!

(1) $\xi(x) = x^2$ MODEL IS RS AT ALL TEMPERATURES.

$H = \langle J\sigma, \sigma \rangle$

(2) $\xi(x) = x^p$, $p \geq 3$ MODEL IS 1RSB FOR ALL $\beta > \beta_c$

$\# \operatorname{supp} \mu^* = 2$ (TALAGRAND '07

(3) $\xi(x) = \frac{5}{6} x^3 + \frac{1}{6} x^{16}$ IS 2-RSB AT ZERO TEMPERATURE

(A. - ZENG '18)

RMK: . BEFORE (3) IT WAS BELIEVED THAT RS, 1RSB, FRSB WERE THE ONLY POSSIBILITIES AT LOW ENOUGH TEMPERATURE.

. FULL CLASSIFICATION IS STILL OPEN.

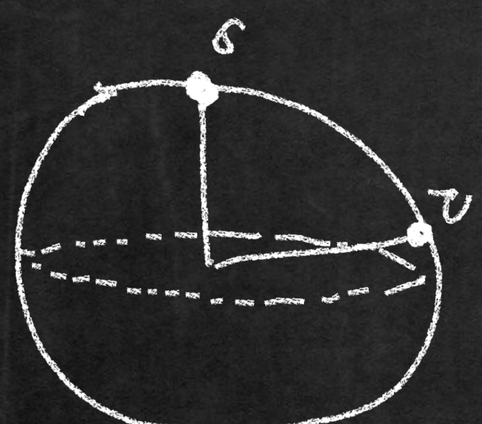
From the Parisi measure to the landscape:

- At finite temperature: set $R_{12} = \frac{1}{N}\langle \sigma, \tau \rangle$, $\xi(x) = \sum \beta_p x^p$

$$\text{If } \beta_p \neq 0, \quad \lim_{N\to\infty} \mathbb{E}\langle R_{12}^p \rangle = \int x^p \, d\mu^*(x)$$

That is, at least for generic models ($\beta_p \neq 0 \ \forall p$) the Parisi measure is the limiting law of the overlap.

RS:



Haar measure $\quad R_{12} \xrightarrow{N\to\infty} \delta_0 \quad (\text{RS } \beta < \beta_c)$

