What's Happening with Dimensional Reduction?

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In this talk I will review the dimensional reduction story and discuss recent exact results on dimensional reduction for branched polymers (BP) and directed branched polymers (DBP).[a] [b]

Outline:

1. Dimensional Reduction and the Random Field Ising Model
2. Dimensional Reduction and Branched Polymers
3. Exact results on BP and DBP
   - Identities between models in different dimensions
   - How it works: Forest-Root formulas and supersymmetry
   - Airy function crossover to the mean field limit

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[a] D. Brydges and J. Imbrie, *Ann. Math.* **158**, 1019 (2003) and *J. Statist. Phys.* **110**, 503 (2003).

[b] J. Imbrie, *Annales Henri Poincare* **4**, S445 (2003) and *J. Phys. A: Math. Gen.* **37**, L137 (2004).

Random Field Ising Model

- Parisi-Sourlas argue[a] that correlations and exponents of the RFIM can be deduced from the pure Ising model in 2 dimensions less.
- Seems to imply no long-range order in 3 dimensions (equivalence with 1-d Ising model)

- Imry-Ma argument[b]: For a contour with diameter L, the random-field fluctuations inside are of order $L^{d/2}$.
- The contour energy L^{d-1} dominates if $d > 2$.
- Implies there is long-range order in 3 dimensions

Loopholes in both arguments

- Multiple extrema wreck the dimensional reduction argument
- Contours may adjust to the random field configuration and wreck the Imry-Ma argument

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[a] G. Parisi and N. Sourlas, *Phys. Rev. Lett.* **43**, 744 (1979)

[b] Y. Imry and S. Ma, *Phys. Rev. Lett.* **39**, 1399 (1975)

Rigorous work goes against dimensional reduction for RFIM

- How to make Imry-Ma argument simultaneously for all contours surrounding a site [a]
- Proof of long-range order at $T = 0$ in $d = 3$ (going beyond the single-contour approximation)[b]
- Proof of long-range order at low T [c]

Recent work attempts to understand the breakdown of dimensional reduction

- Bound states [d] [e]
- Functional Renormalization Group [f]

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[a] D. Fisher, J. Froehlich, T. Spencer, *Jour. Stat. Phys.* **34**, 863 (1984)

[b] J. Imbrie, *Commun. Math. Phys.* **98**, 145 (1985)

[c] J. Bricmont and A. Kupiainen, *Commun. Math. Phys.* **116**, 539 (1988)

[d] E. Brezin and C. de Dominicus, *Europhys. Lett.* **44** 13 (1998)

[e] G. Parisi and N. Sourlas, *Phys. Rev. Lett.* **89** 257204 (2002)

[f] P. Le Doussal, K. Wiese, and P. Chauve, *Phys. Rev.* **E69** 026112 (2004) and references therein

Branched Polymers

Parisi and Sourlas predicted in 1981 that BP falls into the universality class of the Yang-Lee edge in two fewer dimensions.[a]

But can we trust the argument, given the failure of dimensional reduction for the RFIM?

Numerics[b] support dimensional reduction, but is it exact?

We prove an equivalence between BP and the Hard-Core gas at negative activity in two fewer dimensions (which is in the Yang-Lee class).

Also between DBP and the Hard-Core gas at negative activity in one fewer dimension.

