Functional RG for disordered elastic systems:

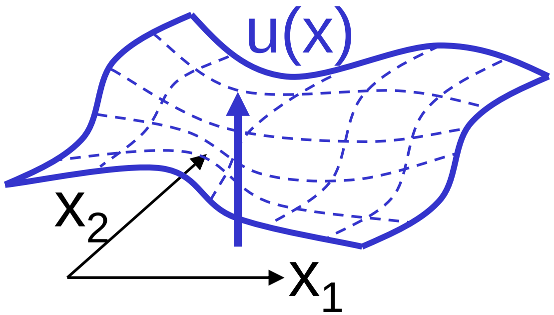
what is it?
what is it useful for?

tests and applications

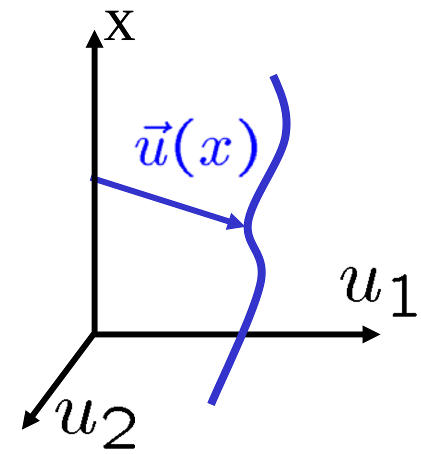
- model: elastic manifolds
- statics
  - Functional RG
  - measuring R(u)
- depinning transition
- activated dynamics (creep)

Elastic manifolds in a random potential

- interface



- directed line



$\in \mathbb{R}^N$ target space

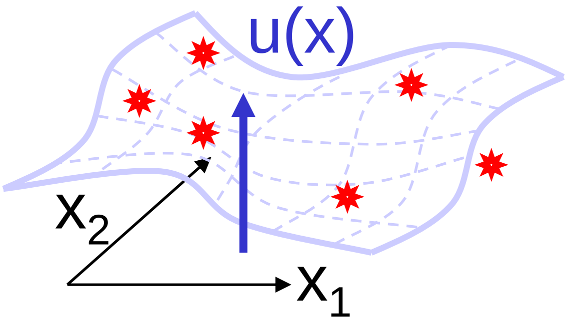
$\vec{u}(\vec{x})$

$\in \mathbb{R}^d$ internal space

$$H = \int d^d x \;\; \frac{c}{2}(\nabla u)^2$$

Elastic manifolds in a random potential

- interface



random potential

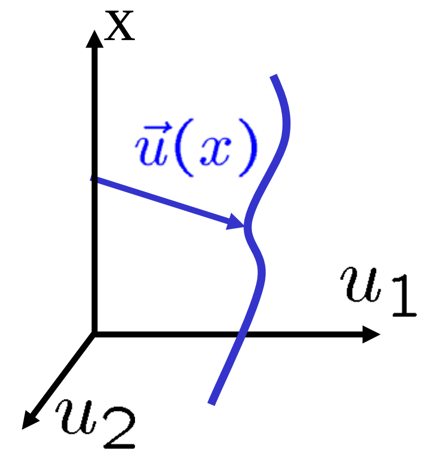
$V(x,u)$

$\vec{u}(\vec{x})$

$\in \mathbb{R}^N$ target space

$\in \mathbb{R}^d$ internal space

- directed line

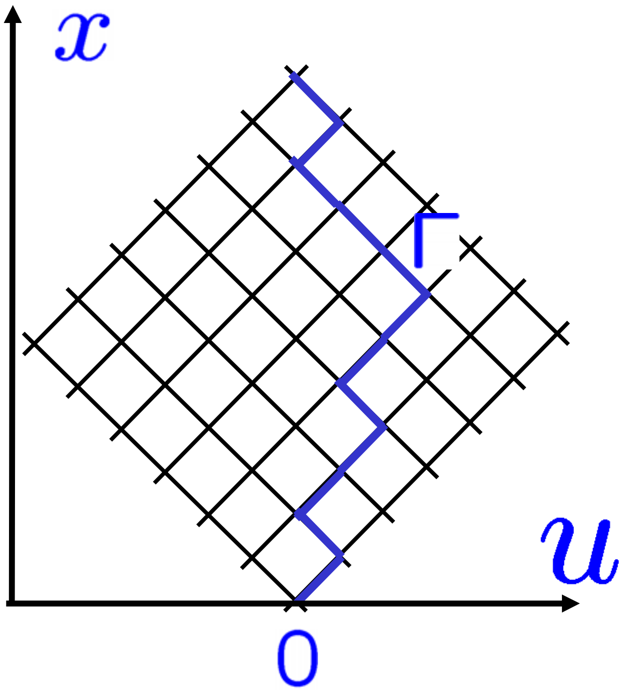


$$H = \int d^d x \quad \frac{c}{2}(\nabla u)^2 + V(x, u(x))$$

$$\overline{(u(x) - u(0))^2} \sim |x|^{2\zeta}$$

critical object

directed polymer $d = 1$



V_i random variables on each bond

$E_\Gamma = \sum_{i \in \Gamma} V_i$ energy of path Γ

find optimal path Γ_{min}

of minimal energy E_{min}

$$\overline{(E_{min} - \overline{E_{min}})^2} \sim x^{2\theta}$$

$$\overline{\langle (u(x) - u(0))^2 \rangle} \sim |x|^{2\zeta}$$

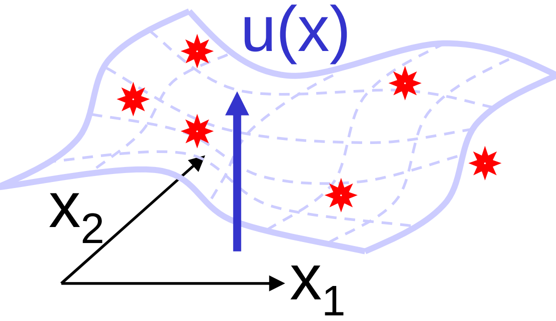
Exact results for N=1 $\zeta = 2/3$

$\theta = 1/3$

Elastic Manifold in Random Potential

$u(x) \in \mathbb{R}^N$

$x \in \mathbb{R}^d$



$$H = \int d^d x \; \frac{c}{2}(\nabla u)^2 + V(x, u(x))$$

$$\overline{V(x,u)V(x',u')} = \delta^d(x-x')\, R(u-u')$$

the function $R(u)$ is

- short range   random bond   *magnetic DW*
- long range   random field   *interfaces*
- periodic   *vortex lattice (Bragg glass)*

$$P[u] \sim e^{-H[u]/T}$$

$$\overline{\langle (u(x)-u(0))^2 \rangle} \sim |x|^{2\zeta}$$

$$T_l \sim L^{-\theta}$$

few univ class

$$\theta = d - 2 + 2\zeta$$

Pinned T=0

$$\zeta(N, d, ..)$$

Periodic object + weak disorder

Abrikosov vortex lattice

Bragg Glass: No dislocation
No translational order

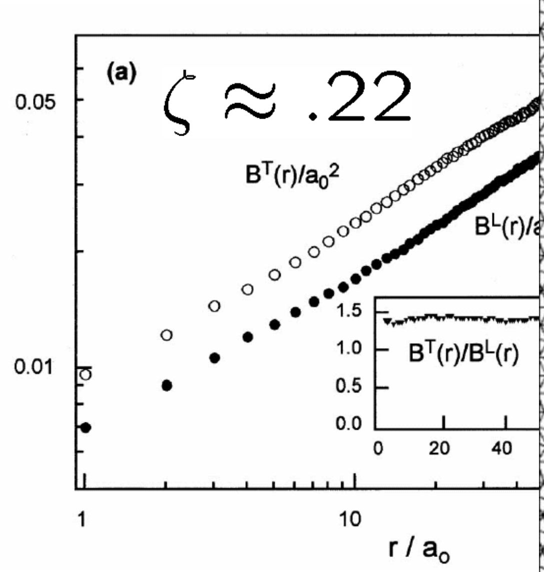
decoration

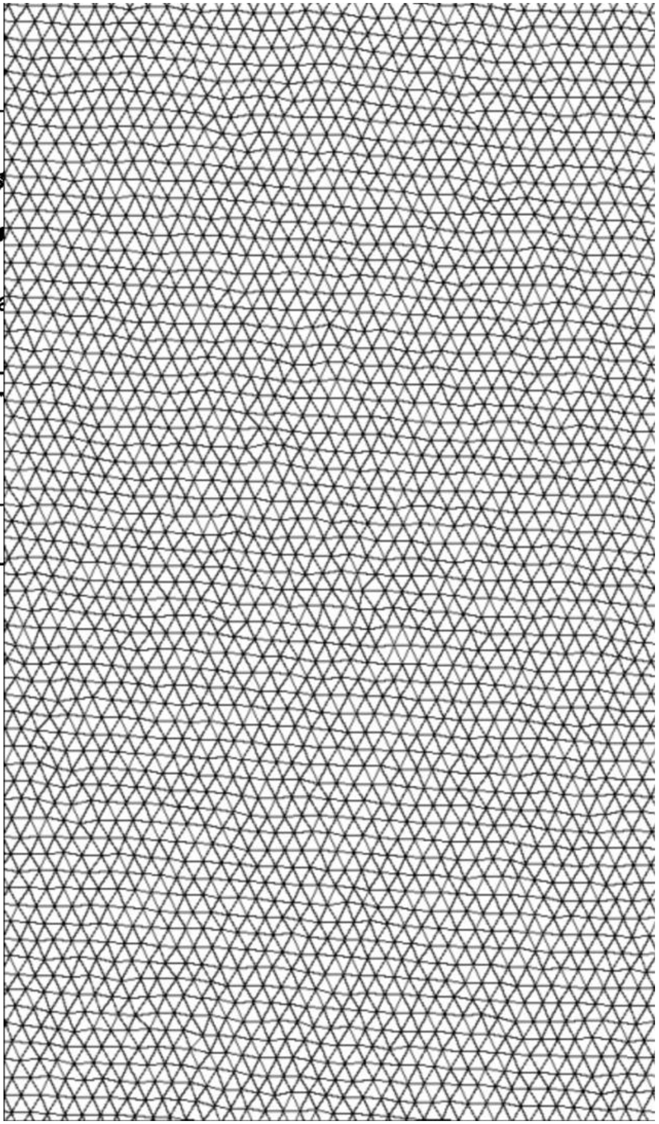
$$B(r) = \overline{(u(r) - u(0))^2}$$

$$B(r) \sim |r|^{2\zeta} \ll a_0^2$$

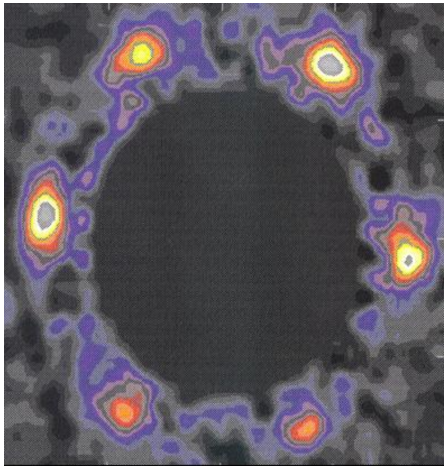
$$d = 3, N = 2$$

$$B(r) \sim A_d \ln |r| \geq a_0^2$$





Neutron diffraction



divergent Bragg peaks

$$\rho_K(x) = \rho_0 e^{iKu(x)}$$

$$\overline{\rho_K(x)\rho_K^*(0)} \sim |x|^{-\eta}$$

T. Klein, TG,PLD
Nature 2001

How to treat the problem

- replicas
- dynamics

$$Z_V = \int Du e^{-\beta H_V[u]} \qquad H_V[u] = \int d^d x \frac{c}{2}(\nabla u)^2 + V(x, u(x))$$

$$\beta = 1/T \qquad \overline{V(x,u)V(x',u')} = \delta^d(x-x')R(u-u')$$

$$\overline{\ln Z_V} = \lim_{n\to 0} \frac{1}{n}(\overline{Z_V^n} - 1) = Tre^{-\beta H_{rep}}$$

$$H_{rep} = \int d^d x \, \frac{1}{2} \sum_a (\nabla u_a)^2 - \frac{1}{2T} \sum_{ab} R(u_a(x) - u_b(x))$$

starting model: interacting field theory

How to average over disorder

- statics

$$H[u] = \int d^d x [\frac{c}{2}(\nabla u)^2 + V(x, u(x))]$$

Replica Field Theory

$$R(u_a - u_b)$$

- dynamics

$$\eta \partial_t u(x,t) = c\nabla^2 u(x,t) + F(x, u(x,t)) + \xi(x,t) + f$$

friction

elastic force

random pinning force

thermal noise

external force

$$F(x,u) = \partial_u V(x,u)$$

$$\overline{F(x,u)F(x',u')} = \delta^d(x-x')\Delta(u-u')$$

$$\Delta(u) = -R''(u)$$

Dynamical Field Theory (MSR,DJ)

$$\Delta(u(t) - u(t'))$$

Peculiar features of field theory for glass

- coupling constant is a function of the field $\begin{matrix} R(u) \\ \Delta(u) \end{matrix} \leftrightarrow g\,\phi^4$