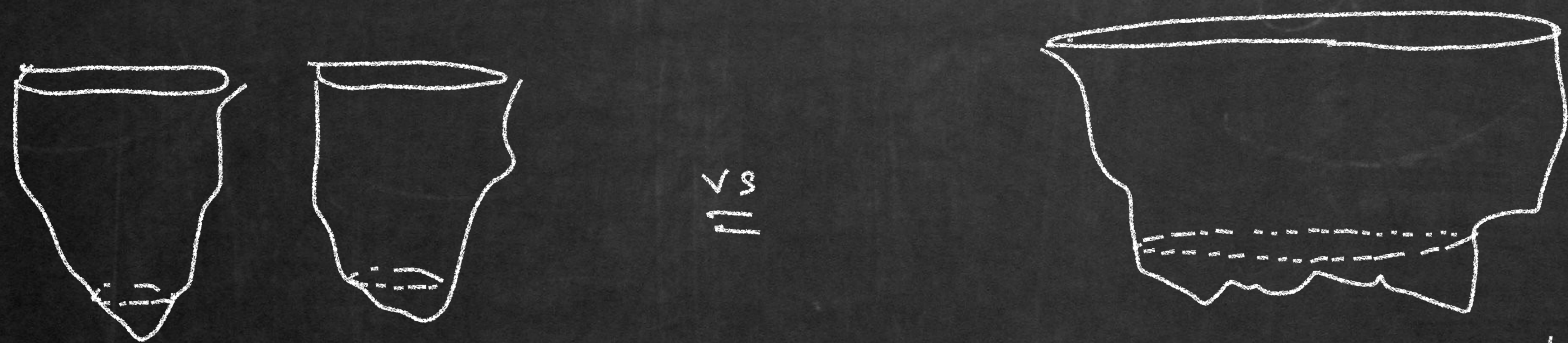
1RSB

$$\mu^* = m\,\delta_0 + (1-m)\,\delta_q$$

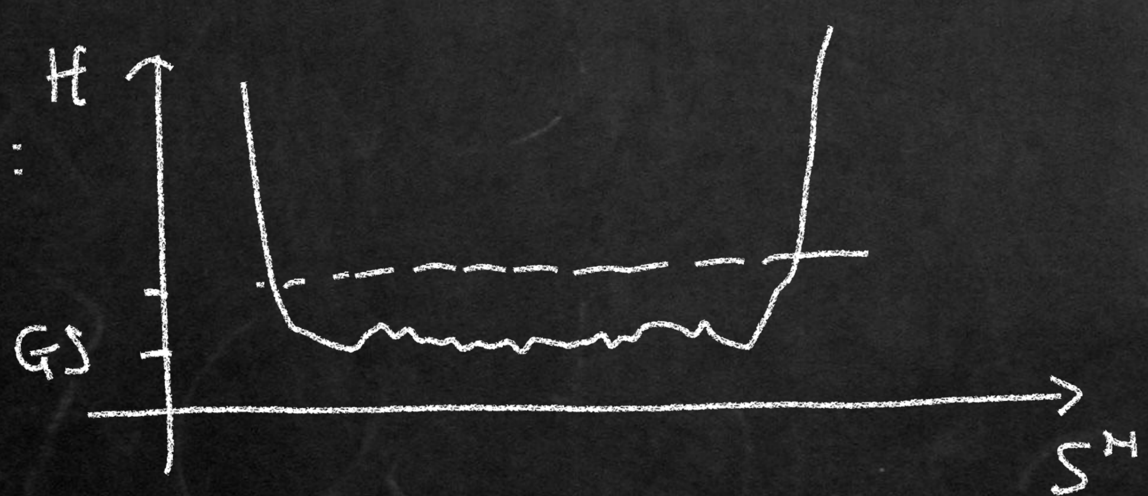
THEOREM (A.-CHEN) Let μ^* be the Parisi measure at zero temperature. Let $u \in [-1,1]$ s.t. $|u| \in \operatorname{supp} \mu^*$.

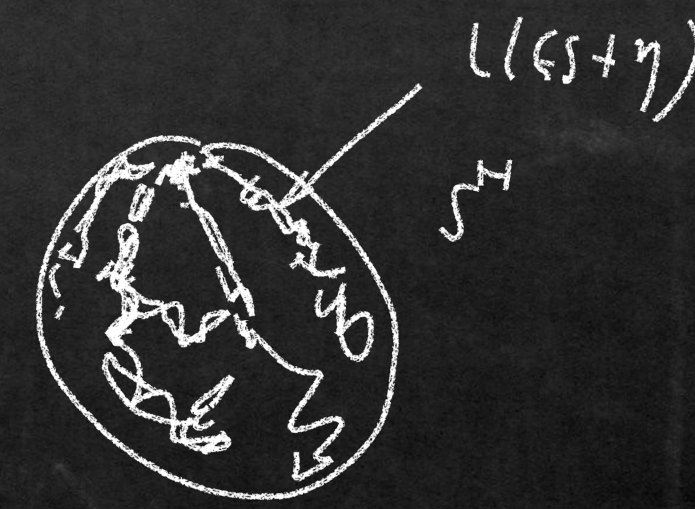
For any $\varepsilon, \eta > 0$, $\exists\, K > 0$ s.t. $\forall\, N \geq 1$

$$\mathbb{P}\left(\exists\, \sigma^1, \sigma^2 \in L(GS+\eta),\ R_{12} \in (u-\varepsilon, u+\varepsilon)\right) \geq 1 - K e^{-N/K}.$$



FRSB:





