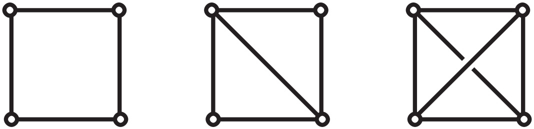
Counting rules for periodic frameworks

Simon Guest

Department of Engineering
University of Cambridge



Outline

Outline

Background

Counting for infinite, repetitive structures

Using symmetry to extend counting rules

A symmetry criterion for ‘equiauxetic’ materials

Conclusions

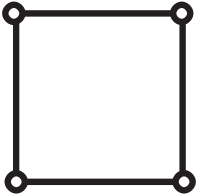
Maxwell counting

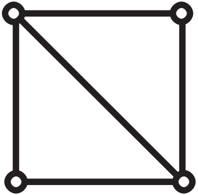
In 1864, James Clerk Maxwell wrote, in *On the calculation of the equilibrium and stiffness of frames,*

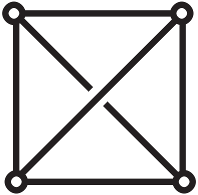
> A frame of s points in space requires *in general* $3s-6$ connecting lines to render it stiff. In those cases in which stiffness can be produced with a smaller number of lines, certain conditions must be fulfilled, rendering the case one of a maximum or minimum value of one or more of its lines. The stiffness of such frames is of an inferior order, as a small disturbing force may produce a displacement infinite in comparison with itself.

Example of Maxwell counting in 2D

For a two-dimensional pin-jointed structure, Maxwell's statement would be that a structure with j joints would require, in general, $2j - 3$ bars to be rigid.







Calladine's extension of Maxwell's Rule

Calladine pointed out, in 1978, that the difference between the number of bars b and $2j - 3$ (in 2D) or $3j - 6$ (in 3D) exactly counts the difference between the number of infinitesimal mechanisms m and the number of states of self-stress s

$$m - s = 2j - 3 \qquad (2D)$$
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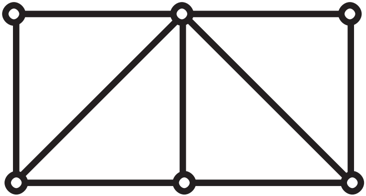
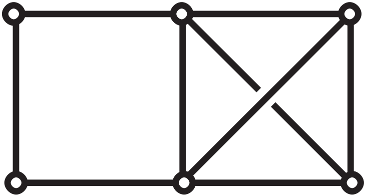
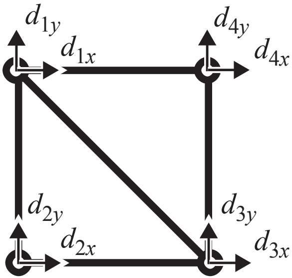


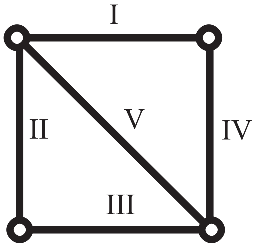


Compatibility/equilibrium relationships

Consider possible nodal displacements of the nodes and extensions of the bars:



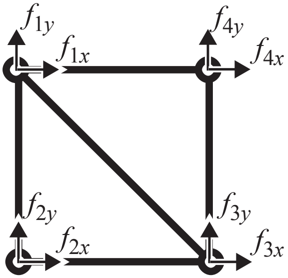
Nodal displacements



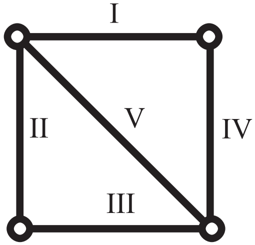
Bar extensions

Compatibility/equilibrium relationships

Consider possible forces at the nodes and tensions in the bars:



Nodal forces



Bar tensions

Compatibility/equilibrium relationships

C is the *compatibility* matrix, describing the (first order) relationship between joint displacements and bar extensions.

$$\mathbf{Cd} = \mathbf{e}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{bmatrix} = \begin{bmatrix} e_{\mathrm{I}} \\ e_{\mathrm{II}} \\ e_{\mathrm{III}} \\ e_{\mathrm{IV}} \\ e_{\mathrm{V}} \end{bmatrix}$$

Any solution to $\mathbf{Cd} = \mathbf{0}$ is either an internal mechanism, or a rigid-body mechanism.

Compatibility/equilibrium relationships

The tranpose of **C** is the equilibrium matrix, describing the relationship between nodal forces, and internal forces in the bars

$$\mathbf{C}^{\mathrm{T}}\mathbf{t} = \mathbf{f}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & -1 & -1/\sqrt{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{bmatrix}$$

Any solution to $\mathbf{C}^{\mathrm{T}}\mathbf{t} = \mathbf{0}$ is a state of self-stress

Proof of the Maxwell Calladine equation (in 2D)

Consider the dimensions of vector spaces associated with **C**:

$$\underbrace{\left.\begin{bmatrix} & & \\ & \mathbf{C} & \\ & & \end{bmatrix}\right\} b}_{2j}$$

If **C** has rank r, nullspace $\mathcal{N}(\mathbf{C})$,

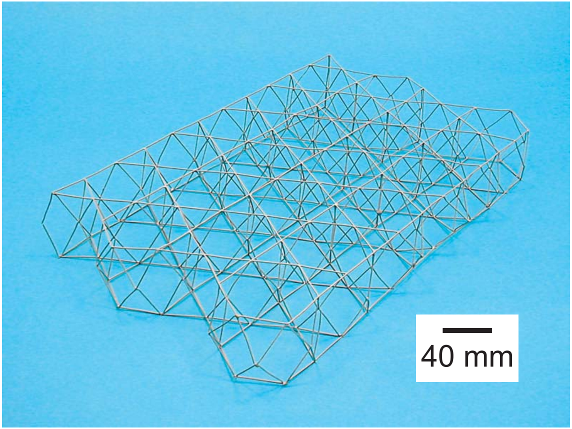
$$m + 3 = \dim(\mathcal{N}(\mathbf{C})) = 2j - r$$
$$s = \dim(\mathcal{N}(\mathbf{C}^{\mathrm{T}})) = b - r$$

and so, for any value of r

$$m - s = 2j - b - 3$$

Outline

Some repetitive structures have the number of constraints equal to the number of freedoms



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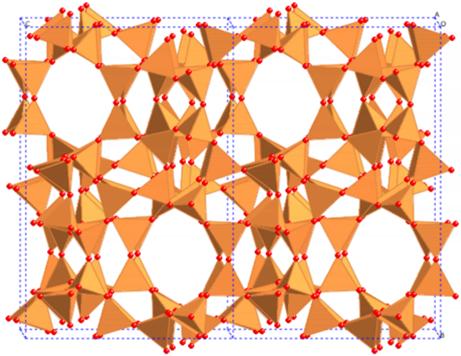

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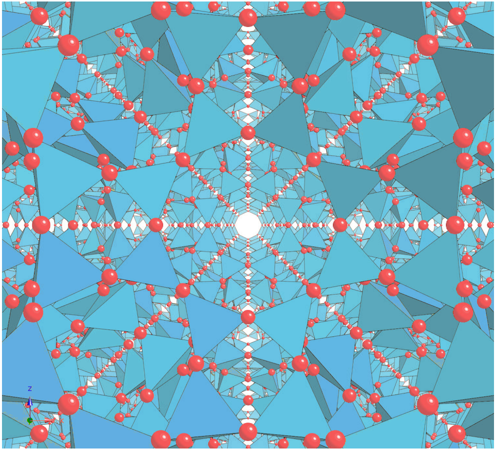