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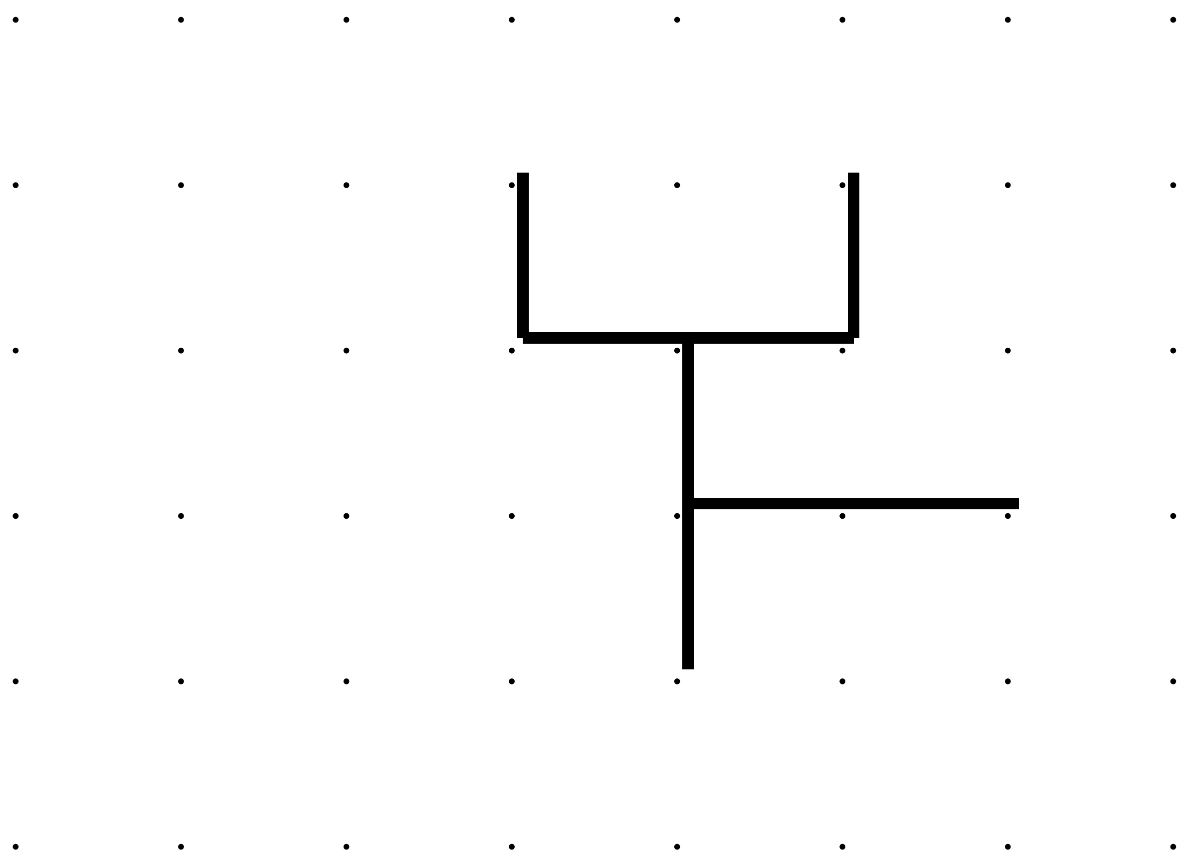
[a] G. Parisi and N. Sourlas, *Phys. Rev. Lett.* **46**, 871 (1981)

[b] U. Glaus, *J. Phys. A: Math. Gen.* **18**, L609 (1985)

Branched Polymers

On the lattice:



A lattice BP is a finite connected set of nearest-neighbor bonds with no cycles.

The generating function can be written as

$$Z_{\mathrm{BP}}(z) = \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \sum_T \sum_{y_2,\ldots,y_N} \prod_{ji \in T} V_{ji} \prod_{ji \notin T} U_{ji},$$

where $V_{ji} := \delta_{|y_j - y_i|,1}$ and $U_{ji} := 1 - \delta_{y_j,y_i}$ enforce the adjacency and loop-free conditions, respectively. Here T is a tree graph on $\{1,\ldots,N\}$ and y_i is the position of the i^{th} vertex.

Branched Polymers in $\mathbb{R}^{D+2}$

A branched polymer consists of

1. A tree graph T on $1, \dots, N$ and
2. An embedding $\mathbf{y} = \{y_i\}_{i=1,\dots,N}$ into $\mathbb{R}^{D+2}$ such that
   - if $ij \in T$ then $|y_{ij}| = 1$ and
   - if $ij \notin T$ then $|y_{ij}| \geq 1$.

The generating function for (rooted) branched polymers is

$$Z_{\mathrm{BP}}(z) = \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \sum_T \int dy_2 \cdots dy_N \prod_{ji \in T} V_{ji} \prod_{ji \notin T} U_{ji},$$

where

$$U_{ji} := U(|y_j - y_i|^2)$$
$$V_{ji} := 2U'(|y_j - y_i|^2).$$

Directed Branched Polymers in $\mathbb{R}_+ \times \mathbb{R}^D$

$Z_{\mathrm{DBP}}(z)$ is given by the same formula but with

$$y_i = (t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}^D$$
$$t_j > t_i \quad \text{for} \quad ji \in T$$
$$U_{ji} = U(|t_j - t_i|, x_j - x_i)$$
$$V_{ji} = U'_{ji} \quad \text{where} \quad U'(t,x) = \frac{\partial}{\partial t} U(t,x)$$