① ASSUME THAT ξ IS <u>RS</u> AT ZERO TEMPERATURE. (SSK or if we add strong external field)

THEN FOR ANY $\varepsilon > 0$ $\exists\ \eta, K > 0$

$$\mathbb{P}\left(\exists\, \sigma^1, \sigma^2 \in L(GS+\eta),\ R_{12} \in [-1, 1-\varepsilon]\right) \le K e^{-N/K}.$$

<u>RS</u>: NEAR THE G.S.:

"ANY TWO NEAR MINIMA MUST BE CLOSE TO EACH OTHER."

$$(*)\quad \xi''(1) \le \xi'(1) + h^2$$

② ASSUME THAT $1/\sqrt{\xi''}$ IS CONCAVE AND $\Sigma_N = S^{N-1}$. THEN ξ is FRSB AT ZERO TEMPERATURE WITH

$$\operatorname{supp} \mu^* = [0,1]$$

3 1-RSB AT ZERO TEMPERATURE $(h=0)$

Let $z \geq 0$ be the unique soln of

$$\frac{1}{\xi'(1)} = \frac{1+z}{z^2}\log(1+z) - \frac{1}{z}.$$

SET

$$\zeta(s) = \xi(s) + \xi'(s)(1-s) + \frac{\xi'(s)}{z} + \frac{(1+z)\xi'(1)}{z^2}\log\left(1 + \frac{z\,\xi'(s)}{\xi'(1)}\right)$$

THM: (A.-CHEN) ξ is 1RSB at 0 temp iff $\zeta(s) \leq 0$ $\forall\, s \in [0,1]$, $\beta_p \neq 0$ for some $p \neq 2$

Furthermore if $\zeta(s) < 0$ $\forall\, s \in (0,1)$ then $\forall\, \varepsilon > 0\ \exists\, K > 0$:

$$\mathbb{P}\left(\exists\, \sigma^1, \sigma^2 \in L(GS+\eta),\ R_{12} \in [-1+\varepsilon, -\varepsilon] \cup [\varepsilon, 1-\varepsilon]\right) \leq K e^{-N/K}$$

NEAR MINIMA CAN BE ONLY
- NEAR EACH OTHER
- ORTHOGONAL

How many?

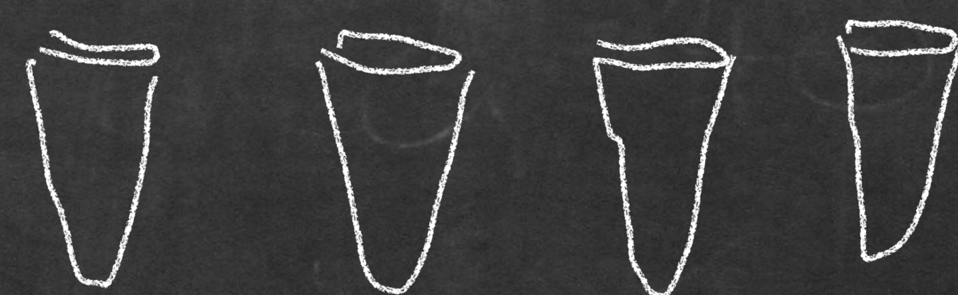
RMK: $\frac{s}{\xi'(s)}$ convex implies $\zeta(s) < 0$.

Multiple Valleys

Thm (A.–Chen) For any $\varepsilon, \eta > 0$ $\exists$ $K > 0$ s.t. $\forall N \geq 1$