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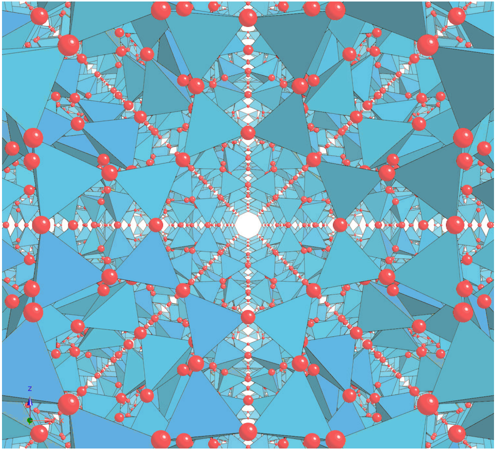
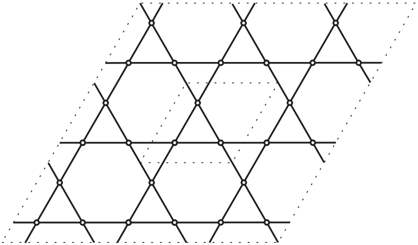
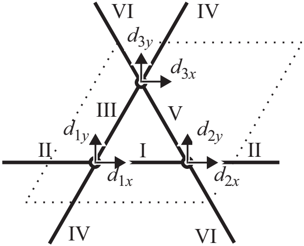


The unexpected properties of these structures can be explored through counting

Example: the kagome lattice





$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 & 0 & 1/2 & \sqrt{3}/2 \\ 1/2 & \sqrt{3}/2 & 0 & 0 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & 1/2 & -\sqrt{3}/2 & -1/2 & \sqrt{3}/2 \\ 0 & 0 & -1/2 & \sqrt{3}/2 & 1/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix} = \begin{bmatrix} e_{\mathrm{I}} \\ e_{\mathrm{II}} \\ e_{\mathrm{III}} \\ e_{\mathrm{IV}} \\ e_{\mathrm{V}} \\ e_{\mathrm{VI}} \end{bmatrix}$$

Counting for repetitive structures

Consider the dimensions of vector spaces associated with $\mathbf{C}$:

$$\underbrace{\left[\begin{array}{c} \\ \mathbf{C} \\ \\ \end{array}\right]}_{2j}\Bigg\}\, b$$

If $\mathbf{C}$ has rank r, nullspace $\mathcal{N}(\mathbf{C})$,

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Rotation is not an allowed rigid-body mode with a fixed unit cell

Counting states of self-stress

Unlike in the finite case, we are not going to define a state of self-stress as any solution $\mathbf{t}$ to $\mathbf{C}^{\mathrm{T}}\mathbf{t} = \mathbf{0}$, because we do not wish to include ‘loads at infinity’, e.g., uniform tensile or shear stress.

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However, this causes difficulties for a general counting rule, because for any particular structure, we do not know if the structure is able to support all possible loads at infinity.

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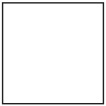
However, this causes difficulties for a general counting rule, because for any particular structure, we do not know if the structure is able to support all possible loads at infinity.

A better approach is to consider an ‘augmented’ compatibility matrix, where uniform deformation of the unit cell is allowed.

Deformations of the unit cell

We consider affine transformations of the unit cell:



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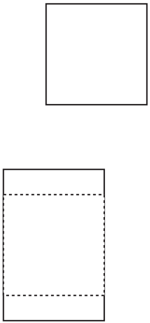

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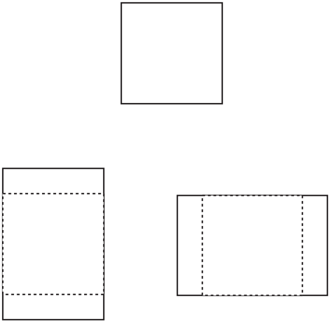

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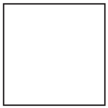

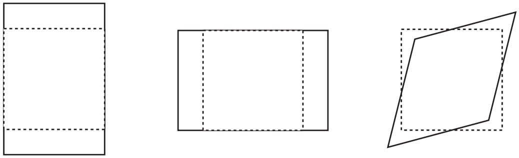

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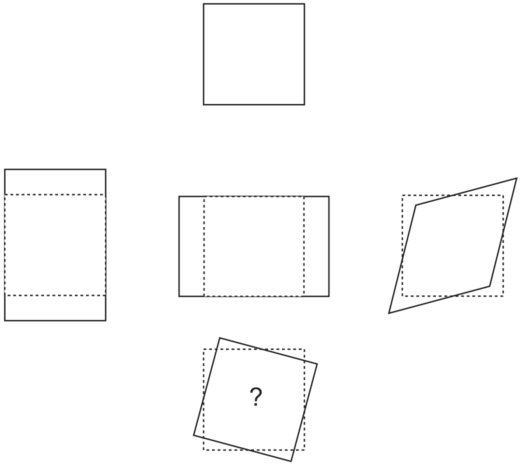
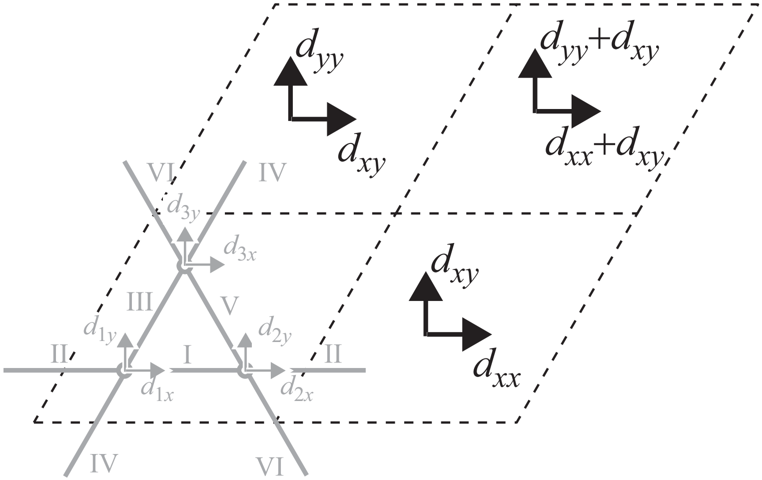


Example: 'augmented' deformation vector for kagome



Augmented compatibility matrix for the kagome

$$\mathbf{C}^*\mathbf{d}^* = \mathbf{e}$$

$$\left[\begin{array}{cccccc|ccc} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 & 0 & 1/2 & \sqrt{3}/2 & 0 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 & 0 & -1/2 & -\sqrt{3}/2 & 0 & \sqrt{3}/2 & 1/2 \\ 0 & 0 & 1/2 & -\sqrt{3}/2 & -1/2 & \sqrt{3}/2 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & \sqrt{3}/2 & 1/2 & -\sqrt{3}/2 & 1/2 & \sqrt{3}/2 & -(1+\sqrt{3})/2 \end{array}\right] \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ - \\ d_{xx} \\ d_{yy} \\ d_{xy} \end{bmatrix} = \begin{bmatrix} e_{\mathrm{I}} \\ e_{\mathrm{II}} \\ e_{\mathrm{III}} \\ e_{\mathrm{IV}} \\ e_{\mathrm{V}} \\ e_{\mathrm{VI}} \end{bmatrix}$$

Counting for repetitive structures (revised)