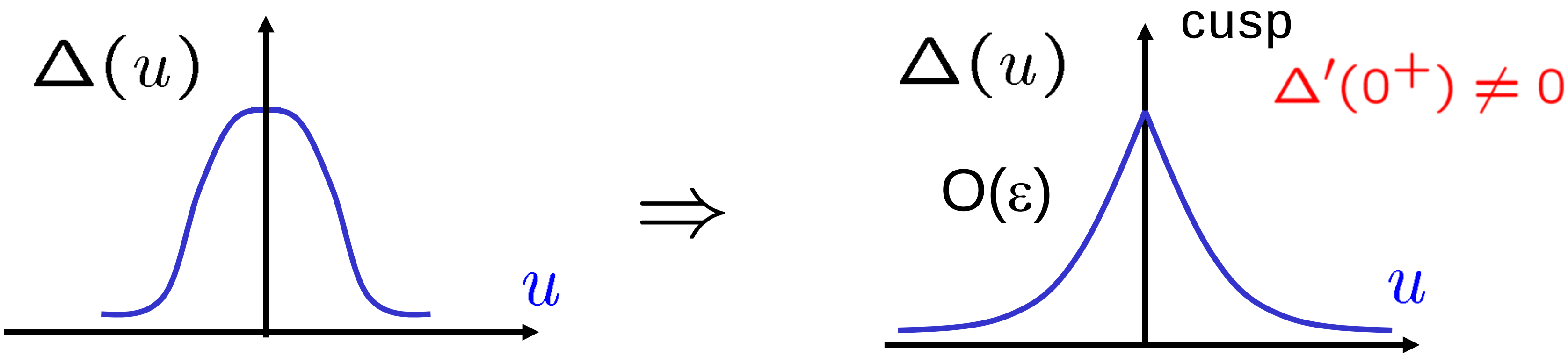
infinity of relevant operators d<4: Functional RG

- since $T_l = L^{-\theta} \to 0$ there should be

  a T=0 fixed point theory, indeed $\Delta_l(u) \to \Delta^*(u) = O(\epsilon)$ $\epsilon = 4 - d$

BUT $\Delta^*(u)$ is non-analytic at u=0 $L > L_c$

why? mode minimization instead of integration : shocks



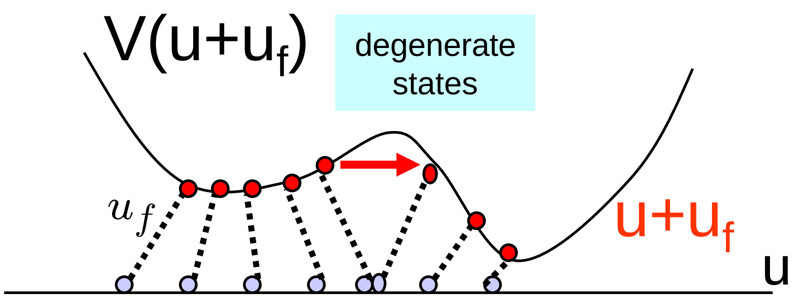
needs rescaling: yields ζ

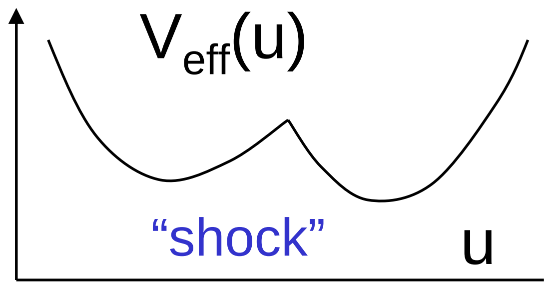
Physics of the Cusp: metastability

- “Toy RG” $$H = \frac{q^2}{2}u^2 + \frac{\Lambda^2}{2}|u_f|^2 + V(u+u_f)$$

$$H_{eff}[u] = \mathrm{Min}_{u_f}\, H[u, u_f]$$

Balents, Bouchaud, Mezard





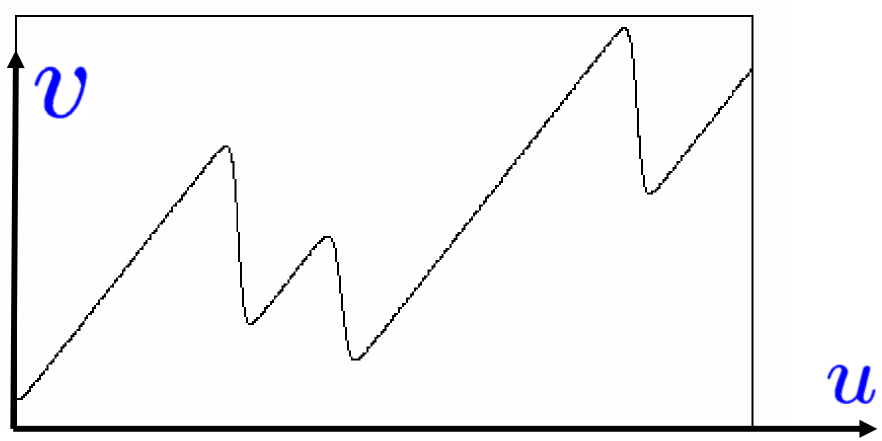
$F(u) = -V'(u)$ jumps

$F(u) - F(u')$ finite if shock between u and u' proba $p \sim |u-u'|$

$$\Delta(u) - \Delta(0) \sim \overline{(F(u) - F(u'))^2} \sim p \sim |u-u'|$$ (uniform density)

- Burgers d=1

$$v(u,t)$$



Functional RG

T=0

$$S = \frac{H_{rep}}{T} = \int_x \frac{1}{2T} \sum_a (\nabla u_a)^2 + m^2 u_a^2 - \frac{1}{2T^2} \sum_{ab} R(u_a(x) - u_b(x))$$

$$\Gamma[u] = \frac{1}{T} \sum_a \Gamma_1[u_a] - \frac{1}{2T^2} \sum_{ab} \Gamma_2[u_a, u_b] + ..$$

$$\Gamma_1[u_a] = \int_q c(q^2 + m^2) u_q^a u_{-q}^a$$

$$R(u_a - u_b) = \Gamma_2[u_a(x) = u_a, u_b(x) = u_b]$$

$$\tilde{R}_l(u) = m^{-\epsilon+4\zeta} R(u m^{-\zeta})$$

$$-m\partial_m \tilde{R}(u) = (\epsilon - 4\zeta)\tilde{R}(u) + \zeta u \tilde{R}'(u)$$

$$= \partial_l \tilde{R}(u)$$

$$l = \ln(1/m)$$

$$+\frac{1}{2}\tilde{R}''(u)^2 - \tilde{R}''(u)\tilde{R}''(0) + O(R^3)$$

Analysis of one loop FRG equation T=0

D.Fisher 86

$$\partial_l \tilde{R}(u) = (\epsilon - 4\zeta)\tilde{R} + \zeta u \tilde{R}' + \frac{1}{2}\tilde{R}''(u)^2 - \tilde{R}''(u)\tilde{R}''(0)$$

start with R(u) analytic

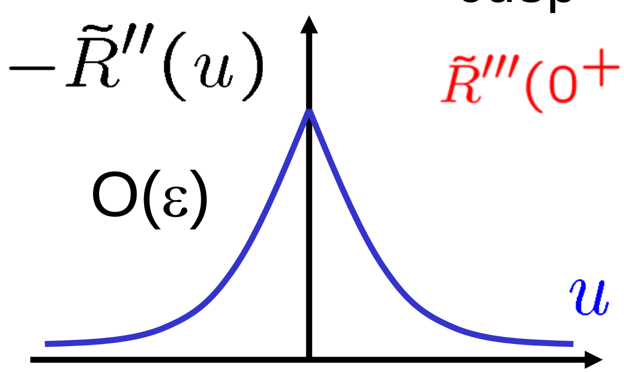
$$\partial_l \tilde{R}''''(0) = \epsilon \tilde{R}''''(0) + \tilde{R}''''(0)^2 \qquad \tilde{R}''''(0) \to \infty \qquad m = m_c^+$$

R(u) becomes non-analytic at u=0 beyond Larkin scale

$\Rightarrow$

cusp

$\tilde{R}'''(0^+) \neq 0$



Unique solution each
Univ class (B.C)

$$\zeta_{RB} = 0.208\epsilon$$

$$\zeta_{RF} = \epsilon/3$$

cusp $\longrightarrow$ *difficulties at two loop*

renormalizable theory for statics at T=0

PLD,Wiese,Chauve 2000

procedure (2 loop): • one loop counterterms are non-ambiguous

• ask for renormalizability cancellation of $1/\epsilon$ poles

$\Longrightarrow$ lift ambiguities

$$\partial_l \Delta(u) = (-\frac{\Delta^2}{2} + \Delta\Delta(0))'' + \frac{1}{2}(\Delta'^2(\Delta - \Delta(0))''$$

$$+\frac{\lambda}{2}\Delta'(O^+)^2\Delta''(u)$$

$$\lambda_{stat} = -1$$

- preserves linear cusp $\Delta(u) = -R''(u)$ $\zeta_{RF} = \epsilon/3$

$$\zeta_{RB} = 0.20829804\epsilon + 0.006858\epsilon^2$$

	one loop	two loop	exact
$d = 1$			
$\epsilon = 3$	0.625	0.687	0.666

- needs testing !

- what is R(u) ?

$$\frac{R''(u)}{R''(0)} = Y(u/\xi) \qquad \int dz Y(z) = 1$$

random field disorder

$$d = 4 - \epsilon$$

$$Y = Y(z) \leftrightarrow z = \frac{\sqrt{Y - 1 - \ln Y - \frac{\epsilon}{3} F(y)}}{\int_0^1 dy \sqrt{y - 1 - \ln y - \frac{\epsilon}{3} F(y)}}$$

$$F(y) = 2y - 1 + \frac{y \ln y}{1-y} - \frac{1}{2} \ln y + \mathrm{Li}_2(1-y)$$

- are there exactly solvable examples ?