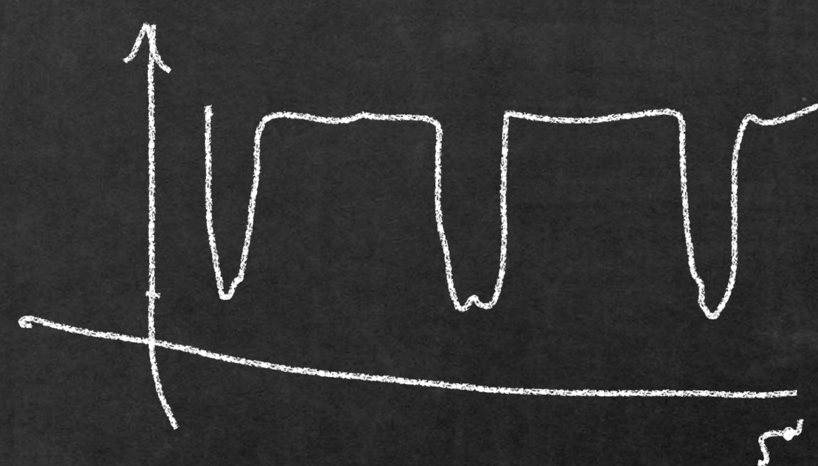
$$(*) \quad \mathbb{P}\left[\exists \text{ a subset } O_N \subseteq L(GS+\eta),\ \#O_N \geq K e^{N/K},\ \frac{1}{N}\langle \sigma, \sigma' \rangle \leq \varepsilon \ \ \forall \sigma \neq \sigma' \in O_N \right] \geq 1 - e^{-N/K}$$

Exponentially many orthogonal valleys



Previous results:

- Chatterjee: $\#O_N \geq \log N$
- Ding, Eldan, Zhai: $\#O_N \geq N^c$.



Thm (A.–Zeng)

(*) also holds in 2-RSB models.

$$\left| \frac{1}{N}\langle \sigma, \sigma' \rangle - q \right| \leq \varepsilon$$

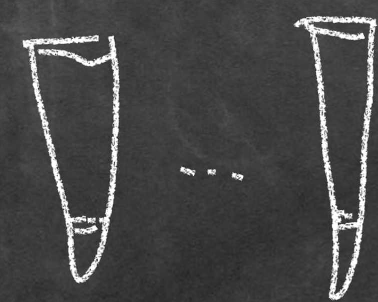
$$\mu^* = m\delta_0 + (1-m)\delta_q .$$

Recap: Parisi measure: implies the following about the landscape

RS

× 1

1RSB



$\times e^{NK}$

orthogonal

no saddles

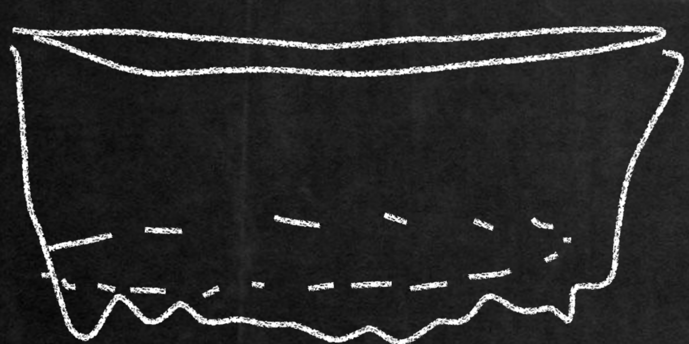
2RSB

(overlap q_1)

orthogonal

no saddles

FRSB



All possible overlaps are present

CONTROLLING THE LANDSCAPE VIA A COUPLED FREE ENERGY

IDEA GOES BACK TO GUERRA/TALAGRAND:

LOOK AT THE COUPLED PARTITION FUNCTION CONSTRAINED AT FIXED OVERLAP.

FIX $A \subseteq [-1,1]$.

$$F_N(A) = \frac{1}{N} \mathbb{E} \log \int \exp\left(\beta H(\sigma^1) + \beta H(\sigma^2)\right) \mathbb{1}\{R_{12} \in A\} \, d\sigma^1 d\sigma^2$$

FOR $A = (u-\varepsilon, u+\varepsilon)$ IT IS HARD (OPEN!) TO DETERMINE $\lim_{N\to\infty} F_N(A)$.