Consider the dimensions of vector spaces associated with $\mathbf{C}^*$:

$$\underbrace{\left[\begin{array}{c} \\ \mathbf{C}^* \\ \\ \end{array}\right]}_{2j+3} \Bigg\} b$$

If $\mathbf{C}^*$ has rank r^*, nullspace $\mathcal{N}(\mathbf{C}^*)$,

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For any value of r^*

$$m - s = 2j - b + 3 - 2$$

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For any value of r^*

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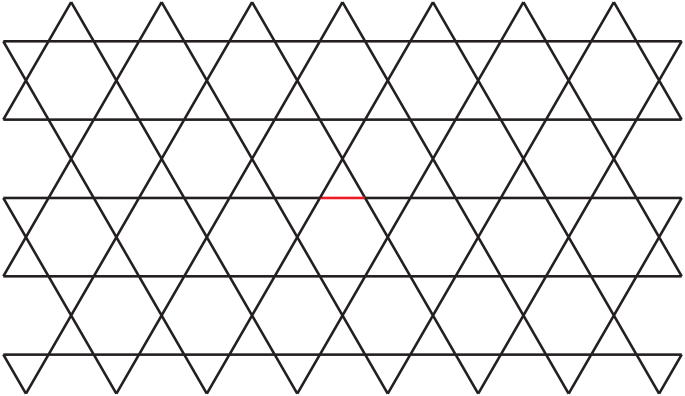
In 3D,

$$m - s = 3j - b + 6 - 3$$

Mechanism for a 'locally isostatic' system ($b = 2j$)



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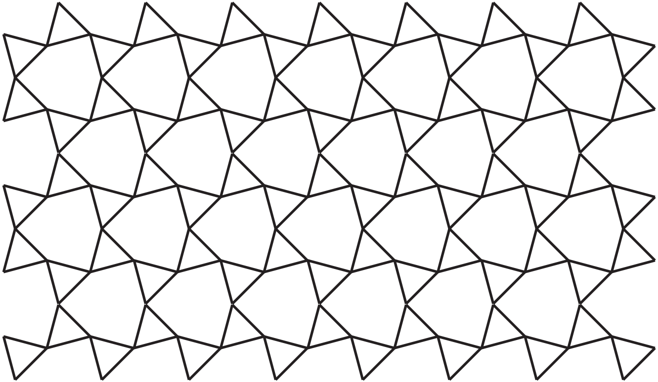

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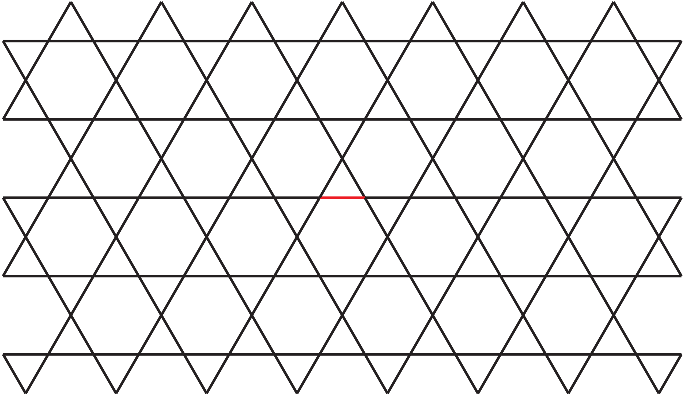

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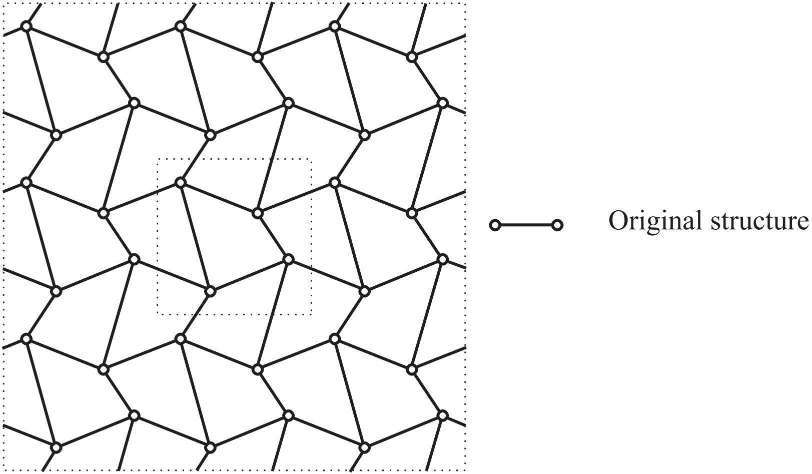

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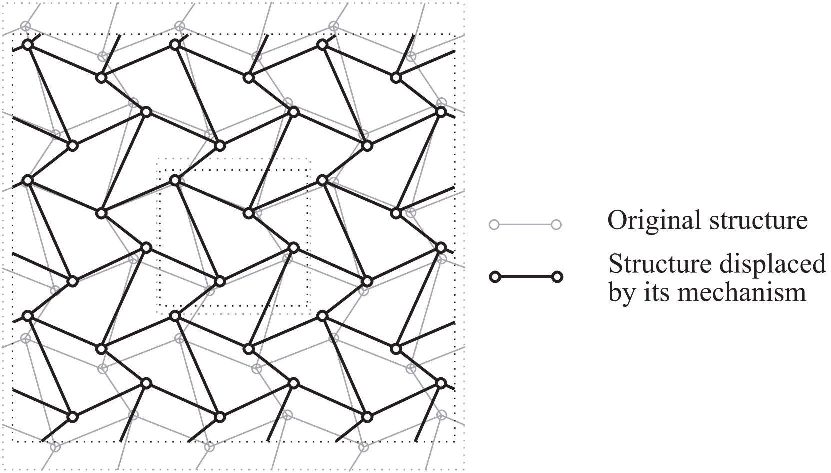

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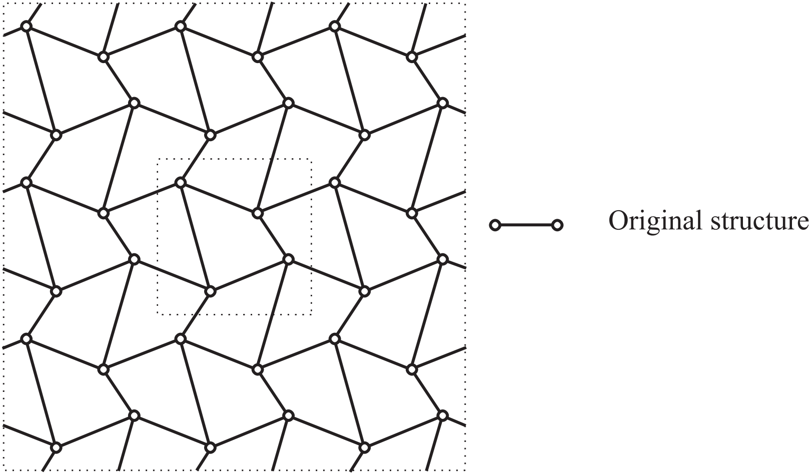
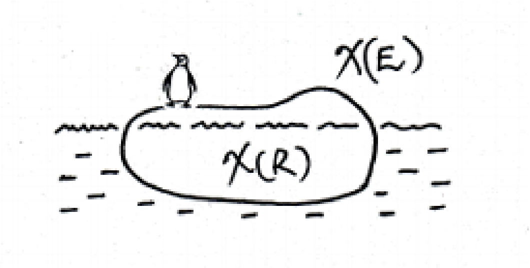


Outline

The Fowler symmetry iceberg proposition

Every counting rule has a symmetry-adapted version.