How to measure R(u)

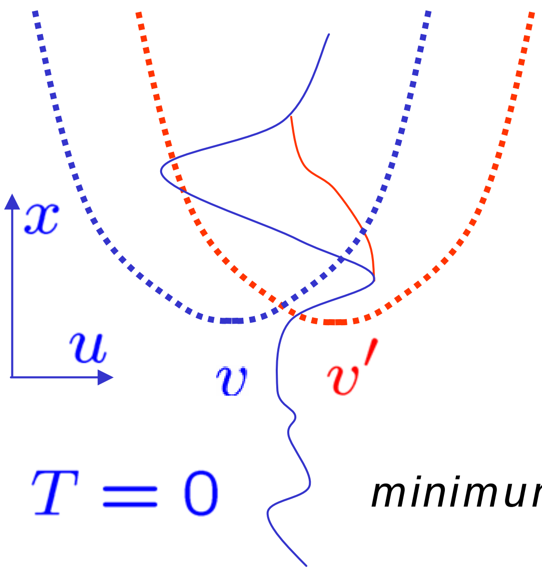
PLD

cond-mat/0605490

$$\exp(-\frac{1}{T}\hat{V}(v)) =$$

$$\int Du \; e^{\frac{1}{T}\int d^dx \frac{1}{2}m^2(u(x)-v)^2+\frac{1}{2}(\nabla u)^2+V(x,u(x))}$$

$$\overline{\hat{V}(v)\hat{V}(v')} = L^d \hat{R}(v-v')$$



one shows that: $\hat{R}(v) = R(v) \longleftrightarrow$ defined from $\Gamma[u]$

differences in higher cumulants

$T = 0$ *minimum energy configuration* $u_0(x;v)$

$$u(v) = L^{-d}\int d^dx \; u_0(x;v)$$

$v - u(v)$ *exhibits shocks*

$$\Delta(u) = -R''(u)$$

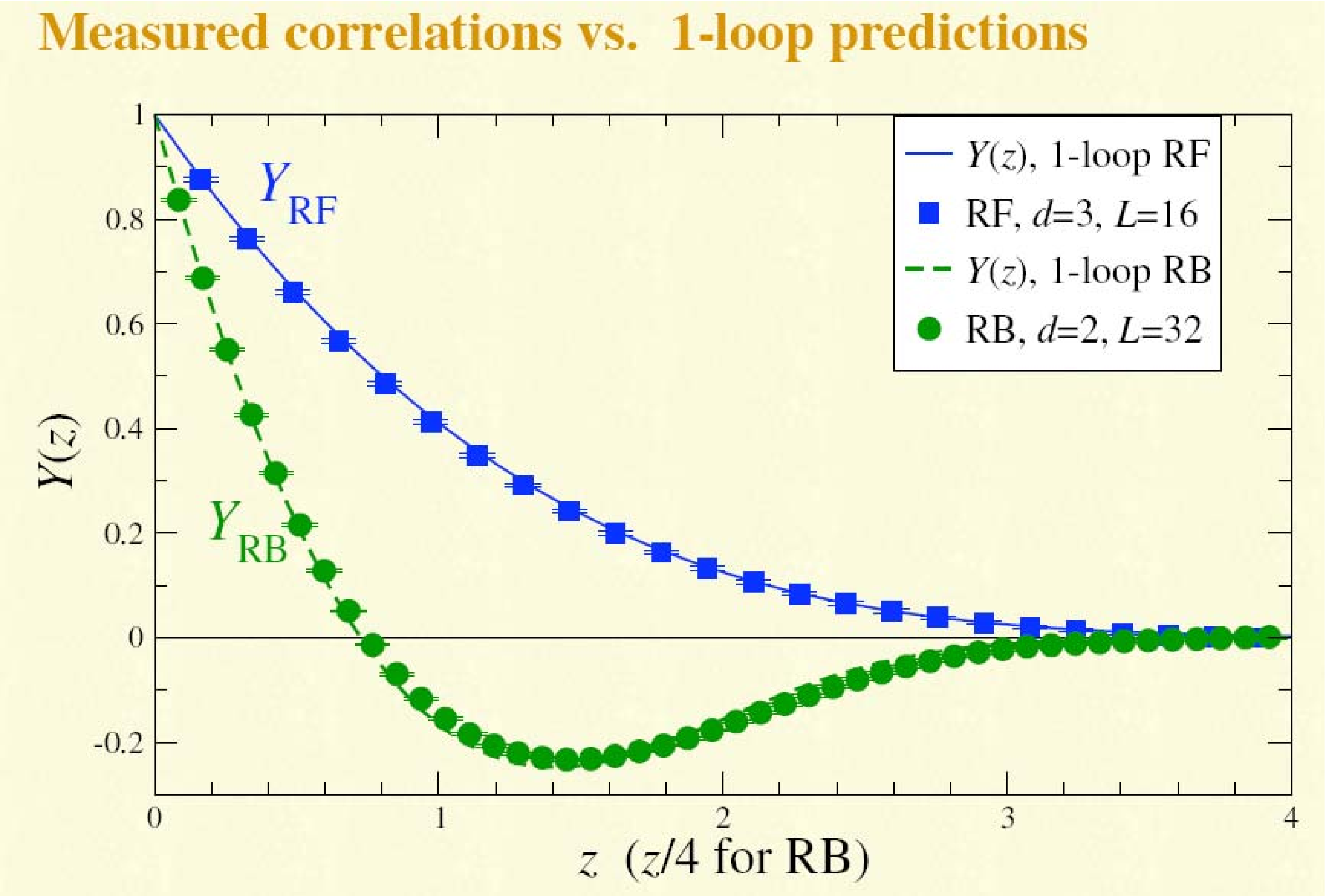
$$\overline{(v-u(v))(v'-u(v'))} = L^{-d}m^{-4}\Delta(v-v')$$

Alan Middleton

PLD,KW

condmat/0606160

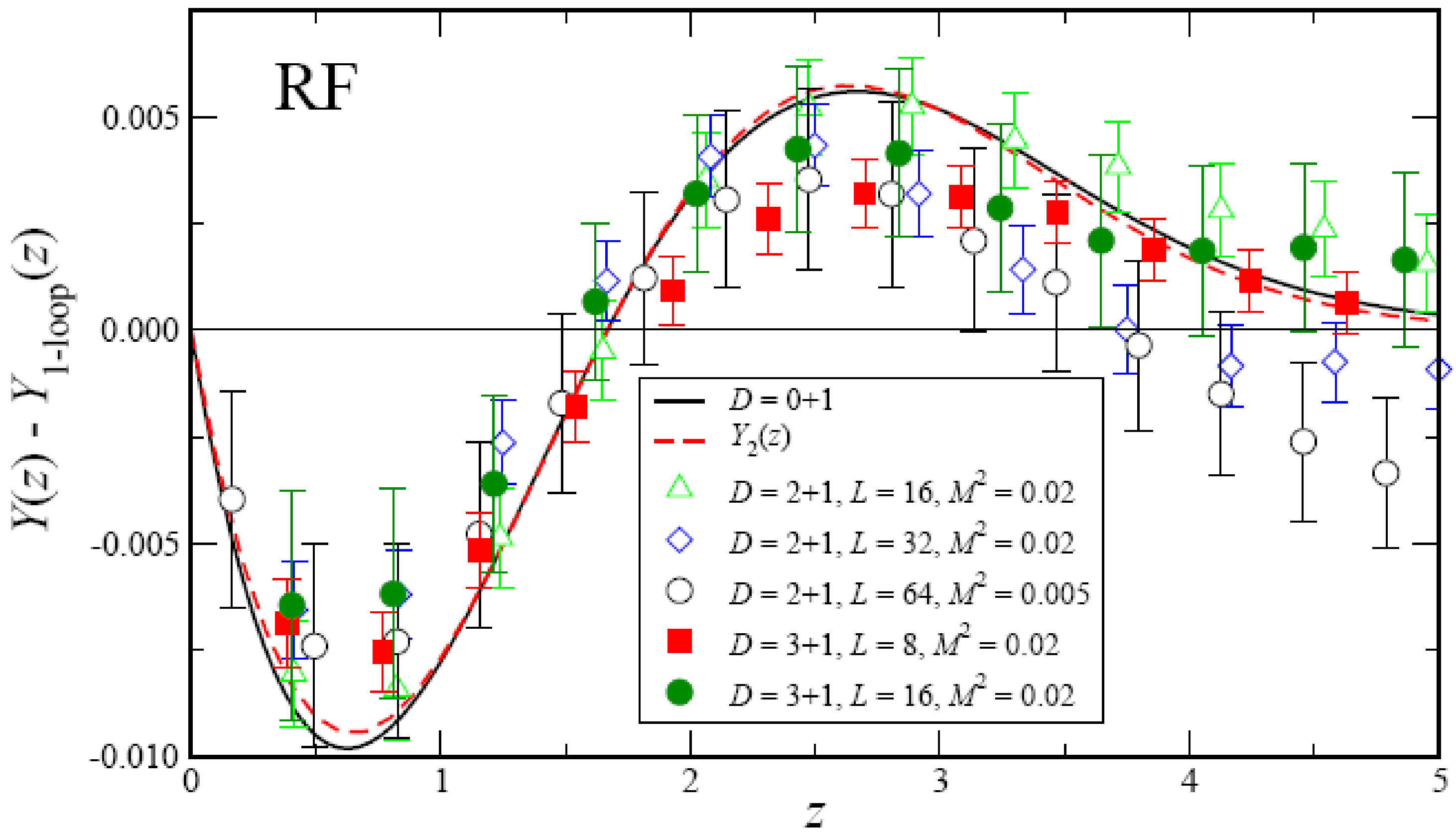
$$Y = \Delta/\Delta(0)$$



$$\overline{(v - u(v))(v' - u(v'))} = L^{-d} m^{-4} \Delta(v - v')$$

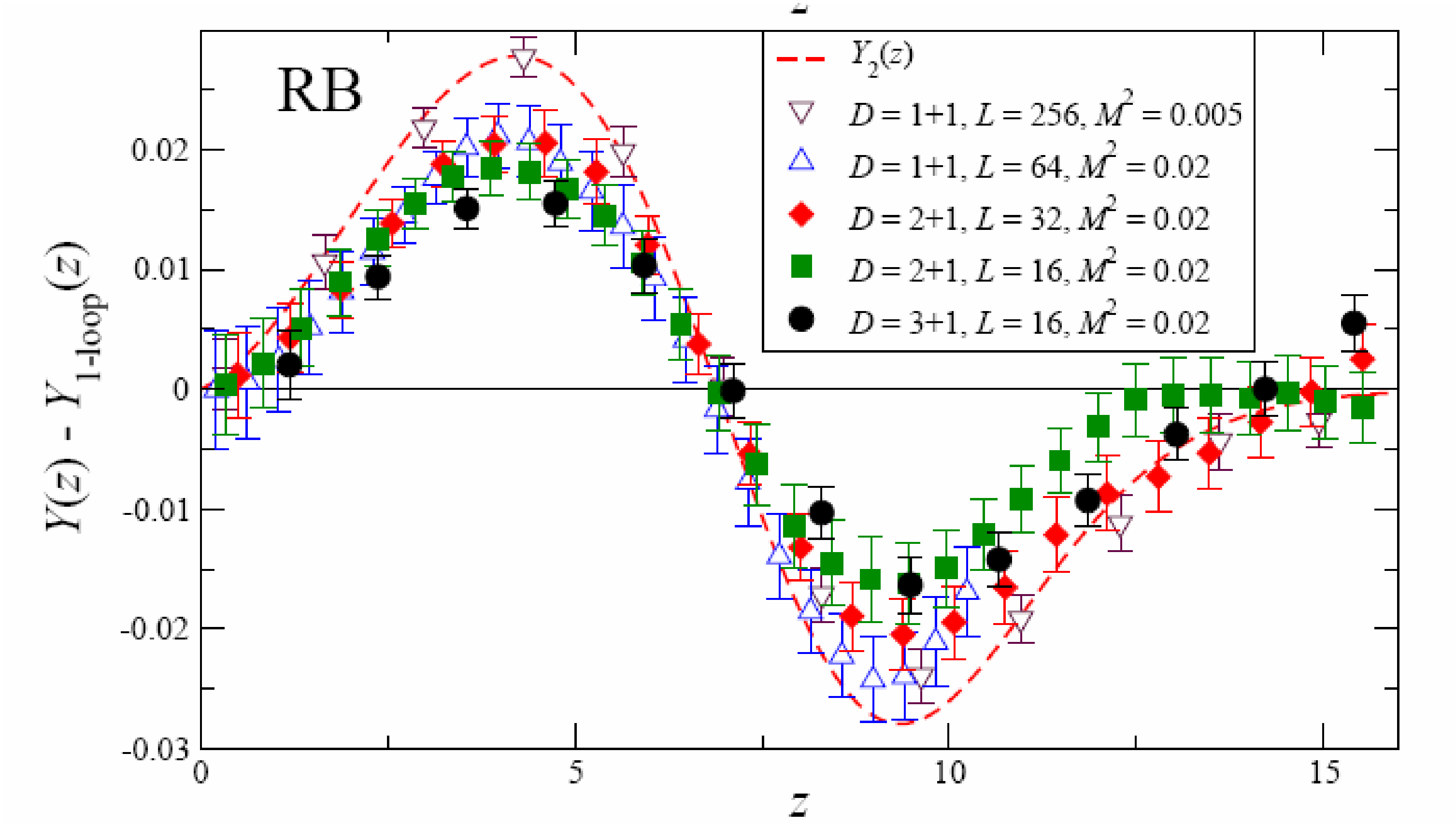
$$\Delta(u) = -R''(u)$$

Deviations from one loop: random field disorder



$$Y(z) = Y_1(z) + \epsilon Y_2(z) + ..$$

Deviations from one loop: random bond disorder



$$Y(z) = Y_1(z) + \epsilon Y_2(z) + ..$$

Decaying Burgers equation and shocks

- *d=0* $\exp(-\frac{1}{T}\hat{V}_m(v)) = \int du\ e^{\frac{1}{T}[\frac{1}{2}m^2(u-v)^2+V(u)]}$