Examples

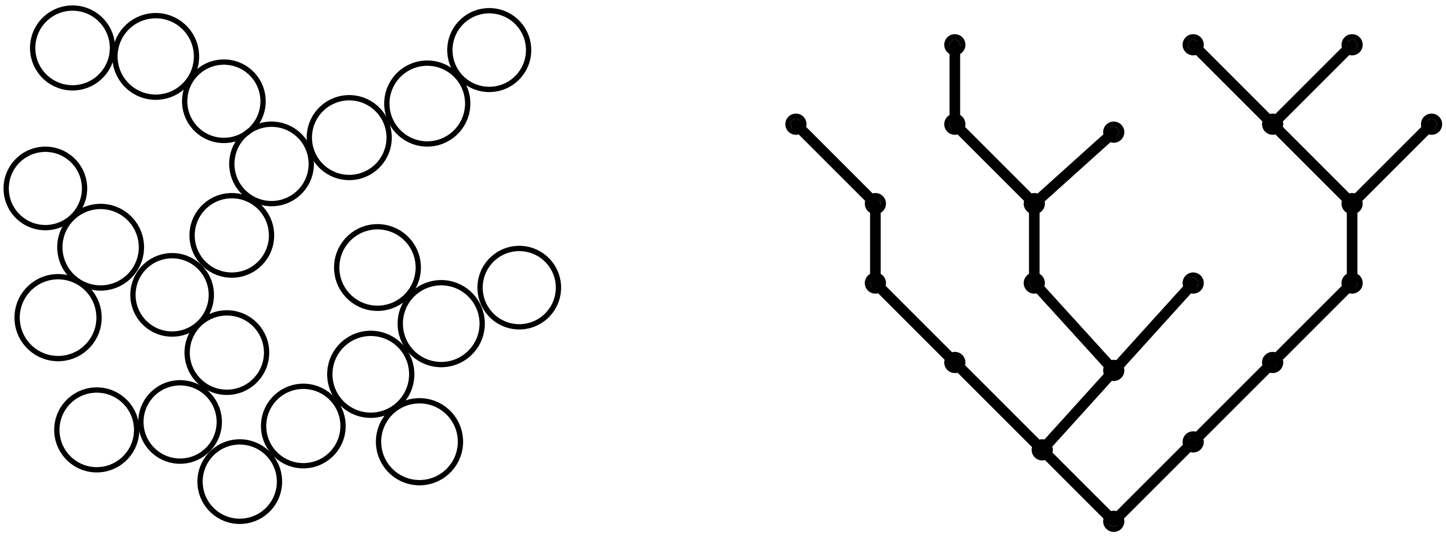


Figure 1: A branched polymer in $\mathbb{R}^2$ (left) and a directed branched polymer in $\mathbb{Z}^2$ (right).

For the BP model on the left, take $U_{ij} = \vartheta(t_{ij} - 1)$.

For the DBP model on the right, take $U_{ij} = 1 - I(x_{ij})\vartheta(1 - t_{ij})$, where $I(x)$ is the indicator function of a set of "neighbors" in the lattice.

Theorem 1

For all z such that the right-hand side converges absolutely,

$$\rho_{\mathrm{HC}}(z) = \begin{cases} -2\pi Z_{\mathrm{BP}}\left(-\frac{z}{2\pi}\right), \\ -Z_{\mathrm{DBP}}(-z). \end{cases}$$

Here

$$Z_{\mathrm{HC}}(z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} dx_1 \cdots dx_N \prod_{1 \le i < j \le N} U_{ij}.$$

$$p(z) = \lim_{\Lambda \nearrow S} \frac{1}{|\Lambda|} \log Z_{\mathrm{HC}}(z); \qquad \rho_{\mathrm{HC}}(z) = z\frac{d}{dz}p(z).$$

S is $\mathbb{R}^D$ or (for some models of DBP) $\mathbb{Z}^D$.

Exact calculations in low dimension:

LHS is computable for $D = 0, 1$ so in dimension $D + 2 = 2$,

$$Z_{\mathrm{BP}}(z) = \frac{z}{1 - 2\pi z}.$$

For continuous space, dimension $D + 2 = 3$ (BP), $D + 1 = 2$ (DBP)

$$2\pi Z_{\mathrm{BP}}\left(\frac{z}{2\pi}\right) = z\frac{d}{dz}T(z) = \sum_{N=1}^{\infty} \frac{N^N z^N}{N!} = Z_{\mathrm{DBP}}(z)$$

On the lattice, $D + 1 = 2$ (DBP)

$$Z_{\mathrm{DBP}}(z) = \frac{1}{2}\left(\frac{1}{\sqrt{1-4z}} - 1\right) = \sum_{N=1}^{\infty} \frac{[2N-1]!!2^{N-1}z^N}{N!}$$