The Fowler symmetry iceberg proposition

Every counting rule has a symmetry-adapted version.

Example: Euler's theorem

$$v + f = e + 2$$

becomes

$$\Gamma_\sigma(v) \times \Gamma_\epsilon + \Gamma_\sigma(f) = \Gamma_\perp(e) + \Gamma_0 + \Gamma_\epsilon$$

Invariant vector subspaces

Consider that a structure has symmetry group $\mathcal{G}$. Then, vector spaces V associated with the structure can be split into subspaces V_i that are invariant with respect to any operation $g \in \mathcal{G}$, and the set of matrices describing the effect of any $g \in \mathcal{G}$ on any vector $v_i \in V_i$ defines a matrix *representation* of the group. The trace of these matrices is called the *character* of the representation.

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If the V_i are as small as possible, these representations are called *irreducible representations*, and these (and the corresponding characters) are known for any group $\mathcal{G}$.

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For any vector space V, we give the set of characters for all $g \in G$ the symbol Γ_V and loosely call this the 'representation'. Most importantly, the structure of V as a summation of irreducible invariant vector spaces can be found directly from Γ_V, and hence we can consider this to count the dimensions of V in terms of dimensions of irreducible representations.

Example: $\mathcal{G} = \mathcal{C}_s$, a single plane of reflection

Character table (with z perpendicular to the plane of symmetry)

$\mathcal{C}_s$	E	σ_h		
A'	1	1	x	'symmetric'
A''	1	-1	z, R_y	'antisymmetric'

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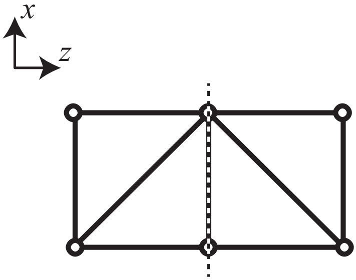

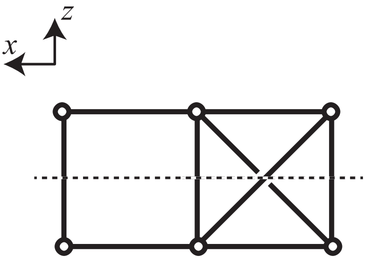

Reminder: counting for finite structures in 2D

Consider the dimensions of vector spaces associated with **C**:

$$\underbrace{\left.\begin{bmatrix} & & \\ & \mathbf{C} & \\ & & \end{bmatrix}\right\} b}_{2j}$$

If **C** has rank r, nullspace $\mathcal{N}(\mathbf{C})$,

$$m + 3 = \dim(\mathcal{N}(\mathbf{C})) = 2 \times j - r$$
$$s = \dim(\mathcal{N}(\mathbf{C}^{\mathrm{T}})) = b - r$$

Eliminating r gives the Maxwell-Calladine equation

$$m - s = 2j - b - 3$$

Counting with symmetry for finite structures

Consider the symmetries of vector spaces associated with **C** for the appropriate symmetry group $\mathcal{G}$:

$$\underbrace{\left.\begin{bmatrix} & & \\ & \mathbf{C} & \\ & & \end{bmatrix}\right\}\Gamma(b)}_{\Gamma_T \otimes \Gamma(j)}$$

Counting with symmetry for finite structures

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$$\underbrace{\left.\begin{bmatrix} & & \\ & \mathbf{C} & \\ & & \end{bmatrix}\right\}}_{\Gamma_T \otimes \Gamma(j)} \Gamma(b)$$

If **C** has a row/column space with 'representation' $\Gamma(r)$, nullspace with representation $\Gamma(N)$ and left-nullspace with representation $\Gamma(N^{\mathrm{T}})$,

$$\Gamma(m) + \Gamma_T + \Gamma_R = \Gamma(N) = \Gamma_T \otimes \Gamma(j) - \Gamma(r)$$
$$\Gamma(s) = \Gamma(N^{\mathrm{T}}) = \Gamma(b) - \Gamma(r)$$

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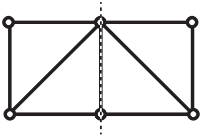
$$\Gamma(s) = \Gamma(N^{\mathrm{T}}) = \Gamma(b) - \Gamma(r)$$

Eliminating $\Gamma(r)$ gives the symmetry-extended Maxwell-Calladine equation

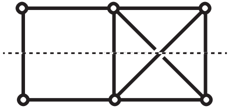
$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) - \Gamma_T - \Gamma_R$$

Symmetry-adapted counting for the example structures

$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) - \Gamma_T - \Gamma_R$$



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$		
$\times\Gamma_T$		
$=$		
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$		
$\times\Gamma_T$		
$=$		
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		

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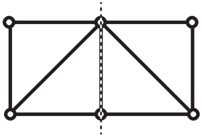
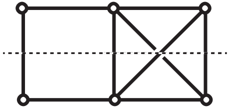


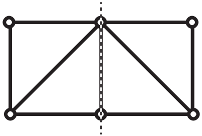
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
$\times\Gamma_T$		
$=$		
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		



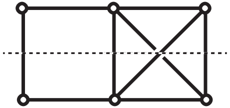
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$		
$\times\Gamma_T$		
$=$		
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		

Symmetry-adapted counting for the example structures

$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) - \Gamma_T - \Gamma_R$$



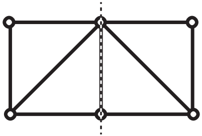
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		



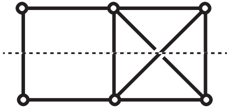
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$		
$\times\Gamma_T$		
$=$		
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		

Symmetry-adapted counting for the example structures

$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) - \Gamma_T - \Gamma_R$$



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	-9	-1
$=$	3	-1
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$		
$\times\Gamma_T$		
$=$		
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		

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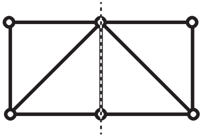

$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
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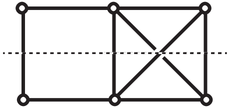
$=$	12	0
$-\Gamma(b)$	-9	-1
$=$	3	-1
$-\Gamma_T$	-2	0
$-\Gamma_R$	-1	1
$\Gamma(m) - \Gamma(s)$	0	0



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$		
$\times\Gamma_T$		
$=$		
$-\Gamma(b)$		
$=$		
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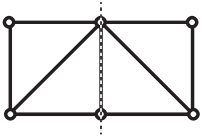
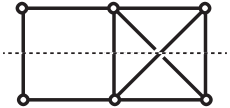


$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	-9	-1
$=$	3	-1
$-\Gamma_T$	-2	0
$-\Gamma_R$	-1	1
$\Gamma(m) - \Gamma(s)$	0	0



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	0
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$		
$=$		
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		

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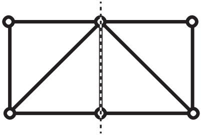
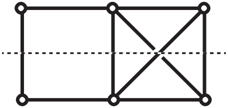
$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) - \Gamma_T - \Gamma_R$$



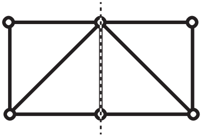
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	-9	-1
$=$	3	-1
$-\Gamma_T$	-2	0
$-\Gamma_R$	-1	1
$\Gamma(m) - \Gamma(s)$	0	0



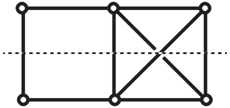
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	0
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	-9	-3
$=$	3	-3
$-\Gamma_T$		
$-\Gamma_R$		
$\Gamma(m) - \Gamma(s)$		