particle in a random potential

$$\overline{V(u)V(0)} = R_0(u)$$

Force $F(v) = \hat{V}'(v)$ *obeys Burgers equation* $F \leftrightarrow$ u

w % y

u(x,t)

any N

n ¢ 3 % u

$$\partial_t \mathrm{u} + \mathrm{u}'_x \mathrm{u} = \nu \mathrm{u}''_{xx}$$

velocity field

$$T \leftrightarrow 2\nu$$

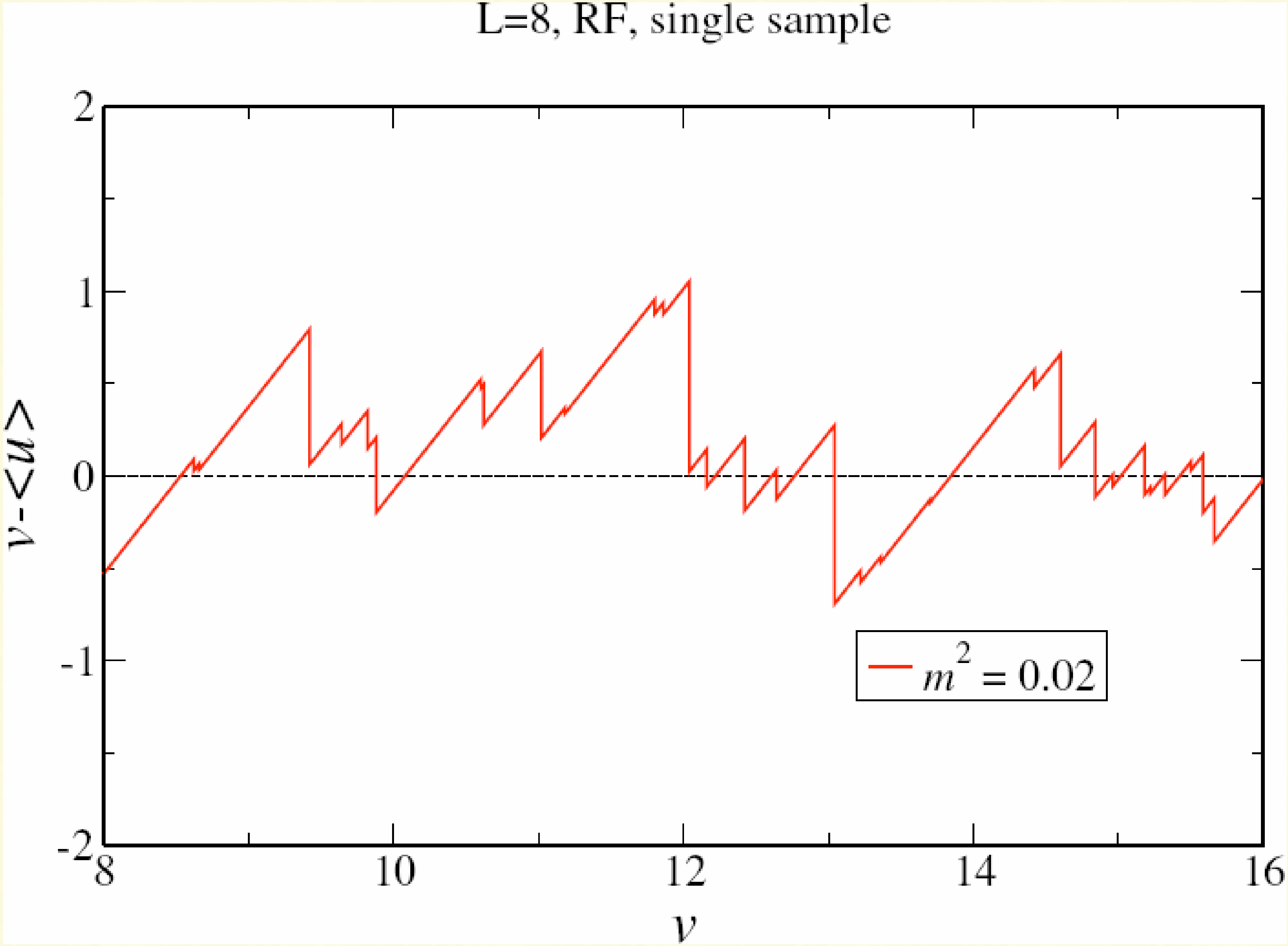
shocks form then merge (N=1: ballistic aggregation)

- *d>0*

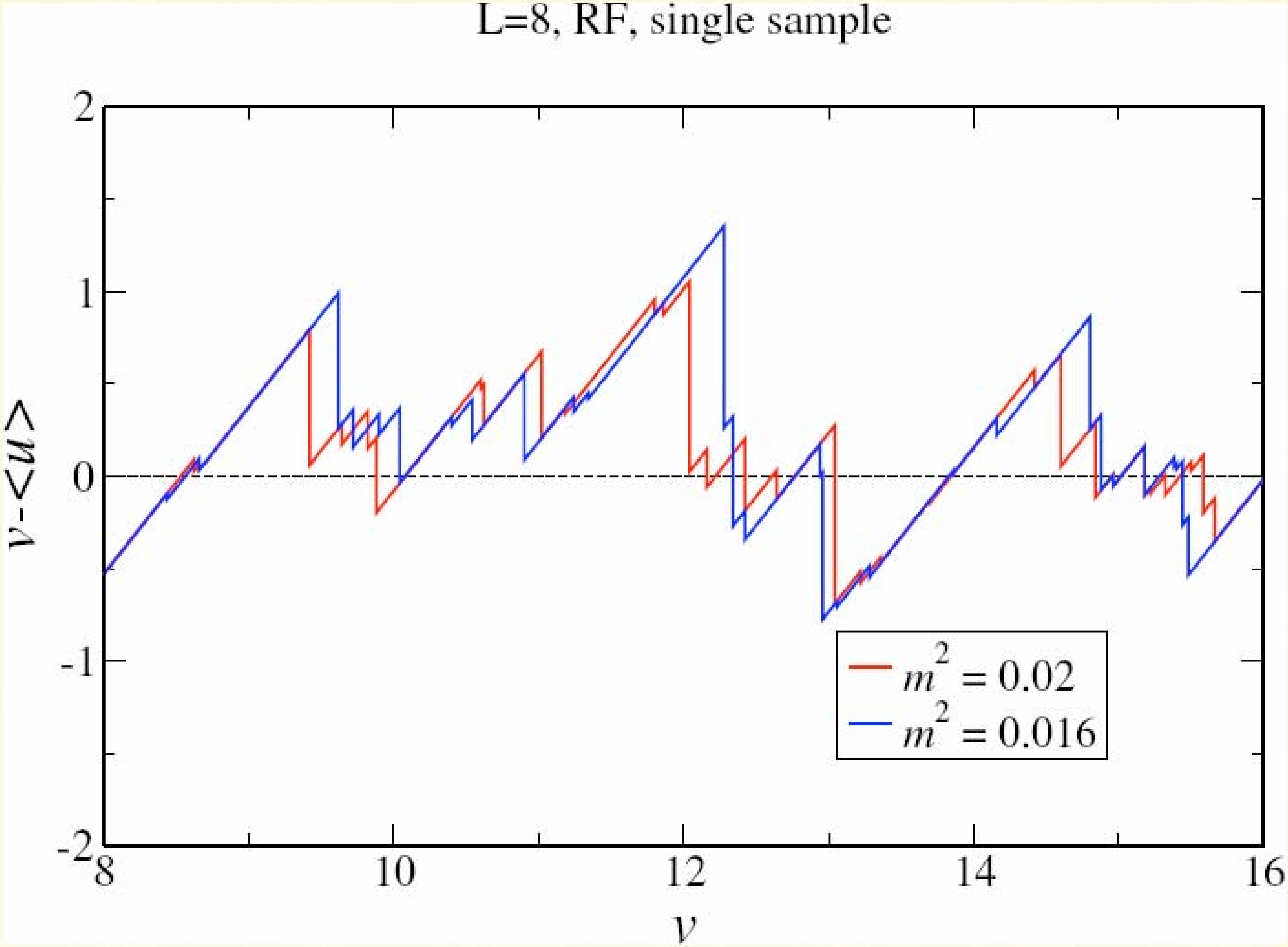
Functional Decaying Burgers $\mathcal{F}_x = \hat{V}'_x[v] \equiv \delta\hat{V}[v]/\delta v_x$

$$-2m\partial_m \mathcal{F}_x[v] = \int_{yz} \partial g_{yz}(T\mathcal{F}''_{xyz}[v] - \mathcal{F}'_{xy}[v]\mathcal{F}_z[v])$$