Consequences for Critical Exponents

Theorem 1 implies that

$$\alpha_{\mathrm{HC}} = \gamma_{\mathrm{BP}}$$

because

$$\begin{array}{ccccc} \text{singularity of } Z_{\mathrm{BP}} & \sim \text{singularity of } \rho_{\mathrm{HC}} \sim & \text{singularity of } Z_{\mathrm{DBP}} \\ (z-\tilde{z}_c)^{1-\gamma_{\mathrm{BP}}} & \sim (z-z_c)^{1-\alpha_{\mathrm{HC}}} \sim & (z-\bar{z}_c)^{1-\gamma_{\mathrm{DBP}}}. \end{array}$$

Theorem 2

$$G_{\mathrm{HC}}(0,x;z) = \begin{cases} -2\pi \int d^2w G_{\mathrm{BP}}\left(0,y;\frac{-z}{2\pi}\right), & y=(w,x) \in \mathbb{C}\times\mathbb{R}^D \\ -\int_0^\infty dt G_{\mathrm{DBP}}(0,y;-z), & y=(t,x)\in\mathbb{R}_+\times S \end{cases}$$

Theorem 2 implies that

$$\nu_{\mathrm{BP}} = \nu_{\mathrm{HC}} = \nu^{\perp}_{\mathrm{DBP}}$$

and for $D \geq 1$ that $\eta_{\mathrm{BP}} = \eta_{\mathrm{HC}}$.

Define θ_{BP}, θ_{DBP} from

$$Z_{\mathrm{BP}}(z) = z\frac{d}{dz}\sum_{N=1}^{\infty} c_N z^N, \qquad Z_{\mathrm{DBP}}(z) = \sum_{N=1}^{\infty} d_N z^N$$

with

$$c_N \sim \tilde{z}_c^{-N} N^{-\theta_{\mathrm{BP}}}, \qquad d_N \sim \bar{z}_c^{-N} N^{-\theta_{\mathrm{DBP}}}$$

Then $\theta_{\mathrm{BP}} = 3-\gamma_{\mathrm{BP}}$, and $\theta_{\mathrm{DBP}} = 2-\gamma_{\mathrm{DBP}}$.

Critical Exponent Results

HC dim D	DBP dim $D+1$	BP dim $D+2$	α_{HC} $=\gamma_{\mathrm{BP}}$ $=\gamma_{\mathrm{DBP}}$	θ_{DBP}	θ_{BP}	ν_{HC} $=\nu_{\mathrm{BP}}$ $=\nu^{\perp}_{\mathrm{DBP}}$	η_{HC} $=\eta_{\mathrm{BP}}$
0	1	2	2	0	1		
1	2	3	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	-1
2	3	4	$\frac{7}{6}$	$\frac{5}{6}$	$\frac{11}{6}$	$\frac{5}{12}$	$-\frac{4}{5}$
MFT $D>6$	$D>7$	$D>8$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{4}$	0

Now rigorous

ν_{HC} can be obtained from α_{HC} by hyperscaling $D\nu = 2-\alpha$

$\alpha_{\mathrm{HC}}(2)$ from Baxter's solution to Hard Hexagon Model

$\eta_{\mathrm{HC}}(2)$ from CFT or $\alpha_{\mathrm{HC}} = (2-\eta_{\mathrm{HC}})\nu_{\mathrm{HC}}$

Note: the Yang-Lee edge exponent σ equals $1-\alpha_{\mathrm{HC}}$. When combined with the relations above, this leads to the Parisi-Sourlas relation

$$\theta_{\mathrm{BP}}(D+2) = 2 + \sigma(D)$$

and its analogue for DBP:

$$\theta_{\mathrm{DBP}}(D+1) = 1 + \sigma(D)$$

Relation with the Yang-Lee Edge

Repulsive gases at negative activity are described by an $i\varphi^3$ or Yang-Lee edge field theory.