Symmetry-adapted counting for the example structures

$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) - \Gamma_T - \Gamma_R$$



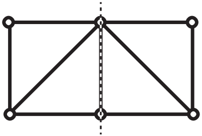
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	−9	−1
$=$	3	−1
$-\Gamma_T$	−2	0
$-\Gamma_R$	−1	1
$\Gamma(m) - \Gamma(s)$	0	0



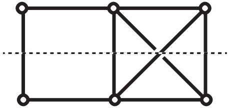
$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	0
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	−9	−3
$=$	3	−3
$-\Gamma_T$	−2	0
$-\Gamma_R$	−1	1
$\Gamma(m) - \Gamma(s)$	0	−2

Symmetry-adapted counting for the example structures

$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) - \Gamma_T - \Gamma_R$$



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	2
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	-9	-1
$=$	3	-1
$-\Gamma_T$	-2	0
$-\Gamma_R$	-1	1
$\Gamma(m) - \Gamma(s)$	0	0



$\mathcal{C}_s$	E	σ_h
$\Gamma(j)$	6	0
$\times\Gamma_T$	2	0
$=$	12	0
$-\Gamma(b)$	-9	-3
$=$	3	-3
$-\Gamma_T$	-2	0
$-\Gamma_R$	-1	1
$\Gamma(m) - \Gamma(s)$	0	-2

$= -A' + A''$

Symmetry of deformations of the unit cell

Affine deformations of the unit cell can be defined as a second-order tensor, which can be written as the symmetric part of a 2×2 (in 2D) or a 3×3 (in 3D) matrix. The antisymmetric parts of the matrix represent the rotations that we do not want.

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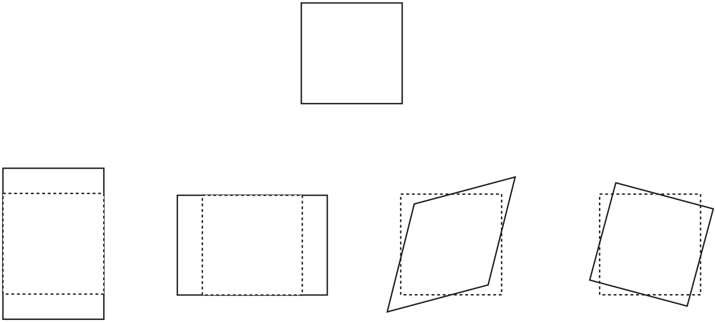

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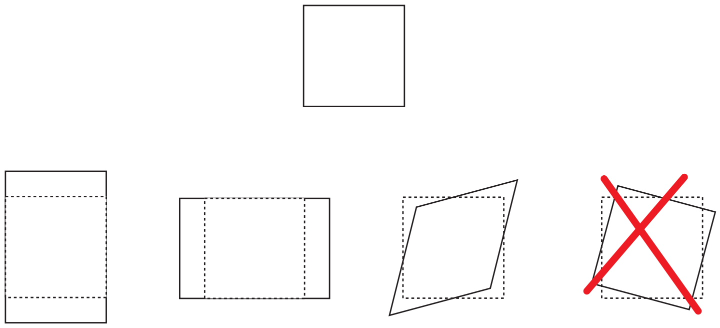

$$\Gamma_a = \Gamma_T^2 - \Gamma_R$$

Counting with symmetry for repetitive structures

Consider the symmetries of vector spaces associated with $\mathbf{C}^*$ for the appropriate symmetry group $\mathcal{G}$:

$$\underbrace{\left.\left[\begin{array}{c} \\ \mathbf{C}^* \\ \\ \end{array}\right]\right\}}_{\Gamma_T \otimes \Gamma(j) + (\Gamma_T^2 - \Gamma_R)} \Gamma(b)$$

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Consider the symmetries of vector spaces associated with $\mathbf{C}^*$ for the appropriate symmetry group $\mathcal{G}$:

$$\underbrace{\left[\begin{array}{c} \\ \mathbf{C}^* \\ \\ \end{array}\right]\left.\vphantom{\begin{array}{c} \\ \\ \\ \end{array}}\right\}\Gamma(b)}_{\Gamma_T \otimes \Gamma(j) + (\Gamma_T^2 - \Gamma_R)}$$

$$\Gamma(m) + \Gamma_T = \Gamma(N) = \Gamma_T \otimes \Gamma(j) + \Gamma_T^2 - \Gamma_R - \Gamma(r^*)$$

$$\Gamma(s) = \Gamma(N^{\mathrm{T}}) = \Gamma(b) - \Gamma(r^*)$$

Counting with symmetry for repetitive structures

Consider the symmetries of vector spaces associated with $\mathbf{C}^*$ for the appropriate symmetry group $\mathcal{G}$:

$$\underbrace{\left[\quad \mathbf{C}^* \quad\right]\Bigg\}\,\Gamma(b)}_{\Gamma_T\otimes\Gamma(j)+(\Gamma_T^2-\Gamma_R)}$$

$$\Gamma(m) + \Gamma_T = \Gamma(N) = \Gamma_T \otimes \Gamma(j) + \Gamma_T^2 - \Gamma_R - \Gamma(r^*)$$
$$\Gamma(s) = \Gamma(N^{\mathrm{T}}) = \Gamma(b) - \Gamma(r^*)$$

Eliminating $\Gamma(r^*)$ gives

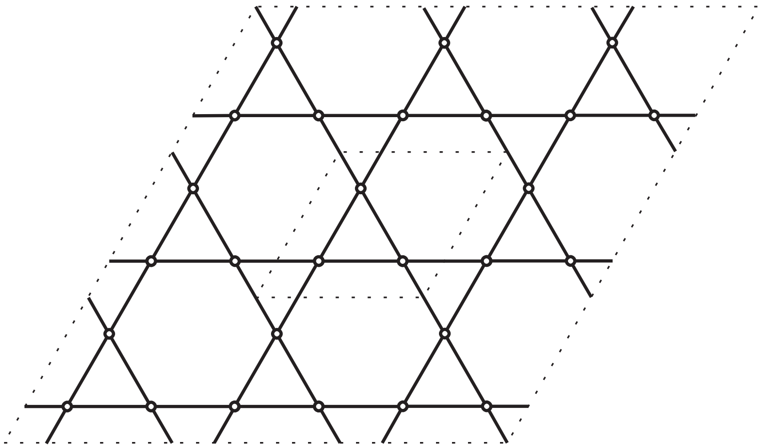
$$\Gamma(m) - \Gamma(s) = \Gamma_T \otimes \Gamma(j) - \Gamma(b) + \Gamma_T^2 - \Gamma_R - \Gamma_T$$

Which group should we use for repetitive structures?

Which group should we use for repetitive structures?

The complete geometric symmetry group of a repetitive structure is a *space group* (plane/wallpaper group in 2D) which has infinite order. However, by identifying components that are equivalent under a displacement, we can factor our the infinite displacement group, leaving us with a point group with which we can work.