HOWEVER IF ONE LOOKS AT A TWO-DIMENSIONAL VERSION OF THE PARISI FUNCTIONAL:

$$\lim_{\varepsilon \downarrow 0} \limsup_{N\to\infty} F_N(u-\varepsilon, u+\varepsilon) \leq \mathcal{P}_{\beta,u}(b,\lambda,x),$$

$b, \lambda \in \mathbb{R}$

x NONDECREASING

$\max\left(1, |\lambda| + \int_0^1 \beta^2 \xi''(s) x \, ds\right) < b$

WHERE

$$\mathcal{P}_{\beta,u}(b,\lambda,x) = \log \sqrt{\frac{b^2}{b^2-\lambda^2}} + \int_0^u \frac{\beta^2 \xi'' \, dq}{b-\lambda-a(q)} + \frac{1}{2}\int_u^1 \frac{\beta^2 \xi''}{b-\lambda-a(q)} + \frac{1}{2}\int \frac{\beta^2 \xi'' \, dq}{b+\lambda-a(q)} - \lambda u + b - 1 - \beta^2 \int_0^1 q \xi''(q) x(q) \, dq$$

$$a(q) = \int_q^1 \beta^2 \xi'' x \, dq.$$

IN THE SAME WAY AS WE OBTAINED A PARISI FORMULA FOR THE GROUND STATE

THE PREVIOUS BOUND LEADS TO:

<u>Prop</u>: FOR ALL $u \in [-1,1]$ and any choices of $(\hat{b}, \hat{\lambda}, \hat{x})$ satisfying $(*)$

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{N\to\infty}} \frac{1}{N} \mathbb{E} \max_{R_{12}\in(u-\varepsilon,u+\varepsilon)} \left(H_N(\sigma^1) + H_N(\sigma^2)\right) \leq \mathcal{P}_{u,\infty}(\hat{b}, \hat{\lambda}, \hat{x})$$

How to use it? ①: If $u \in \operatorname{supp} \mu^*$ then $\forall \varepsilon > 0$

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_{R_{12}\in(u-\varepsilon,u+\varepsilon)} \left(H_N(\sigma^1) + H_N(\sigma^2)\right) = -2\,GS.$$

EASY PART:

②: Assume ①. Concentration of extrema $\Rightarrow$ w.h.p.

$$\begin{cases} \dfrac{1}{N} \max\limits_{R_{12}\approx u} H(\sigma^1) + H_N(\sigma^2) \geq -2GS - \eta/2 \\[2ex] \dfrac{\max H}{N} \leq -GS + \eta/4 \end{cases}$$

$\Rightarrow \exists\, \sigma^1, \sigma^2$ with $\dfrac{H(\sigma^1) + H(\sigma^2)}{N} \geq -2GS - \eta/2$

If $H(\sigma^1) \leq N(GS - \eta)$:

$$\Rightarrow \quad -2GS - \eta/2 \leq \frac{H(\sigma^1) + H(\sigma^2)}{N} \leq -2GS + \frac{\eta}{4} - \eta \leq -2GS - \frac{3\eta}{4} \quad \text{↯}$$

HOW TO GET (1). DO IT FOR GENERIC MODELS FIRST law of $R_{12} \to \mu^*$

SEND $\beta \to \infty$ AND APPROXIMATE.

TO OBTAIN THE OTHER ENERGY LANDSCAPE RESULTS: FIND PARAMETERS S.T.