(same universality class)[a]

To see this, consider the repulsive weight

$$U(|x_{ij}|^2) = e^{-v(|x_{ij}|^2)} = e^{-u(x_{ij})} \text{ with } \hat{u}(k) > 0$$

u is a repulsive, smooth, decaying two-body potential

By the Sine-Gordon transformation, the repulsive gas

$$Z_{\mathrm{HC}}(z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} dx_1 \cdots dx_N \exp\left[-\beta \sum_{1\le i<j\le N} u(x_{ij})\right]$$

can be written as a $-\tilde{z}e^{i\varphi}$ field theory. With $\tilde{z} = ze^{v(0)/2}$, we have

$$Z_{\mathrm{HC}}(z) = \int \exp\left[\int_{\Lambda} dx\, \tilde{z} e^{i\varphi(x)}\right] \frac{e^{-\frac{1}{2}\langle \varphi, u^{-1}\varphi\rangle}}{\mathcal{N}} [d\varphi].$$

If $\tilde{z}$ is negative enough, this action looks critical.

Lowest order term in action is $i\varphi^3$.

(Compare with Ising model in sufficiently large imaginary field)

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[a] S.-N. Lai and M. E. Fisher, *J. Chem. Phys.* **103**, 8144 (1995); Y. Park and M. E. Fisher, *Phys. Rev.* **E60**, 6323 (1999), cond-mat/9907429.

Crossover to Mean-Field BP[a]

Take the mean field limit for two-dimensional BP by letting $v = v(0)$ be small and z be large so that a redefined $\tilde{z} = zve^{v/2}$ is fixed. Then

$$\log \int_{-\infty}^{\infty} \exp\left[-\frac{1}{v}\left(\tilde{z}e^{i\varphi} + \frac{1}{2}\varphi^2\right)\right]\frac{d\varphi}{\sqrt{2\pi v}} = -2\pi Z_{\mathrm{BP},v}\left(\frac{z}{2\pi}\right)$$

Using the method of steepest descent, the integral is

$$\log \int \exp\left[\frac{1}{v}\left(\frac{i\varphi^3}{6} + it\varphi + t - \frac{1}{2}\right)\right]\frac{d\varphi}{\sqrt{2\pi v}},$$

where $t = \frac{\tilde{z}_c - \tilde{z}}{\tilde{z}_c}$ (neglecting terms that are unimportant in the crossover regime $t \sim v^{2/3}$). The singular part is an Airy function, the same as in Cardy's analysis of the crossover from area-weighted SAL to SAL.[b]

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[a] J. Z. Imbrie, *Ann. Henri Poincaré* **4**, 421 (2003), math-ph/0303015.

[b] J. L. Cardy, *J. Phys.* **A34**, L665 (2001), cond-mat/0107223.

Forest-Root Formula

Let $f(\mathbf{t})$ depend on $\mathbf{t} = (t_{ij}), (t_i)$ for $1 \le i < j \le N$.

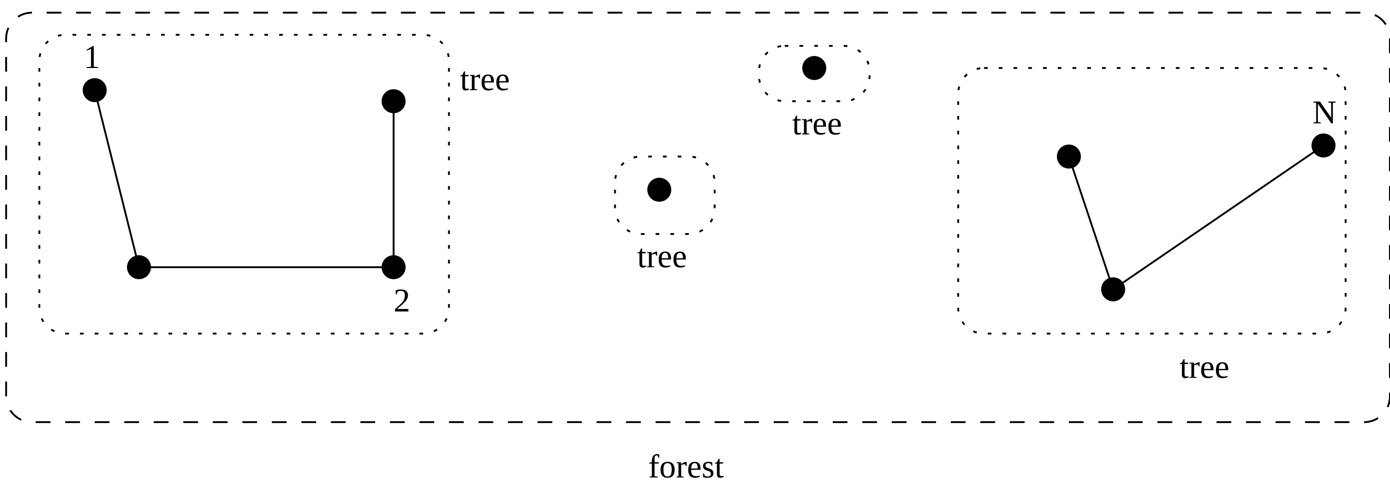
Assume $f \to 0$ as $t_i \nearrow \infty$.