Example: counting symmetries for the kagome lattice

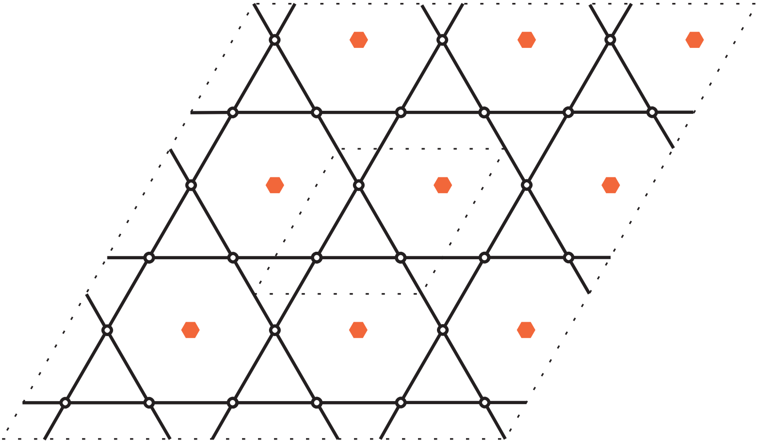


- Plane Group $p6m$
- Point Group C_{6v}

Components preserved by E

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma(j)$	3					
$\times\Gamma_T$	2					
$=$	6					
$-\Gamma(b)$	-6					
$=$	0					
$-\Gamma_T$	-2					
$-\Gamma_R$	-1					
$+\Gamma_T^2$	4					
$\Gamma(m) - \Gamma(s)$	1					

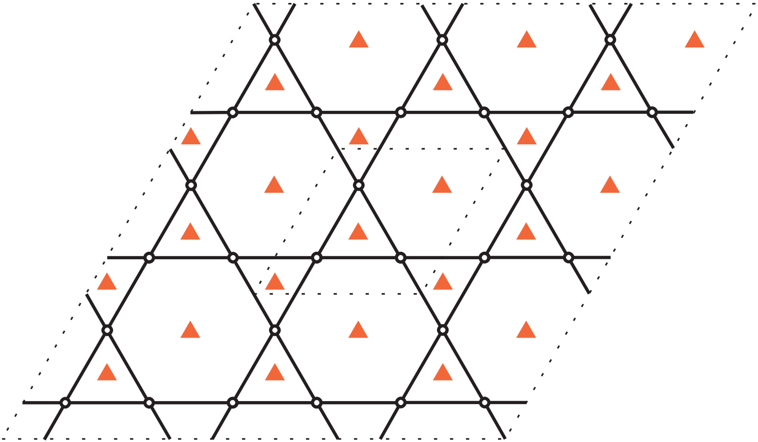
The kagome lattice — C_6 axes



Components preserved by C_6

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma(j)$	3	0				
$\times\Gamma_T$	2	1				
$=$	6	0				
$-\Gamma(b)$	-6	0				
$=$	0	0				
$-\Gamma_T$	-2	-1				
$-\Gamma_R$	-1	-1				
$+\Gamma_T^2$	4	1				
$\Gamma(m)-\Gamma(s)$	1	-1				

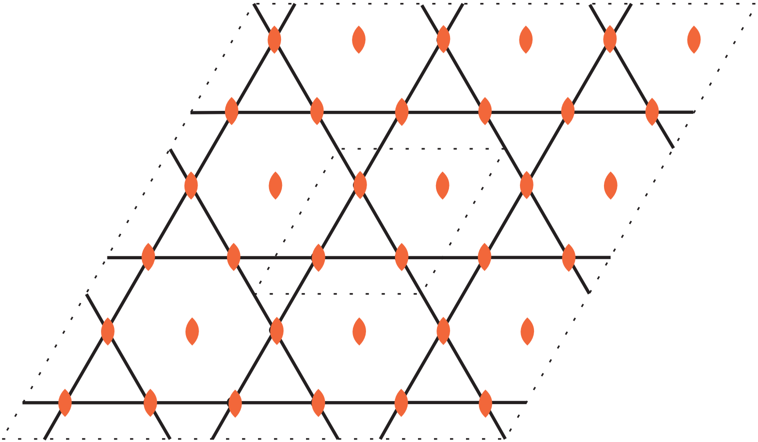
The kagome lattice — C_3 axes



Components preserved by C_3

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma(j)$	3	0	0			
$\times\Gamma_T$	2	1	-1			
$=$	6	0	0			
$-\Gamma(b)$	-6	0	0			
$=$	0	0	0			
$-\Gamma_T$	-2	-1	1			
$-\Gamma_R$	-1	-1	-1			
$+\Gamma_T^2$	4	1	1			
$\Gamma(m) - \Gamma(s)$	1	-1	1			

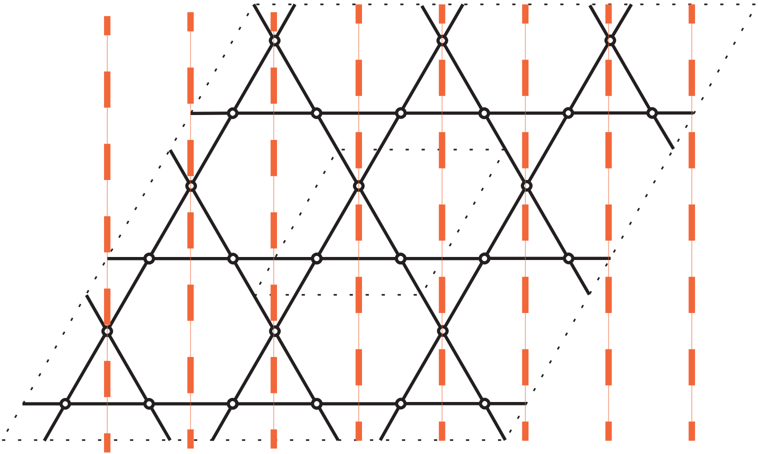
The kagome lattice — C_2 axes



Components preserved by C_2

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma(j)$	3	0	0	3		
$\times\Gamma_T$	2	1	-1	-2		
$=$	6	0	0	-6		
$-\Gamma(b)$	-6	0	0	0		
$=$	0	0	0	-6		
$-\Gamma_T$	-2	-1	1	2		
$-\Gamma_R$	-1	-1	-1	-1		
$+\Gamma_T^2$	4	1	1	4		
$\Gamma(m) - \Gamma(s)$	1	-1	1	-1		

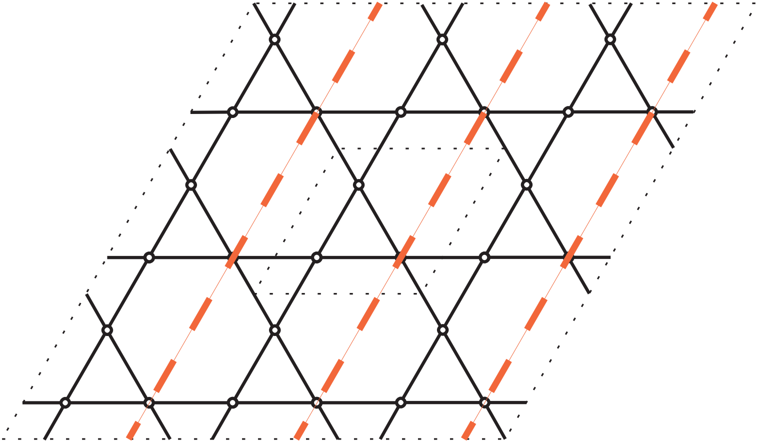
The kagome lattice — σ_v lines



Components preserved by σ_v

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma(j)$	3	0	0	3	1	
$\times\Gamma_T$	2	1	-1	-2	0	
$=$	6	0	0	-6	0	
$-\Gamma(b)$	-6	0	0	0	-2	
$=$	0	0	0	-6	-2	
$-\Gamma_T$	-2	-1	1	2	0	
$-\Gamma_R$	-1	-1	-1	-1	1	
$+\Gamma_T^2$	4	1	1	4	0	
$\Gamma(m) - \Gamma(s)$	1	-1	1	-1	-1	