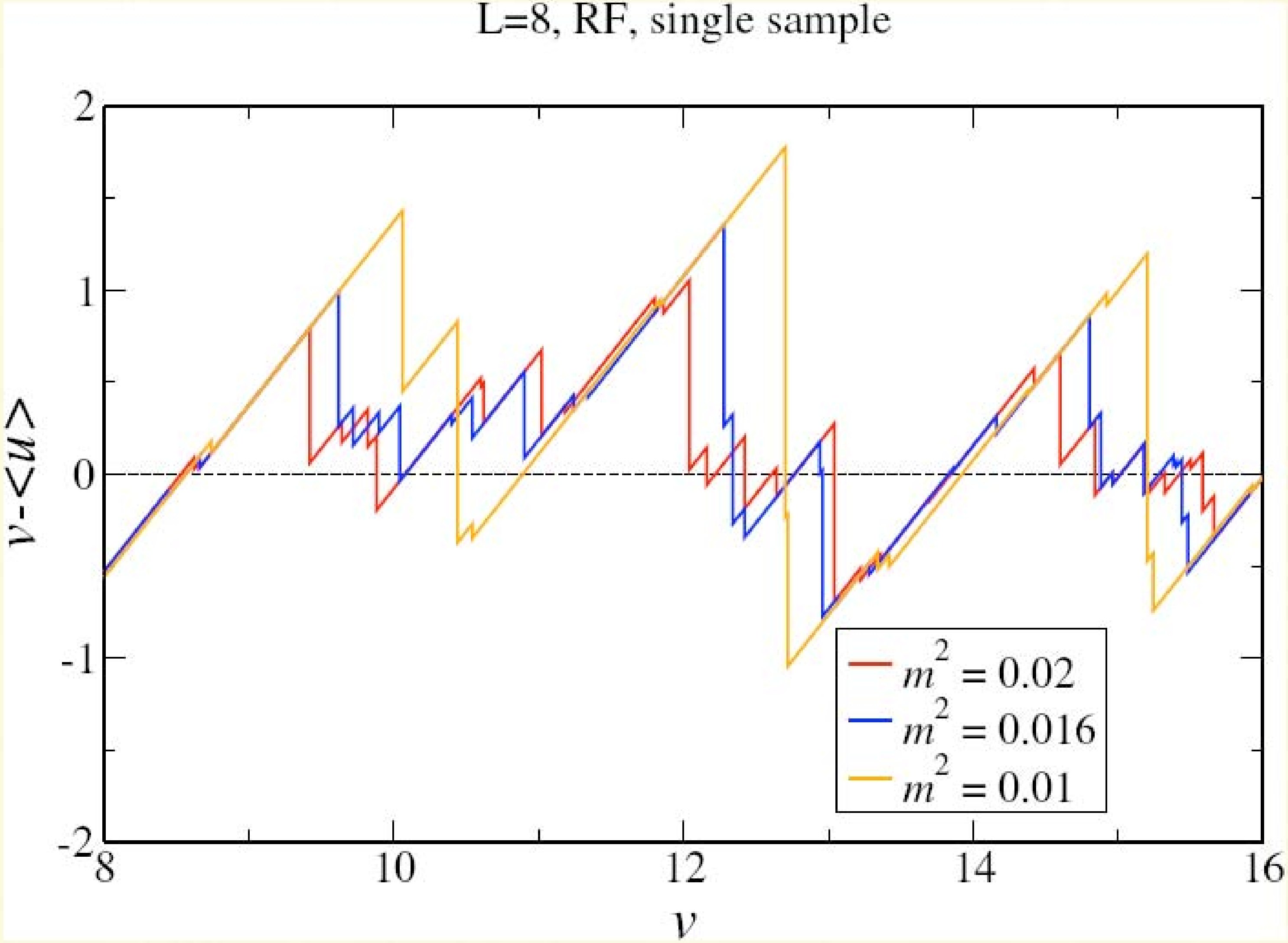
Sequence of m^2 in a single sample



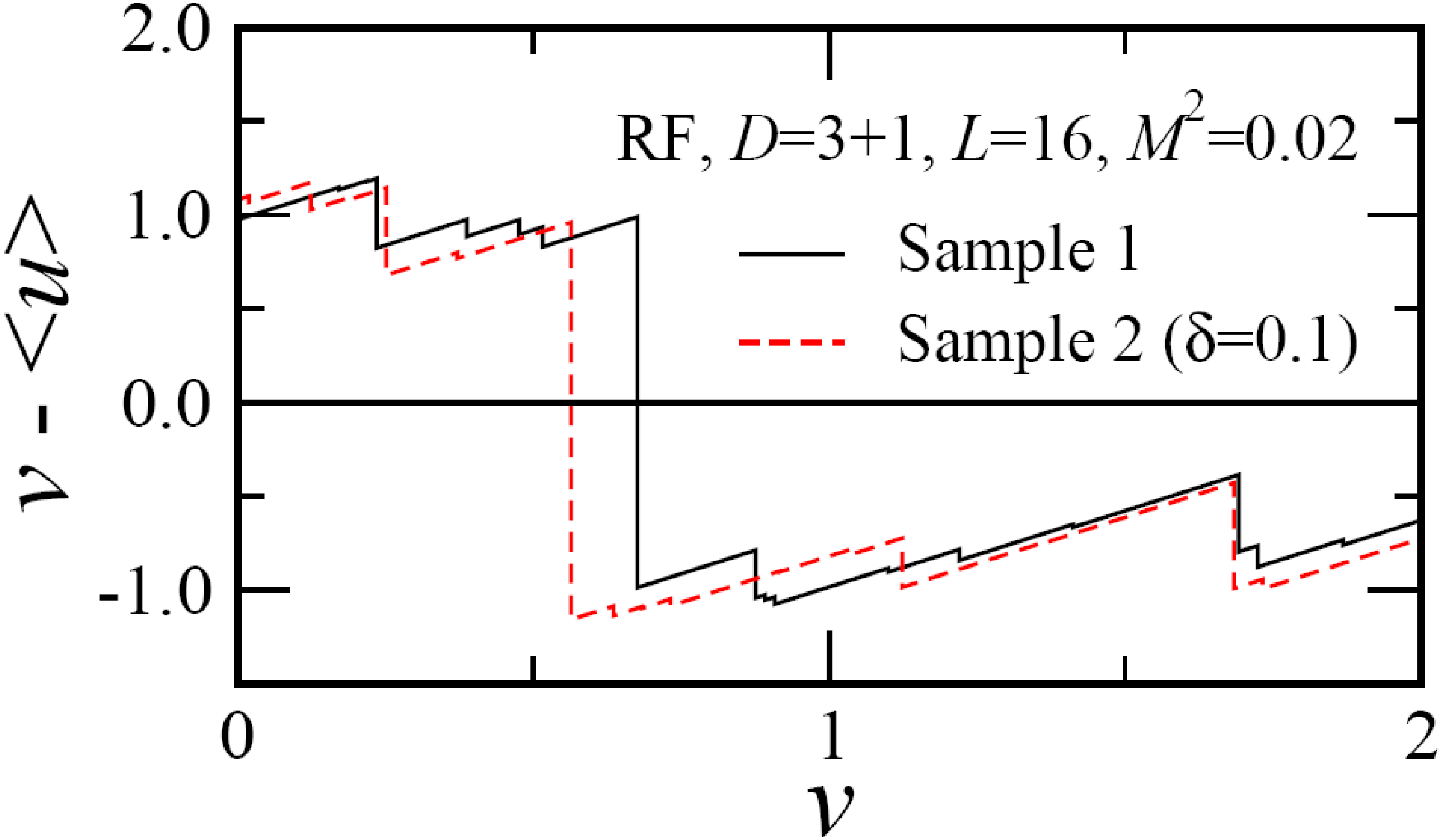
Sequence of m^2 in a single sample



Sequence of m^2 in a single sample



Chaos



Solution for Sinai model: *particle in Brownian energy lansdcape*

random field disorder d=0

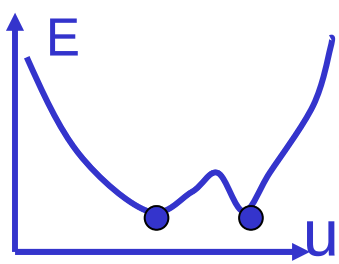
$$\overline{(V(u)-V(0))^2} \sim 2|u|$$

$$R_0(u) \sim -|u|$$

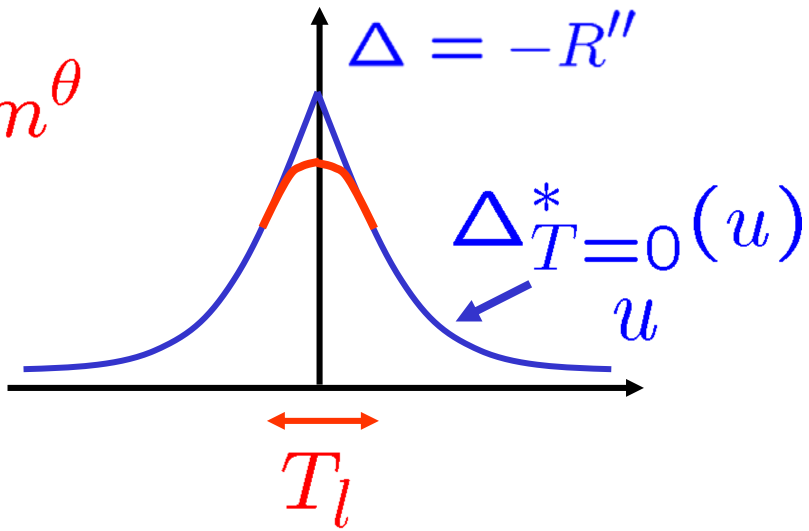
$$R^*_{T=0}(0) - R^*_{T=0}(v) = 2\sqrt{\pi v} e^{-\frac{1}{48}v^3} \int_{-\infty}^{+\infty} \frac{dz_1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dz_2}{2\pi i} [v + 2(z_2 - z_1)^2]$$

$$\times e^{\frac{v}{2b}(z_1+z_2) + \frac{(z_2-z_1)^2}{v}} \left[\frac{1}{v Ai(z_1) Ai(z_2)} + \frac{\int_0^\infty dV e^{\frac{v}{2}V} Ai(V+z_1) Ai(V+z_2)}{Ai(z_1)^2 Ai(z_2)^2}\right]$$

Thermal boundary layer $u \sim T_l = T m^\theta$



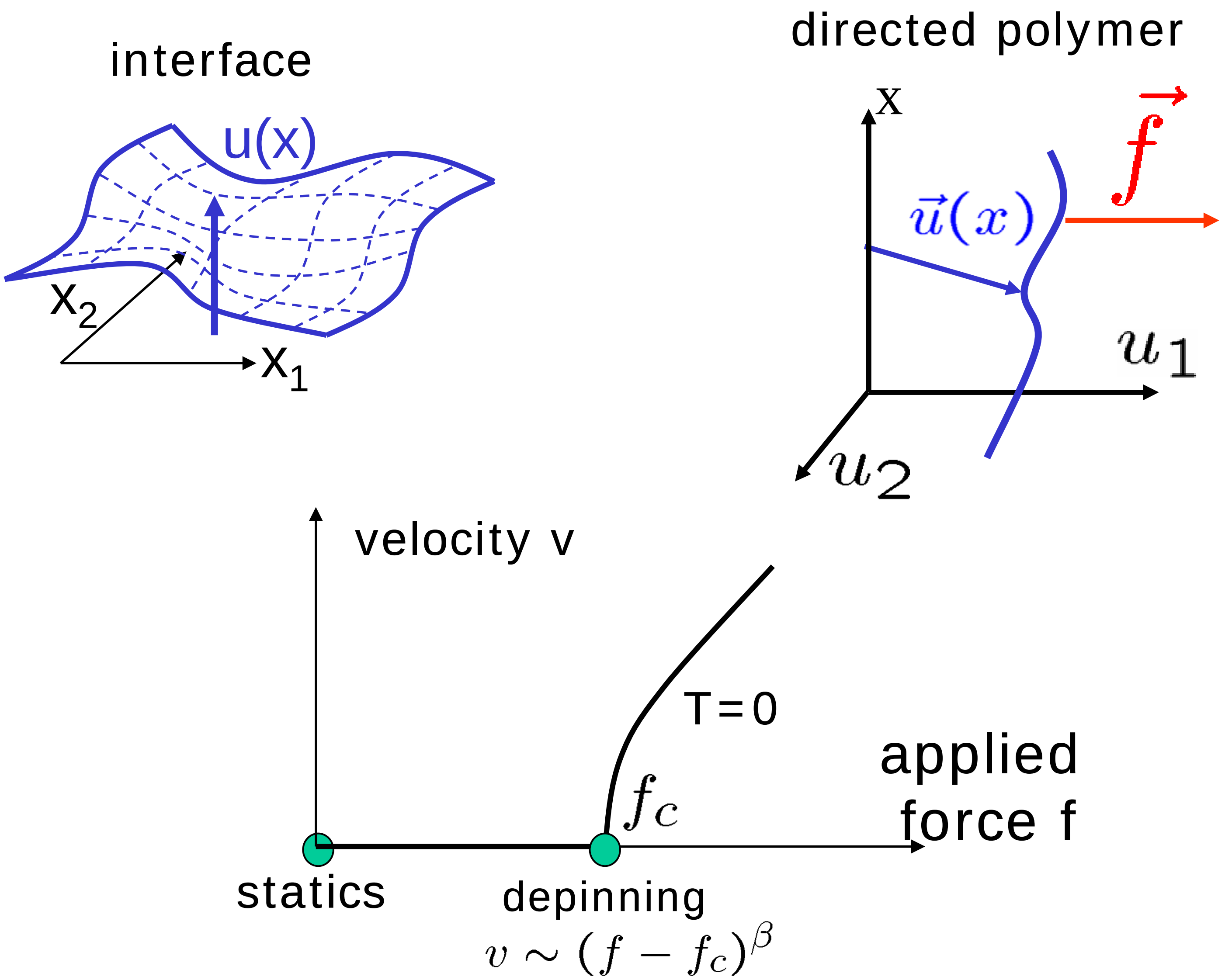
is probability density of
2 degen. minima distant of
u=y