Let $t_{ij} = |w_i - w_j|^2$, $t_i = |w_i|^2$ with $w_i \in \mathbb{C}$.

Then

$$f(\mathbf{0}) = \sum_{(F,R)} \int_{\mathbb{C}^N} f^{(F,R)}(\mathbf{t}) \left(\frac{d^2w}{-\pi}\right)^N .$$

The sum is over *forests* F and *roots* R (collections of *bonds* ij, and *vertices* i, respectively) such that each tree of F has exactly one root R.



Example: $N = 1, F = \emptyset, R = \{1\}$.

$$f(0) = \int_{\mathbb{C}} f'(t) \frac{d^2w}{-\pi} = -\int_0^\infty f'(t)dt$$

Supersymmetry

Replace

$$t_i \text{ with } \tau_i = w_i \bar{w}_i + \frac{dw_i \wedge d\bar{w}_i}{2\pi i}$$

and

$$t_{ij} \text{ with } \tau_{ij} = w_{ij} \bar{w}_{ij} + \frac{dw_{ij} \wedge d\bar{w}_{ij}}{2\pi i}.$$

(Recall that $w_{ij} = w_i - w_j$).

$f(\underline{\tau})$ is defined by its Taylor series.

Then I claim that a "localization" formula holds:

$$\int_{\mathbb{C}^N} f(\underline{\tau}) = f(\mathbf{0}).$$

This formula becomes the Forest-Root formula when expanded out.

The absence of loops comes from the fact that $(dw \wedge d\bar{w})^G = 0$ if G has a loop.

This can be proven by deforming the problem to the independent case $(N = 1)$, using ideas from [a].

[a] E. Witten, *J. Geom. Phys.* **9**, 303 (1992), hep-th/9204083.

Decoupling in Two Extra Dimensions

Use the Forest-Root formula to decouple the spheres in the HC gas by moving them apart in two extra dimensions.

At fixed N we have an integral over $x \in \mathbb{R}^D$ of

$$f(\mathbf{0}) = \prod_{1 \leq i < j \leq N} U(|x_{ij}|^2)$$

where $t_i = |w_i|^2 = 0$.

Extend this to $w \neq 0$ by writing

$$f(\mathbf{t}) = \prod_{1 \leq i < j \leq N} U(|x_{ij}|^2 + t_{ij}) \times (\text{ large } t \text{ cutoff})$$

Apply the Forest-Root formula,

$$f(\mathbf{0}) = \sum_{(F,R)} \int_{\mathbb{C}^N} f^{(F,R)}(\mathbf{t}) \left(\frac{d^2 w}{-\pi} \right)^N .$$

- Each $\dfrac{d}{dt_{ij}}$, when applied to one of the U's, becomes $\frac{1}{2}$ surface measure for the combined integrals over $y_{ij} = (x_{ij}, w_{ij})$.
- Spheres are stuck together according to the forest F.
- The trees of the forest decouple in the limit as the large t cutoff is removed. All but one cancel the normalization $Z_{\mathrm{HC}}(z)$.
- ρ_{HC} is evaluated as a sum/integral over branched polymers.

A Forest-Root Formula on $\mathbb{R}_+^N$

The following one-dimensional Forest-Root formula does the same job for DBP [a]:

Let $t_i \in \mathbb{R}_+,\ i = 1, \ldots, N$. and put $t_{ij} = |t_i - t_j|$

For any $f(\mathbf{t})$ which vanishes as $t_i \nearrow \infty$,

$$f(\mathbf{0}) = \sum_{(F,R)} \int_{\mathbb{R}_+^N} \prod_{r \in R} [-dt_r] \prod_{ji \in F} [-d(t_j - t_i)] f^{(F,R)}(\mathbf{t}).$$

Each link of T connects a vertex j to a vertex i which is one step closer than j to the root along T.

The integration region is $\{t_r \geq 0, r \in R$ and $t_j - t_i \geq 0, ji \in F\}$

$N = 1$: $f(0) = -\int_0^\infty f'(t)dt$. $N = 2$: chevron interpolation in $\mathbb{R}_+^2$

Theorems 1 and 2 for DBP follow as in the BP case.

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[a] J. Imbrie, *J. Phys. A: Math. Gen.* **37**, L137 (2004)