The kagome lattice — σ_d lines



Components preserved by σ_d

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma(j)$	3	0	0	3	1	1
$\times\Gamma_T$	2	1	-1	-2	0	0
$=$	6	0	0	-6	0	0
$-\Gamma(b)$	-6	0	0	0	-2	0
$=$	0	0	0	-6	-2	0
$-\Gamma_T$	-2	-1	1	2	0	0
$-\Gamma_R$	-1	-1	-1	-1	1	1
$+\Gamma_T^2$	4	1	1	4	0	0
$\Gamma(m)-\Gamma(s)$	1	-1	1	-1	-1	1

Character table for C_{6v}

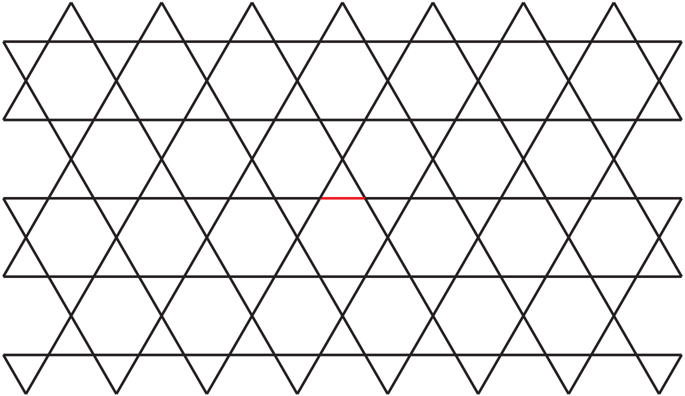
C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$	
A_1	1	1	1	1	1	1	z
A_2	1	1	1	1	-1	-1	R_z
B_1	1	-1	1	-1	1	-1	
B_2	1	-1	1	-1	-1	1	
E_1	2	1	-1	-2	0	0	$(x, y)\,(R_x, R_y)$
E_2	2	-1	-1	2	0	0	

Decomposition into irreducible representations for each representation

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$	
$\Gamma(j)$	3	0	0	3	1	1	$A_1 + E_2$
$\times\Gamma_T$	2	1	-1	-2	0	0	E_1
$=$	6	0	0	-6	0	0	$B_1 + B_2 + 2E_1$
$-\Gamma(b)$	-6	0	0	0	-2	0	$-A_1 - B_1 - E_1 - E_2$
$=$	0	0	0	-6	-2	0	$-A_1 + B_2 + E_1 - E_2$
$-\Gamma_T$	-2	-1	1	2	0	0	$-E_1$
$-\Gamma_R$	-1	-1	-1	-1	1	1	$-A_2$
$+\Gamma_T^2$	4	1	1	4	0	0	$A_1 + A_2 + E_2$
$\Gamma(m) - \Gamma(s)$	1	-1	1	-1	-1	1	B_2

Kagome mechanism has representation B_2

The deformation is preserved by E, C_3, σ_d; reversed by C_6, C_2, σ_v.



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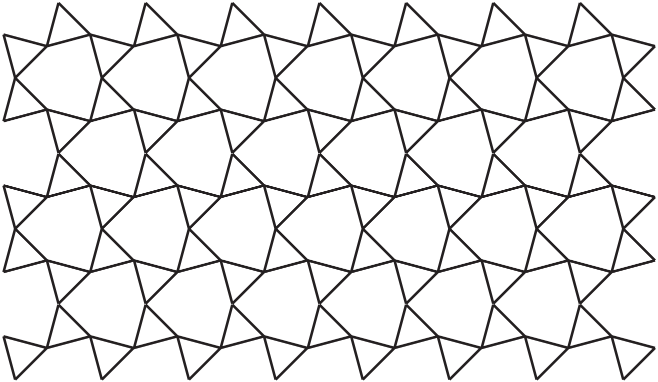


Outline

Background

Counting for infinite, repetitive structures

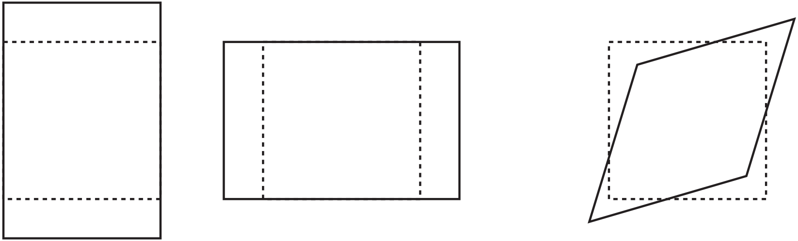
Using symmetry to extend counting rules

A symmetry criterion for ‘equiauxetic’ materials

Conclusions

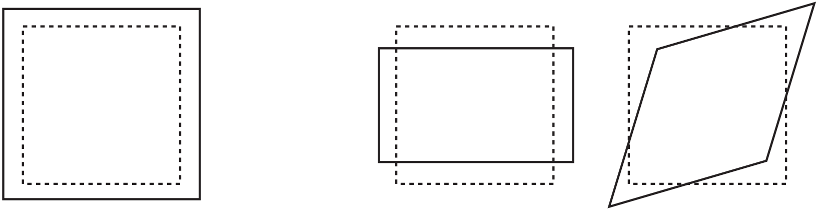
Basic *equiauxetic* concept in 2D

There are three affine deformation modes associated with the deformation of a network



Basic *equiauxetic* concept in 2D

There are three affine deformation modes associated with the deformation of a network



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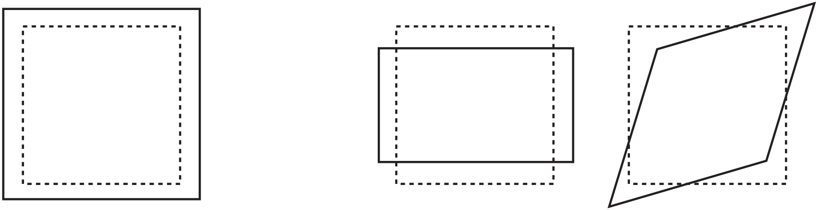

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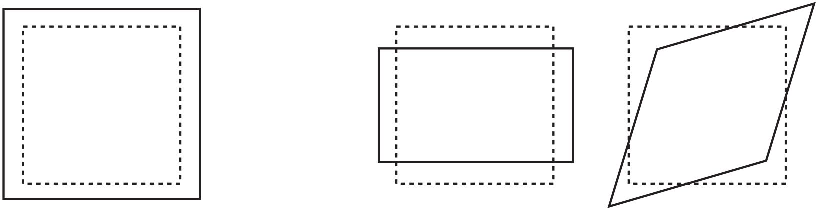

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We call such a mode *equiauxetic*

2D Symmetry condition 1: the network must have enough symmetry

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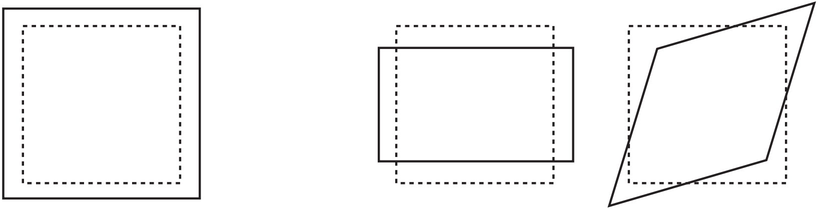
(2-fold symmetry cannot distinguish between shear and contraction — they both appear as ‘totally symmetric’ in a symmetry analysis)



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To give a Poisson's ratio close to -1, we should also ensure that the equiauxetic mode is the *only* mode that doesn't require stretching of bars.