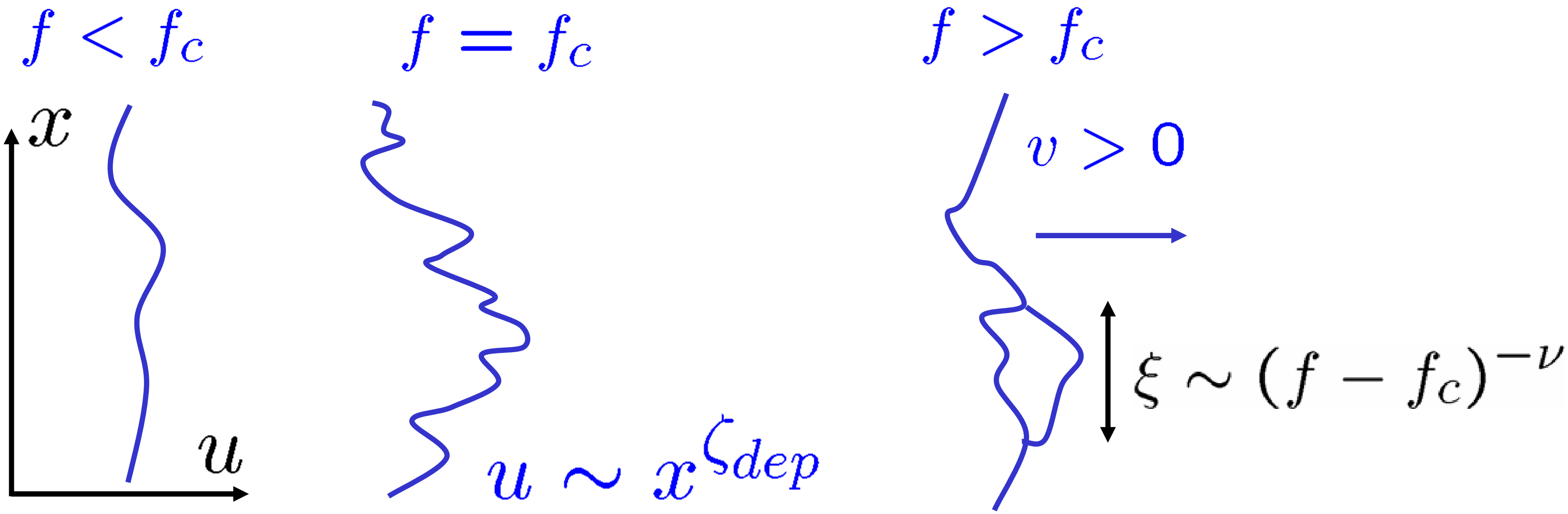
$$D(y) = 2 \int \frac{d\lambda}{2\pi} e^{i\lambda y} \frac{Ai'(i\lambda)}{Ai(i\lambda)} \int \frac{d\mu}{2\pi} e^{-i\mu y} \frac{1}{Ai(i\mu)^2}$$

$$\tilde{R}''(u) - \tilde{R}''(0) = T_l \int dy D(y) y^2 \frac{yu}{4T_l} (\coth \frac{yu}{2T_l} - \frac{1}{2})$$

Driven elastic manifolds in random potential



Scaling picture of depinning



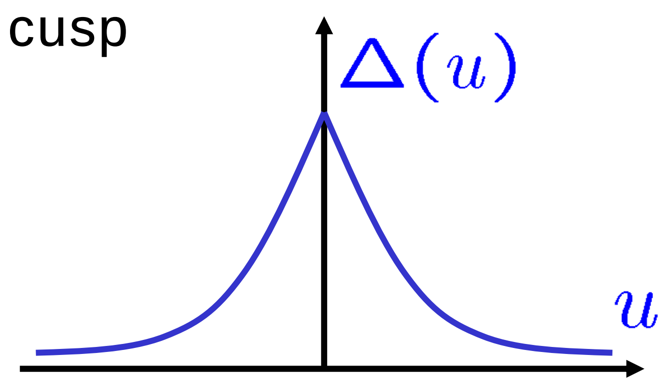
critical blocked configuration

moving

$$v \sim u/\tau \sim \xi^{\zeta - z} \sim (f - f_c)^{\beta}$$

Puzzle with one loop FRG for depinning

One loop: Nattermann et al. 92 Narayan Fisher 92



$f_c \sim |\Delta'(0^+)|$

$$\partial_l \Delta(u) = -\frac{d^2}{du^2}\partial_l R(u)$$

$$\longrightarrow \quad \Delta_l(u) = -R_l''(u)$$

- depinning fixed point same as statics ?

  $v \to 0^+$ where is irreversibility ?

- Narayan-Fisher conjecture $\zeta_{dep} = \epsilon/3$

  numerics ?

one loop theory is NOT consistent!

Two loop depinning

PLD,K.Wiese,P.Chauve 2001

$$\partial_l \Delta(u) = (\epsilon - 2\zeta)\Delta + \zeta\Delta' - (\frac{\Delta^2}{2} + \Delta\Delta(0))''$$
$$+\frac{1}{2}(\Delta'^2(\Delta - \Delta(0))'' + \frac{\lambda}{2}\Delta'(O^+)^2\Delta''(u)$$

$$\lambda_{dep} = 1$$

$$\lambda_{stat} = -1$$

- different from statics: irreversibility recovered

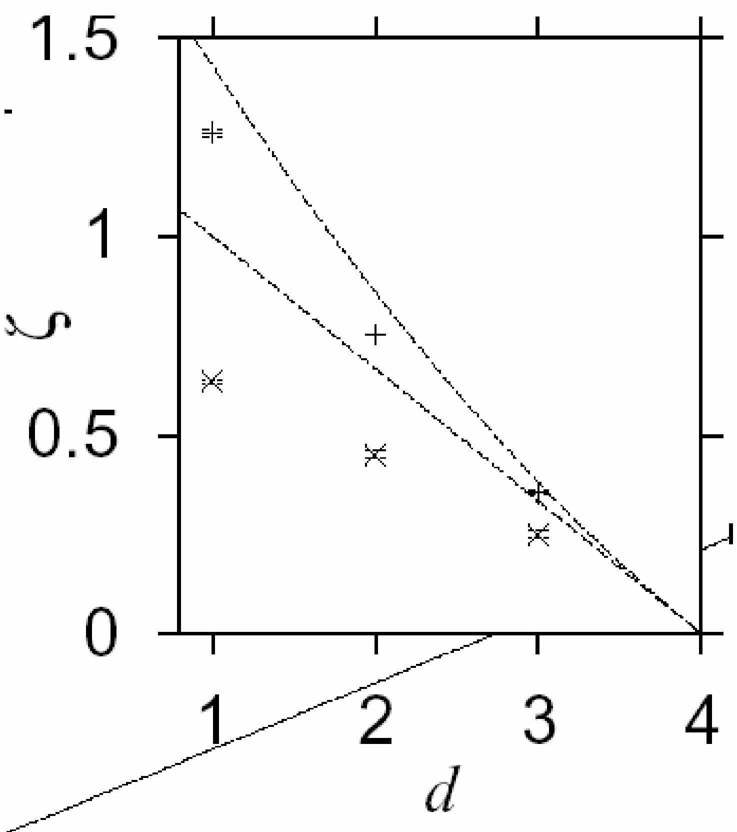
$$\zeta_{dep} = \frac{\epsilon}{3}(1 + 0.1433\epsilon + ..) > \zeta_{NF}$$

- single universality class

Numerics new high precision algorithm by Rosso and Krauth

Find exact critical string configuration on cylinder $L^d \times M$

	Analytic		
d	One-loop	Two-loop	ζ_{Δ^2}
1	1	1.44	1.26 ± 0.01
2	2/3	0.86	0.753 ± 0.002
3	1/3	0.38	0.355 ± 0.01



Numerical calculation of FRG fixed point at depinning

Alberto Rosso

PLD,KW

condmat/0610821

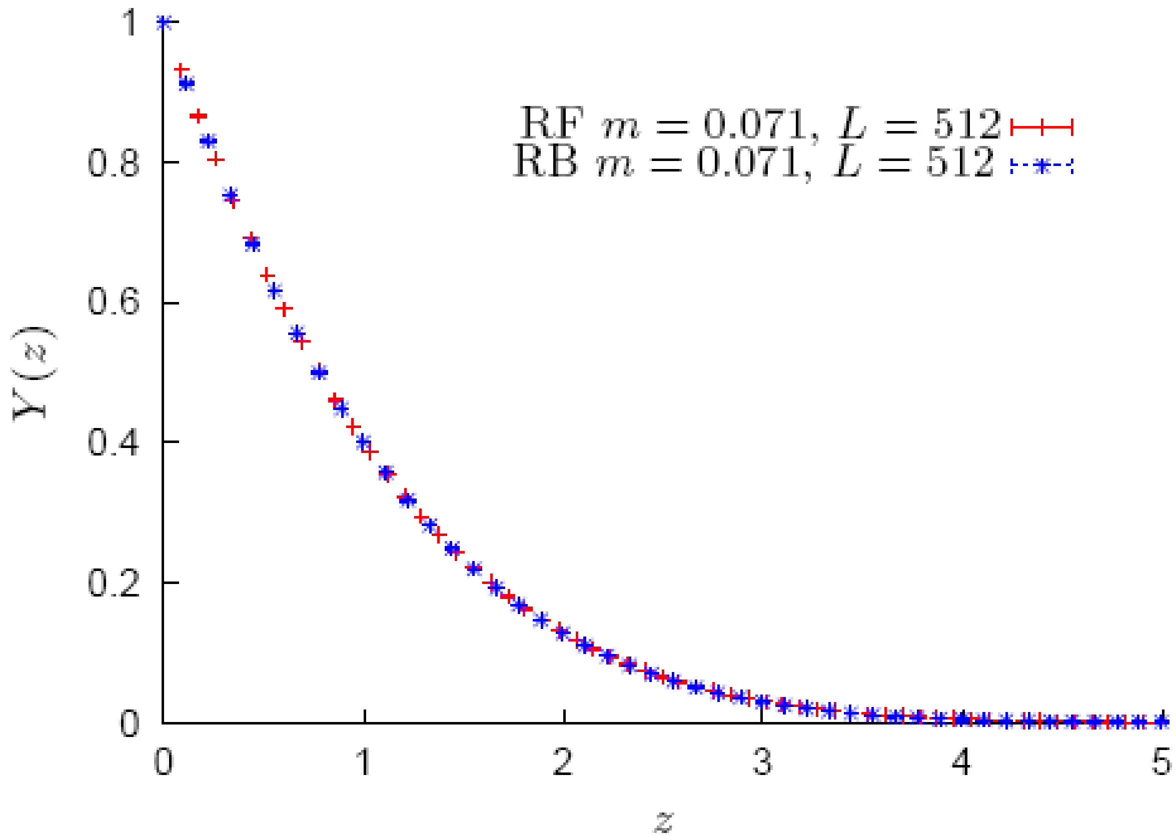
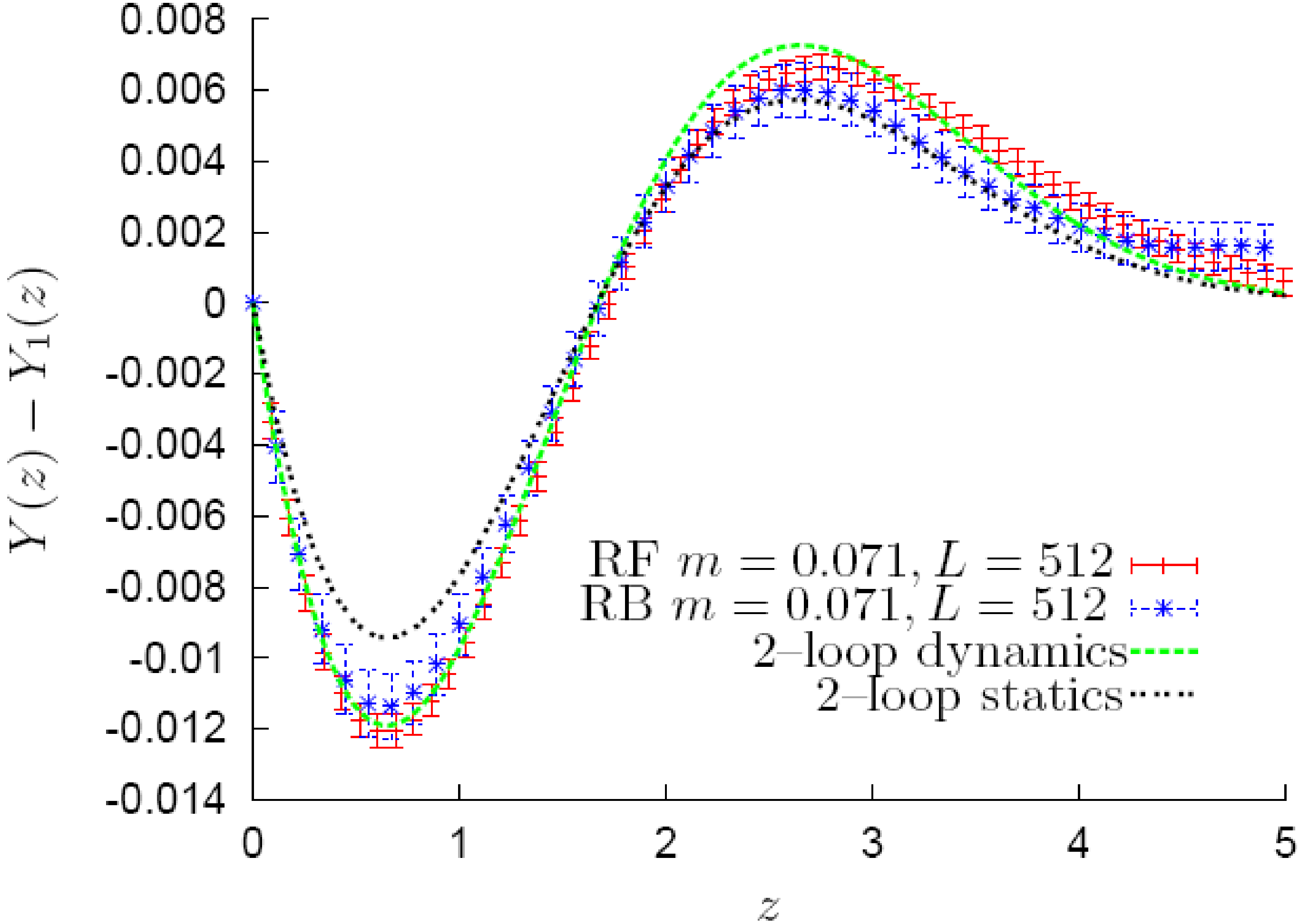