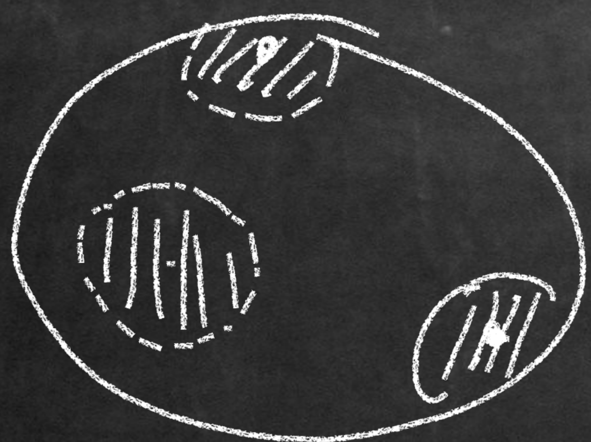
$$\mathcal{P}_{\infty,n}(\hat{\lambda}, \hat{b}, \hat{x}) < -2GS - \eta.$$

(FINER ANALYSIS OF THE 2-dim. PARISI FUNCTIONAL)

OPEN QUESTIONS:

- GIVEN ξ DETERMINE IF 1RSB / k-RSB / FULL
- RELATE BACK TO $\# Crt_k$.
- VOLUME OF STABLE MANIFOLDS OF LOCAL MINIMA
-

GOING BACK TO THE SPHERICAL PERCEPTRON: $k \in \mathbb{R}$



let $p_1, \dots, p_M$ IID UNIFORM SPHERE

$$A_{N,M} = \bigcap_{k=1}^{N} \{ \sigma : \langle \sigma, p_k \rangle \geq k \}$$

$$= \left[\bigcup_{k=1}^{M} \{ \sigma : \langle \sigma, p_k \rangle < k \} \right]^c$$

$M = \alpha N$

Q: FOR EACH VALUE OF k, α

DETERMINE IF

- $\mathbb{P}[A_{N,M} = \emptyset] \to 1$
- $\frac{\mathrm{vol}(A_{N,M})}{\mathrm{vol}(S^{N-1})} \approx ?$

FOR $k \geq 0$ THE PROBLEM IS RS:

$A_{N,M}$ IS CONVEX!

GARDNER '88

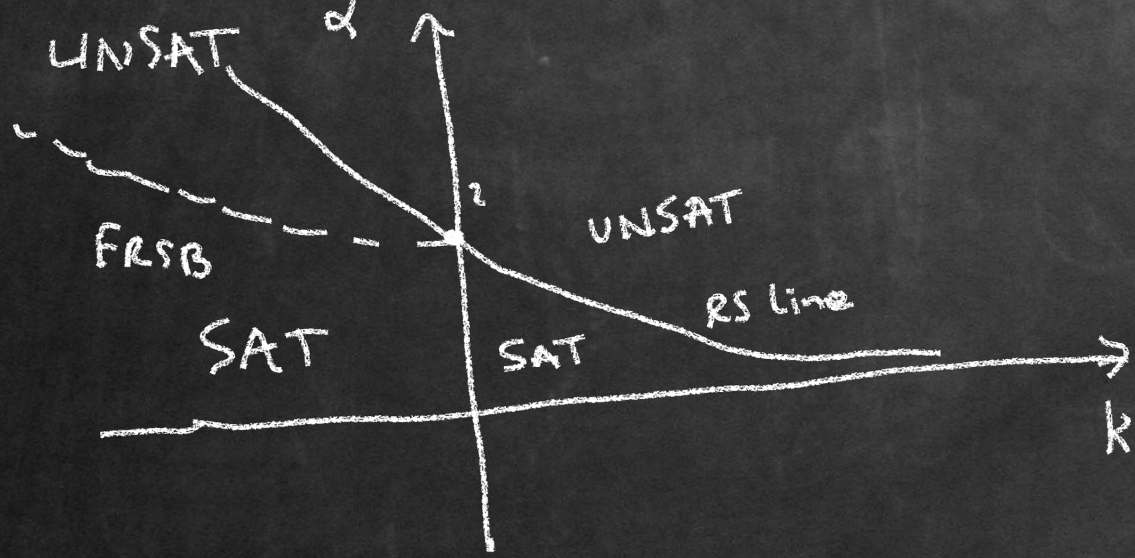
SHCHERBINA - TIROZZI '02

$$\exists \, \alpha_c = \left(\frac{1}{\sqrt{2\pi}} \int_{-k}^{\infty} (z+k)^2 e^{-z^2/2} \, dz \right)^{-1}$$