Example from Mitschke catalogue

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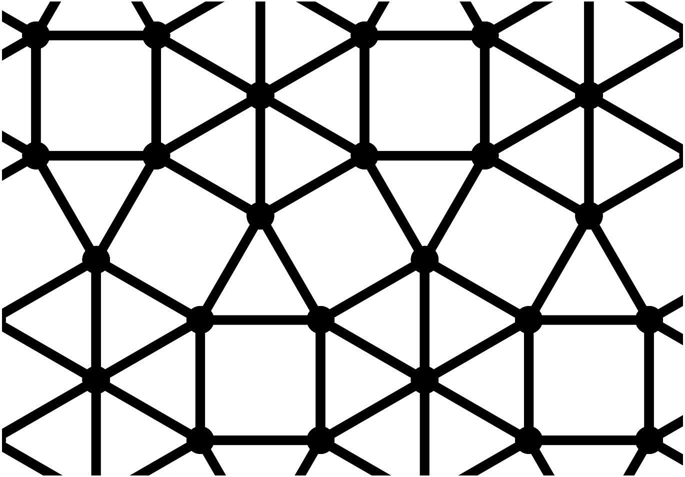

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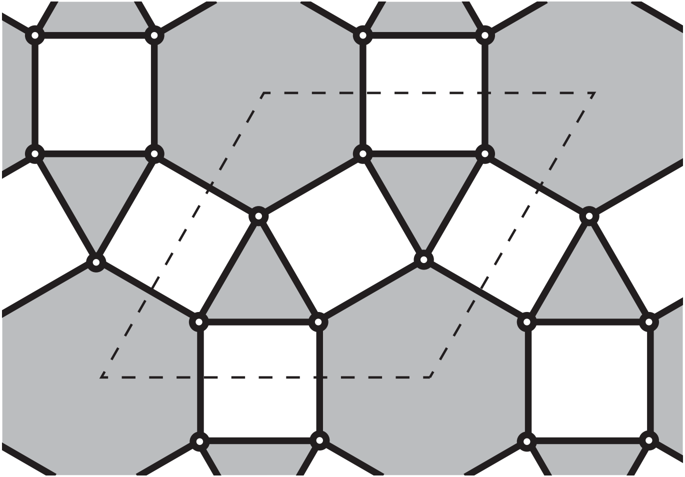

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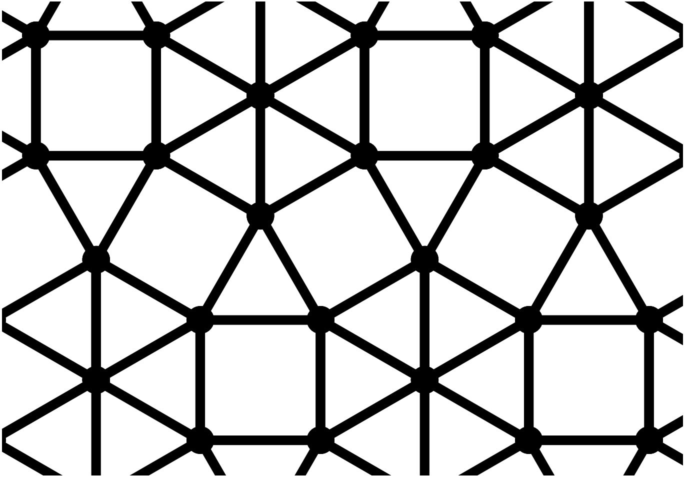

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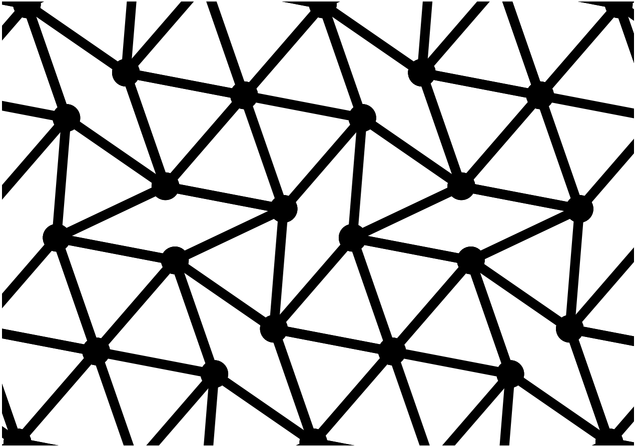

Extension to 3D: symmetry requirement

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$$m - s = 3j - b + 3$$

and hence for $b = 3j$ there are three modes, rather than the unique mode in 2D.

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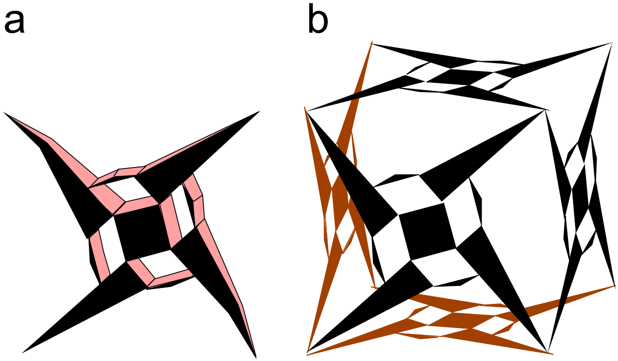
and hence for $b = 3j$ there are three modes, rather than the unique mode in 2D.

The only possibility is to find some special geometry that overconstrains the structure, but still allows one totally symmetric mode.

One possible 3D example

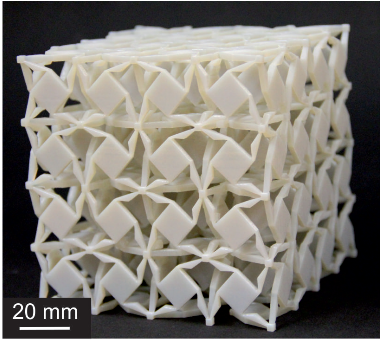
Given by Milton[1].

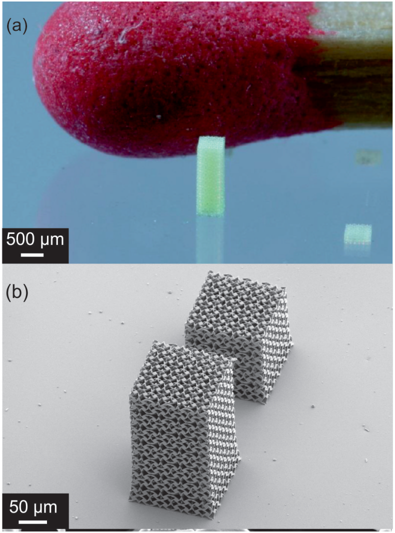


[1] Journal of the Mechanics and Physics of Solids, 61 (2013) 1543–156

A meta-material based on the Milton example

Bückmann et al.[2] have manufactured the meta-material shown, and measured a Poisson's ratio of -0.76.





[2] New Journal of Physics (2014) 16 033032

Outline

Conclusion

- Any counting rule will have a symmetry-extended counterpart.
- These rules allow strong statements to be made about placement of structural components to achieve particular structural behaviour.
- Symmetry helps show where we should look to achieve limiting *equiauxetic* behaviour, with ν close to -1.