FIG. 3: Universal scaling form $Y(z)$ for $\Delta(u)$ for RB and RF disorder.

+ quadratic well

driven quasi-statically

Deviations from one loop



Contact line depinning

viscous fluid   water,glycerol

Moulinet,Guthmann,Rolley 2002

overdamped
quasi-static
} checked

$$\zeta = 0.51 \pm 0.03$$

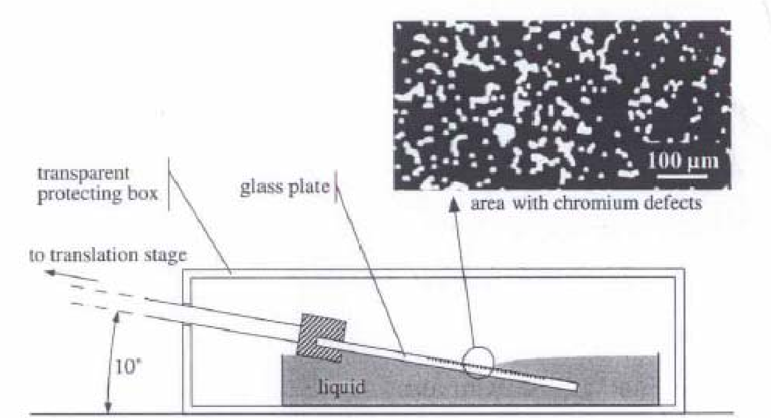


Fig. 2. Sketch of the experimental setup. Inset: photograph of the disordered substrate, the chromium defects appear as white square spots.

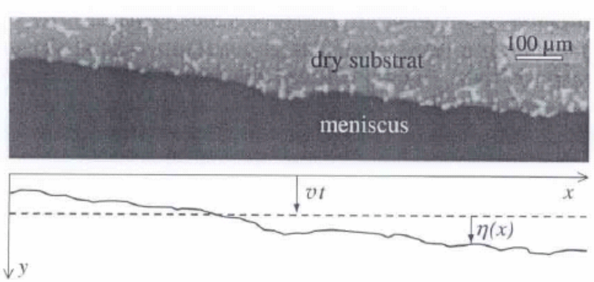


Fig. 1. Upper part: image of the contact line obtained with an ordinary CCD camera. Lower part: the position $\eta(x,t) \equiv y(x,t) - vt$ of the CL is defined with respect to its average position vt.

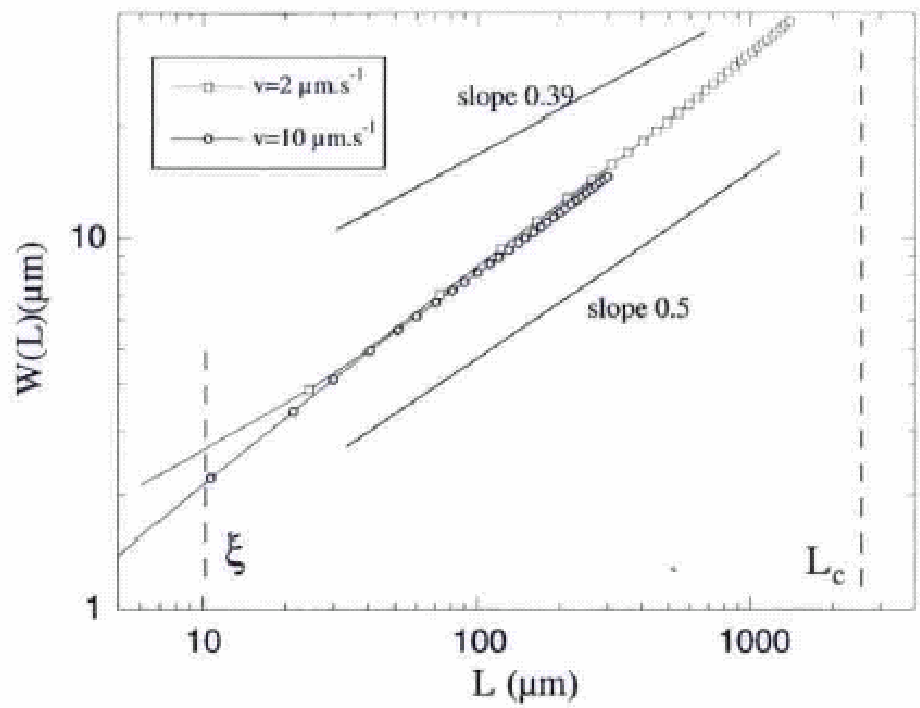


Fig. 3. Roughness W as a function of distance L for different drift velocities. The upper (respectively, lower) graph corresponds to data obtained with water (respectively, water-glycerol mixture). For both graphs, the data ∘ have been obtained with a larger magnification (resolution 2.1 μm) than the others (resolution 6.1 μm).

contact line, cracks $\zeta_{exp} \approx 0.55$

for long range elasticity $cq^2 \to c|q|$ Joanny De Gennes

$\epsilon = 2 - d$ $\zeta_{1loop} = \frac{\epsilon}{3} \approx 0.33$

- 2loop FRG we find $\zeta_{dep} = \frac{\epsilon}{3}(1 + 0.397\epsilon + ..)$

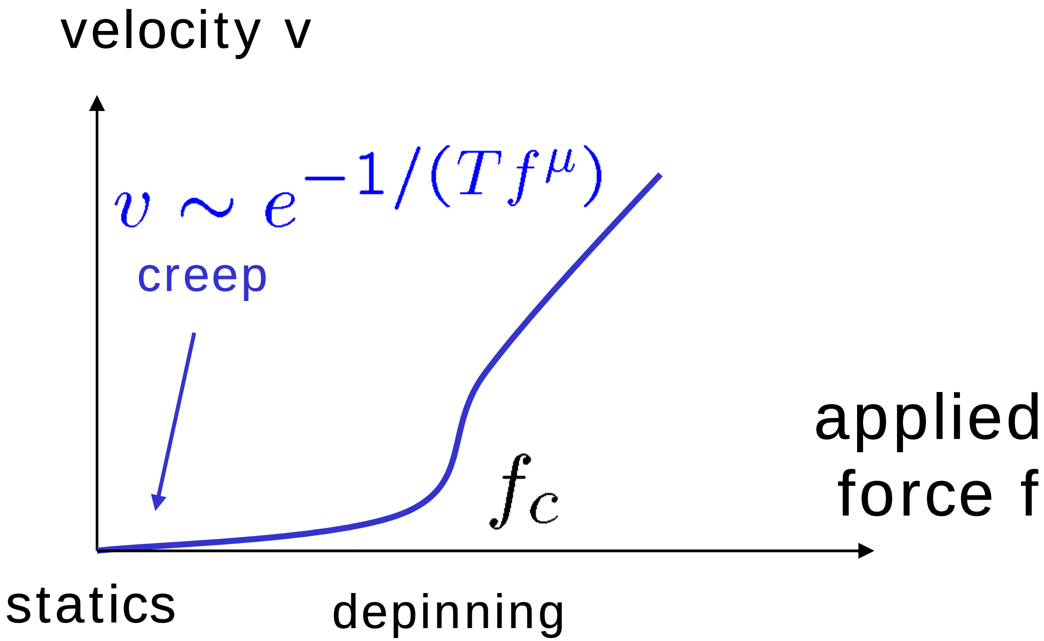
$\longrightarrow$ $\zeta_{2loop} \approx 0.47$

BUT Rosso Krauth $\zeta = 0.390 \pm 0.002$

is this elastic overdamped model

the correct one for these systems ?

T>0 : creep



- equilibrium dynamics

  ‘‘near equilibrium’’ dynamics: creep

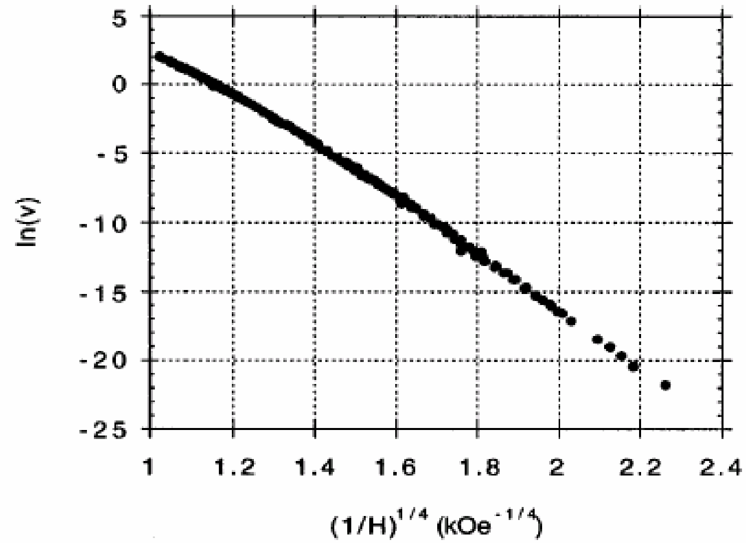


FIG. 3. Natural logarithm of MDW velocity as a function of $(1/H)^{1/4}$ (room temperature, $H \le 955$ Oe).