$\alpha_c(0) = 2$

$$\frac{1}{N} \log \mathrm{vol}(A_{N,M}) \longrightarrow \begin{cases} f(\alpha) & \text{if } \alpha < \alpha_c \\ -\infty, & \text{if } \alpha > \alpha_c \end{cases}$$

SPHERICAL PERCEPTRON



FRANZ - PARISI: $K<0$ NON-CONVEX

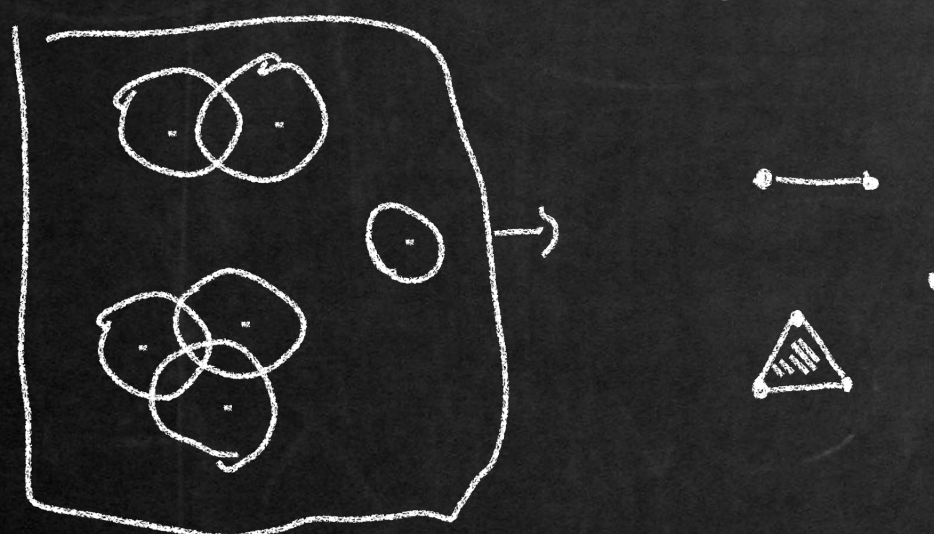
$K<0$ α_c IS GIVEN BY A FRSB SOLUTION.

$r = r(K)$

INTERESTED IF $\displaystyle\bigcup_{i=1}^{\alpha N} B(p_i, r) = S^{N-1}$

DIFFERENT QUESTION

ČECH COMPLEX



$A = \bigcup B(p_i, r) \longmapsto \check{C}(A)$

STATISTICS OF EACH GEOMETRIC CELL OF THE ČECH COMPLEX

FOR N FIXED: LET [S] BE THE TYPE OF A SIMPLICIAL COMPLEX

$$\mu_N = \frac{1}{b_0(\check{C}(A))} \sum_S \delta_{[S]}$$

· EXPLORED BY NAZAROV - SODIN, SARNAK - WIGMAN
ADLER - BOBROWSKI,

THM (A. - LERARIO - LUNDBERG) $\exists$ A MEASURE ν_d ON THE SPACE OF GEOMETRIC SIMPLICIAL COMPLEXES SUCH THAT IF $R = \alpha N^{-1/d}$

$$\mu_N \longrightarrow \nu_{d,\alpha} \quad \text{a.s.}$$

FURTHERMORE, $\nu_d([s]) > 0 \quad \forall [s]$.

RMK:

· CONNECTIVITY HAPPENS AT LARGER RADIUS. $R = \alpha N^{-1/d} \log N$ (BOBROWSKI - WEINBERGER)

· PROOF BASED ON INTEGRAL GEOM. SANDWICH ⊕ SARNAK - WIGMAN POSITIVITY THM.

THANK YOU.