Ising magnetic film Co

Lemerle Ferre et al. 98

domain wall creep, d=1+1, RB

$$\mu = 1/4$$

$$v \sim e^{-1/T f^{\mu}}$$

Zeldov et al. BSCCO

Bragg glass

$$\mu = 1/2$$

$$R \sim v \sim e^{-U(j)/T}$$

$$U(j) \sim j^{-\mu}$$

Qualitative argument for creep

Assume: • small f: limited by typical nucleation event

• near equilibrium, activated dynamics over optimal barrier

$$\tau \sim e^{U_b/T} \qquad U_b = R^{\psi} - fuR^{d} \sim_{sp} f^{-\mu} \qquad u \sim R^{\zeta}$$

$$R_T = R_{opt} \sim f^{-1/(2-\zeta)}$$

$$\mu = \frac{\psi}{d+\zeta-\psi}$$

$$\psi = \theta = d - 2 + 2\zeta_{eq}$$

$$\zeta = \zeta_{eq}$$

$$\longrightarrow \quad \mu = \frac{\theta}{2-\zeta}$$

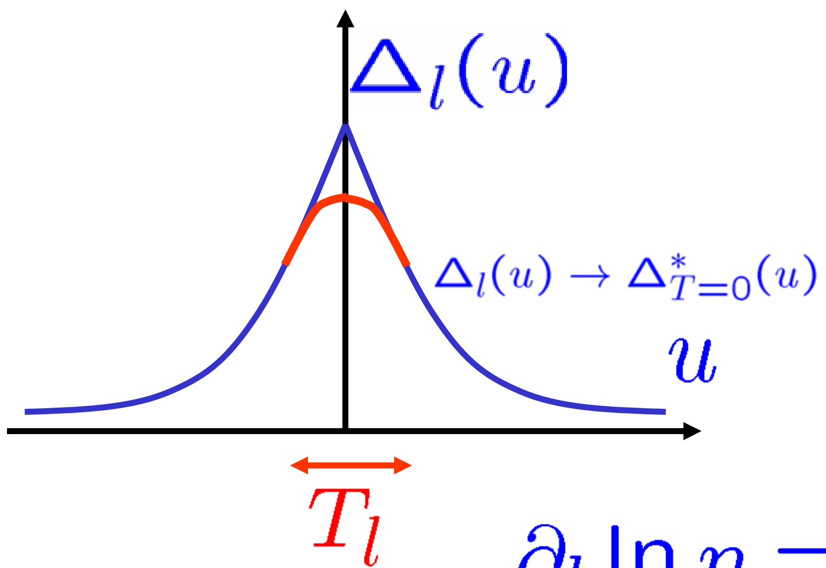
Q: What happens after jump ?

FRG in temperature

Balents,Chauve,TG,PLD

$$\partial_l \tilde{\Delta}(u) = \epsilon \tilde{\Delta} - \tilde{\Delta}'^2 - \tilde{\Delta}''(\tilde{\Delta} - \tilde{\Delta}(0)) + T_l \tilde{\Delta}''(u)$$

$$T_l = T_0 e^{-\theta l}$$



$\Delta(u)$ remains analytic

but curvature is blowing up

$$\partial_l \ln \eta = -\tilde{\Delta}''(0) \sim \frac{1}{T_l} = \frac{e^{\theta l}}{T_0} = \frac{L^\theta}{T_0}$$

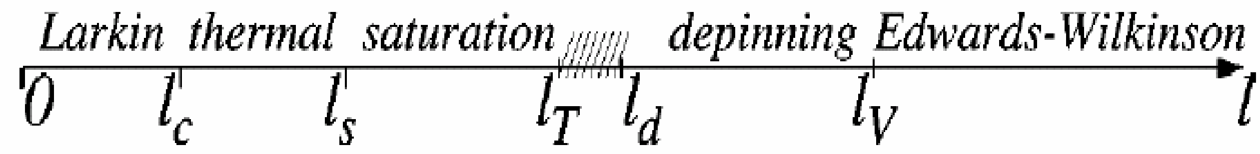
exponential growth of time scale

$$\Rightarrow \quad \tau_L = \eta_l = e^{L^\theta / T}$$

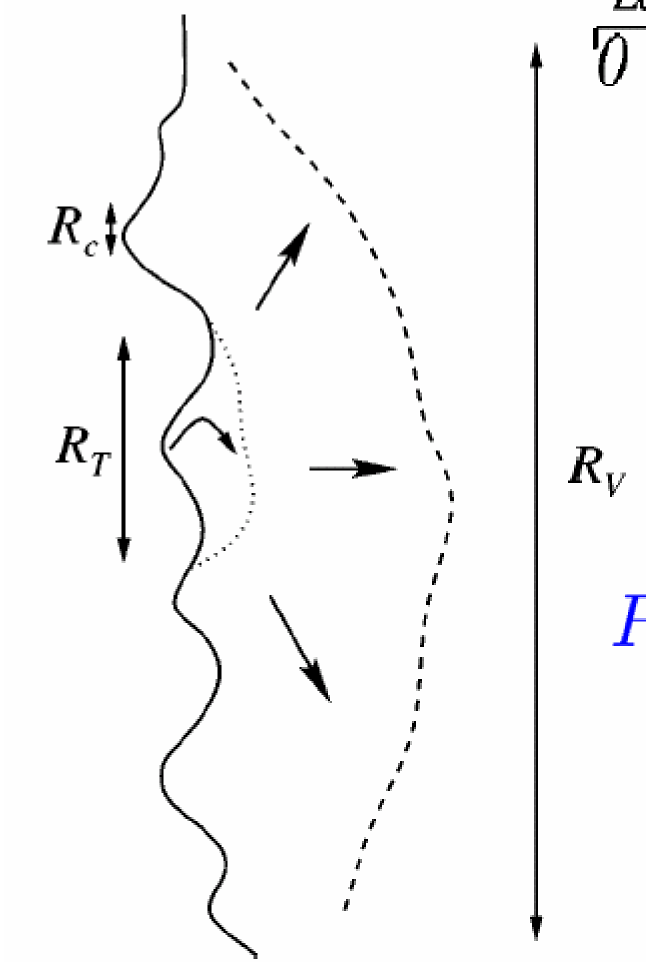
barriers grow as $U_L \sim L^\psi \quad \psi = \theta$ CREEP LAW

one loop FRG, v>0 and T>0 : creep physics

Chauve,TG,PLD 98



$l = \ln R$



follow RG flow of $\Delta_l(u), \eta_l, \tilde{T}_l, f_l$

- derive creep law
- new depinning regime

$R_V \sim T^{-\sigma} f^{-\lambda}$ $\qquad R_T < R < R_V$

suggest fast deterministic motion

- statistics of nucleation events
- which scales equilibrate ?

Conclusion

- *field theory of pinning: statics, depinning and creep*

  Functional RG

- *method to measure the fixed point function of the FRG in numerics*

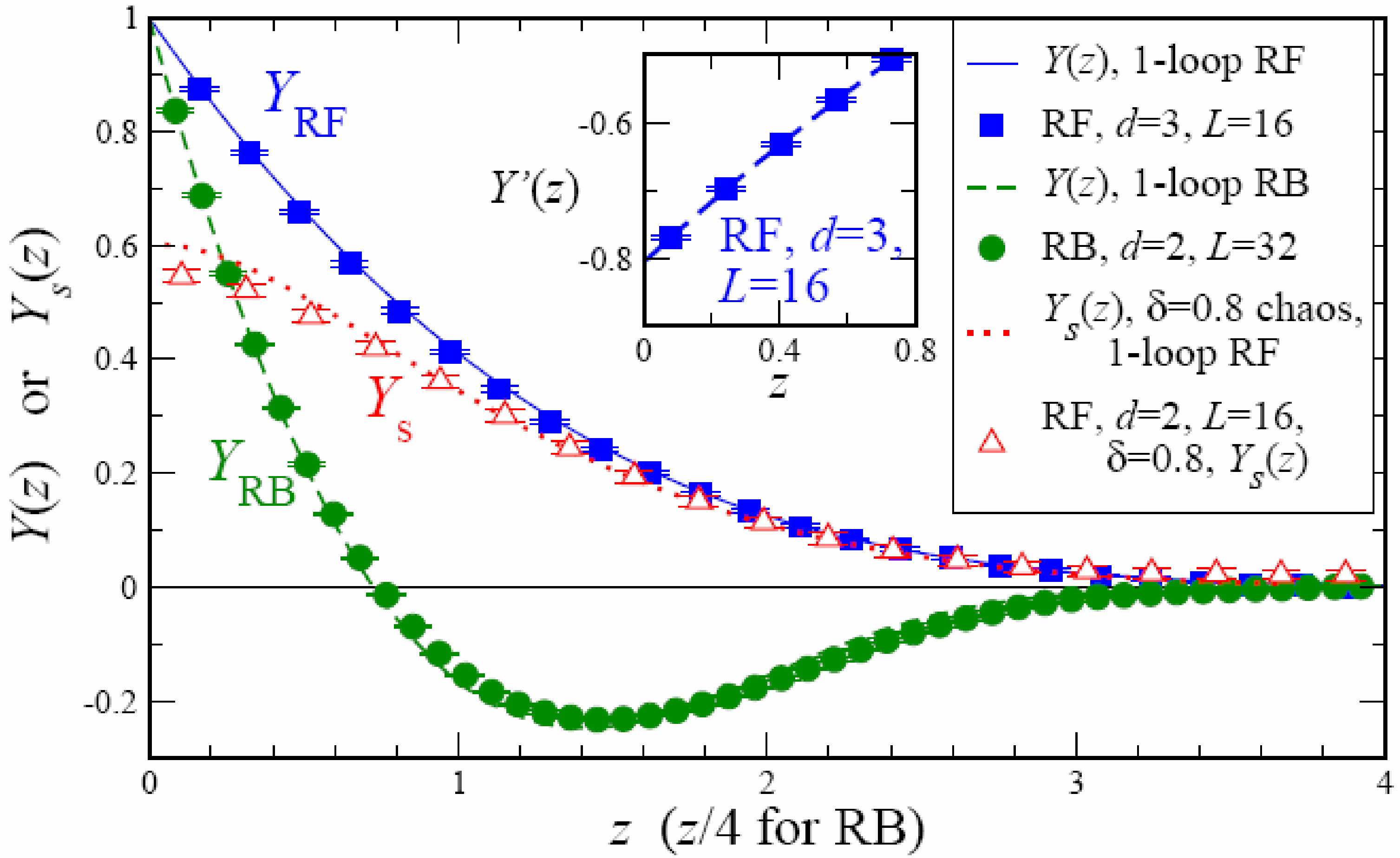
  for statics and depinning: confirms main features

    - *experimental tests ?*
        - *DW in magnetic film w/field gradient*
        - *contact line of fluid in partial wetting/capillarity*

E. Rolley, S. Moulinet

- *random field O(N) model*
- *chaos*
- *2D connections to fermions, nearly conformal FT*

chaos



$Y(z)$ or $Y_s(z)$
z (z/4 for RB)
Y_{RF}
Y_s
Y_{RB}
$Y'(z)$
RF, d=3, L=16
z
$Y(z)$, 1-loop RF
RF, d=3, L=16
$Y(z)$, 1-loop RB
RB, d=2, L=32
$Y_s(z)$, δ=0.8 chaos, 1-loop RF
RF, d=2, L=16, δ=0.8, $Y_s(z)$